Topology Project Ideas

TG 1

Spring 2021 Mentor: Wesley Hamilton

Text: *Discrete Morse Theory* by Scoville

Project: Morse theory studies the interplay between nice functions on a surface, and the topology/features of said surface; the features we are usually interested in are the connected components, holes, cavities, etc. Discrete Morse theory is a discrete version, where surfaces are replaced by triangulations, but the features we care about are the same. Recently, connections between discrete Morse theory and a number of applied problems were discovered, including: how do you know if a network of sensors completely covers a region to be sensed? what is the structure of a rat's brain? what's an efficient way to play 20 questions?
Through this project, students will learn the basic notions of simplicial complexes and topology of surfaces. Depending on the student's interest, they can then either focus on the more computational aspects, or start to explore Morse theory, towards the end of the project.

Suggested Prerequisits: 381, Helpful: 547, 521, 550.

Keywords: *topology, morse theory, homology, persistent homology, analysis, discrete*

TG 2

Text: *Perspectives on Projective Geometry: A Guided Tour Through Real and Complex Geometry* by Richter-Gebert

Project: In this project you will first learn about projective transformations and the equivalence of the axiomatic definition and the definition in homogeneous coordinates. Then we will look at some geometric constructions and how to use determinants/wedge products to determine incidence relations.

Suggested Prerequisites: Basics of linear algebra: linear transformations, determinants and wedge products

Keywords: *projective geometry, geometry, linear algebra*

TG 3

Text: *Topology and Groupoids* by Ronald Brown

Project: Brown's text is a modern take on point-set topology that emphasizes ideas and constructions that are important in algebraic topology. The standard tools are all here: product, quotient, compact, metric, and connected spaces are covered. But these are treated very categorically by Brown, and that is on purpose as the latter half of the book focuses on an important invariant called the Fundamental Groupoid. This project differs from the traditional course (Math 550) by looking at point-set topology with a different perspective, by learning about categories, and by learning about the Fundamental Groupoid. A great final presentation would be the Seifert-van Kampen theorem and a proof that the fundamental group of a circle is the additive group of integers. After reading this text and learning some group theory the student would be ready to read Hatcher's Algebraic Topology.

Suggested Prerequisites: Math 381 is highly recommended, but if the student is already comfortable working with sets then that will be enough. No previous experience with topology, groups, or categories will be required as that will be covered in the reading.

Keywords: *topology, categories, fundamental group, fundamental groupoid, Seifert- van Kampen theorem, algebraic topology*

TG 4

Text: *Undergraduate Algebraic Geometry* by Miles Reid

Additional Materials: *Algebraic Geometry* by Harris

Project: We will work through some of the most classical parts of the theory of smooth curves and surfaces over the complex numbers. The culmination of this project will be an understanding of the 27 lines on the cubic surface theorem.

Suggested Prerequisites: Linear algebra, discrete math, MATH 233

Keywords: *algebraic geometry*

TG 5

Text: *An Introduction to Algebraic Topology* by Rotman

Additional Materials: *Topology* by Munkres; *Algebraic Topology* by Hatcher

Project: Two groupings of topics in algebraic topology lend themselves well to a semester-long reading project, homology theory and homotopy groups. A potential reading project on homology theory covers homology and cohomology and culminates in a final presentation on Poincare duality. In order to minimize algebraic prerequisites, one can work over field coefficients. In this case, the only algebraic prerequisite would be linear algebra. Alternatively, students with a strong background in algebra could consider arbitrary coefficients. In both cases, knowledge of basic point-set topology is necessary and any additional point-set topics can be learned as needed. A project on homology theory should be relatively accessible with minimal background in topology and algebra. A project on homotopy groups, on the other hand, would likely be more demanding of the student.

A potential reading project on homotopy groups includes a quick review or overview of the fundamental group and covering spaces before generalizing to arbitrary homotopy groups and fibrations. Some candidates for a final project are Whitehead's theorem and connections between homotopy theory and homology theory (provided, of course, that the student knows homology theory). For such a project, a strong background in algebra is required. In terms of topological background the content of MATH 550 is sufficient. However, a good understanding of the fundamental group and covering spaces would help the student to progress through the material faster. Again, specific results in point-set topology (partitions of unity, Urysohn's lemma, etc.) can be learned as needed.

Suggested Prerequisites: point-set topology (MATH 550 is sufficient), linear algebra (preferably MATH 577)

Keywords: *algebra, topology, algebraic topology*

TG 6

Text: *Topology from the Differentiable Viewpoint* by John Milnor

Additional Materials: Analysis II, Tao

Project: Differential topology is a field of mathematics concerned with topological spaces that carry additional analytic structure. If you have taken a multivariable calculus course, you will have already encountered such spaces: curves and surfaces in $\mathbb{R}^3$. In a typical multivariable calculus course, we extend the familiar analytic notions of differentiation and integration to curves and surfaces by means of a parametrization, an identification of a curve to $\mathbb{R}$ and a surface to $\mathbb{R}^2$. In differential topology, we consider more general spaces admitting parametrizations that allow us to transfer onto them the analytic structure of $\mathbb{R}^n$. The resulting analytic structure on the space is deeply intertwined with the underlying topology. A central theme in differential topology is that analytic properties force topological features to appear and vice versa. This project is a broad survey of these connections between analysis and topology. The main text, Topology from the Differentiable Viewpoint, explores a varied selection of classical topics in differential topology. The text is mostly self contained, but a foundation in multivariable analysis (MATH 522) would be extremely beneficial. The project can, however, be approached with nothing more than a multivariable calculus (MATH 233) background. In this case, we will supplement the reading with selections from Tao's Analysis II. Despite the nomenclature, knowledge of topology is helpful but not essential. The project will culminate in a final presentation on a central idea or result to one topic of the main text of your choosing.

Suggested Prerequisites: MATH 233

Keywords: *topology, analysis, manifold, differential topology*

TG 7

Text: *An Introduction to Manifolds* by Loring Tu

Project: A term that many undergraduates will hear thrown around frequently without clear understanding is the term "manifold." While the general, intuitive idea is clear after briefly perusing Wikipedia, the precise mathematical description of manifold can prove somewhat esoteric for many students for years. This leads to the question asked by many undergraduates: "What exactly is a manifold?" This is the question that we aim to answer. We will start by discussing the concept of a manifold from both an intuitive and precise viewpoint. From there, we will select from the variety of objects fundamentally studied on manifolds (functions, vector fields, pushforwards, pullbacks, exterior derivative, Lie derivatives, differential forms, etc.) and provide a brief overview, depending on the background of the student. An example schedule, for a student with the listed prerequisites, could be something like studying manifolds, functions between manifolds, differential forms on manifolds, and the integration of differential forms. A student lacking some of the prerequisites might pursue something less advanced and more tractable, with prerequisites filled in as needed.

Suggested Prerequisites: MATH 381 is required, MATH 521 is highly recommended, MATH 547/577 and/or MATH 550 would be useful

Keywords: *topology, manifolds*

TG 8

Text: *From Geometric Calculus to Clifford Algebra* by David Hestenes

Additional Material: *Geometric Algebra* by Eric Chisholm

Project: This reading course will lead to a new way of understanding the unifying themes behind basic geometric operations (reflection, projection, rotation), Grassmann algebras and complex numbers. We will learn about the so called geometric product, sometimes called the Clifford product, of multivectors (and what a multivector is). The algebra we will study has found favor among some in the physics community for, among other things, aiding in the geometric interpretation of the Dirac equation.

Suggested Prerequisites: MATH 233 and nothing else

Keywords: *geometric algebra, Clifford Algebras*

TG 9

Text: *Applications of Lie Groups to Differential Equations* by Peter Olver

Project: "Geodesics" are the shortest path between two points in a curved space. They can be determined by solving a differential equation the depends on the metric of the space, but these same differential equations can be framed in a physical context as well. The physical way of thinking allows us to tap into Noether's theorem, which relates symmetry and conserved quantities. This project aims to answer the question, "when is a space 'symmetric enough' to solve the geodesic equations exactly?" In so doing, we will see a number of classical results that tie physics to the geometry of curved spaces.

Suggested Prerequisites: Calculus III (Math 233), differential equations (Math 383) recommended.

Keywords: *calculus of variations, geometry, physics, symmetry*

TG 10

Spring 2021 Mentor: Wesley Hamilton

Text: *A Topological Picture Book* by George Francis

Project: The goal of this reading project is to understand the theory and historical development of sphere eversions: how can one turn a sphere inside-out, where the surface of the sphere is allowed to pass through itself but the sphere isn't allowed to develop any creases. This project will be a deep reading of chapter 6 in Francis' "A Topological Picture Book." The primary goal is to understand the main techniques for producing sphere eversions, including the corrugation method and halfway-models. Two directions this project could go are either 1) (more theoretical) dive deeper into eversion techniques for higher-dimensional spheres or other kinds of surfaces, or 2) (more practical) explore computational methods used for visualizing and displaying eversions.

Suggested Prerequisites: Topology (Math 550) or Geometry (Math 551) is preferred, but not necessary.

Keywords: *topology, surgery, spheres, sphere eversion, homotopy theory*