Algebra Project Ideas

ALG 1

Spring 2021 Mentor: Wesley Hamilton

Text: *Ideals, Varieties, and Algorithms* by Cox, O'Shea, Little

Additional Materials: *Groups of Prime Power Order* by Berkovich, and some online articles

Project: The fundamental theorem of algebra states that any single variable polynomial factors into linear terms over the complex numbers, i.e. a polynomial is completely determined by its roots. Does the same hold for multivariable polynomials? In some sense, yes; the result is called Hilbert's Nullstellensatz, and relates the zero set of a family of polynomials to the algebraic structure (the ideal) generated by the polynomials.
In this DRP, we will follow the algorithmic approach in the book Ideals, Varieties, and Algorithms to 1) state and prove Hilbert's Nullstellensatz, 2) explore the geometric intuition underlying this theorem, and 3) utilize Mathematica (or Python) to carry out the various polynomial algorithms given in the text. The topics explored in this DRP will provide a solid foundation for any future algebraic explorations, including commutative algebra and algebraic geometry.

Suggested Prerequisites: Math 381 is necessary. Math 533, 534, or 578 are helpful but not necessary.

Keywords: *algebra, geometry, ideals, computational algebra, algebraic geometry*

ALG 2

Text: *Basic Category Theory* by Tom Leinster

Project: Leinster's book is a great introduction to Category Theory for those not familiar with it already. Category theory is sometimes called "the mathematics of math" because it is a framework that seeks to understand mathematical objects not by their intrinsic properties but by their relation to similar objects. This is a highly abstract subject, so understanding comes from working with concrete structures but observing that the idea is more general. This project will sharpen your proof reading and writing skills, and give you perspective on the constructions and ideas you'll see in more advanced algebraic courses. Examples are what give Category Theory life, so the student will potentially be exposed to mathematical ideas and subjects they've never seen nor heard; this makes this a great project for those who want to see a wider array of mathematical theories but without the depth. The student should expect to learn about functors, natural transformations, Yoneda lemma, adjoint functors, limits and colimits, and representable functors. The final presentation could be on almost any application.

Suggested Prerequisites: Math 381 and Math 521 for proof skills, and some knowledge of sets, functions, vector spaces, and linear maps for motivating examples.

Keywords: *category theory, functors, natural transformations, (co)limits, adjoint functor (theorems), Yoneda lemma, representability*

ALG 3

Text: *A Course in Arithmetic* by Jean-Pierre Serre

Project: Student can expect to learn about several number theory topics: quadratic reciprocity, p-adic numbers, classification of quadratic forms, and/or modular forms. Possible presentation topics: Lagrange's four-square theorem (and maybe an alternate proof with quaternions), Weil conjectures/Petersson conjecture/Ramanujan conjecture.

Suggested Prerequisites: Basic linear algebra (matrix multiplication, determinant), familiarity with the notions of a group, a ring, and a field.

Keywords: *number theory, modular forms, quadratic forms, p-adic, modular arithmetic*

ALG 4

Text: *Classical Galois Theory* by Lisl Gaal

Project: Student can expect to learn about field extensions of the rationals, groups of field automorphisms, roots and properties of polynomials, the correspondence between field extensions and Galois groups, unsolvability of the quintic, quartic and cubic formulas, roots of unity, and cyclotomic polynomials. Presentation topics could be: the main theorem of Galois theory, unsolvability of the quintic.

Suggested Prerequisites: Familiarity with the notion of a field, of a group, basic knowledge of complex numbers

Keywords: *Galois theory, polynomials, fields, groups, quintic, roots, complex numbers*

ALG 5

Text: *Rational Points on Elliptic Curves* by Silverman and Tate

Project: Student can expect to learn about plane curves defined by rational polynomial equations (both quadratics and cubics), elliptic curves in particular, group law on an elliptic curve, points of finite order on an elliptic curve, integer points, Diophantine approximation. Possible presentation ideas: proof of the Nagell-Lutz theorem, applications to cryptography, relation to Fermat's last theorem.

Suggested Prerequisites: Familiarity with the notion of a group, modular arithmetic

Keywords: *elliptic curve, groups, abelian, integer solutions, Fermat's last theorem, cryptography, Diophantine equations, number theory, finite order*

ALG 6

Text: *A Classical Introduction to Modern Number Theory* by Ireland & Rosen

Additional Materials: *An Introduction to the Theory of Numbers* by Nivens

Project: Through the first five chapters of the book, we will discuss unique factorization and congruence, culminating in a proof of the famous Law of Quadratic Reciprocity (the first major goal). From here, we move on to studying as many Diophantine equations as time permits (Pell equations, Fermat equations, etc) and elementary algebraic geometry, supplementing with the theory as necessary. Finally, we will study L-functions and some analytic number theory, culminating with a discussion of elliptic curves.

Suggested Prerequisites: MATH 381 and/or familiarity with proofs. MATH 676/578 highly recommended (we need basic results from group theory, field theory, and complex analysis) but not necessary.

Keywords: *number theory, algebra*

ALG 7

Text: *Quantum Calculus* by Kac & Cheung

Project: Quantum Calculus by Kac is short and approachable for a motivated first or second year undergrad, and most of our time will be spent working on understanding and writing proofs. The material itself is intriguing on its own, but has its uses in representation theory of Lie algebras. "Quantum" here is the mathematical usage of the term, which typically means a "deformation" of something you already know or non-commutativity. In a sentence, quantum calculus is "calculus without taking limits," and it has two types that are related. The main goal will be to cover as much material as possible, and to understand the relationship between these two types. We will explore the early chapters in depth, specifically the q-derivative and q analogues of concepts the student will have seen in calculus, like Taylor's formula, binomial coefficients, and trigonometric and exponential functions. From there if the student has interest in special functions, combinatorics, or number theory, then applications of q-calculus can be explored.

Suggested Prerequisites: This project is designed for a younger undergrad who has a strong grasp of the 231, 232, and 233 material. Any linear algebra we may need can be picked up along the way.

Keywords: *calculus, combinatorics, special functions, gamma function*

ALG 8

Text: *Character Theory of Finite Groups* by M. Isaacs

Additional Materials: *Groups of Prime Power Order* by Berkovich, and some online articles

Project: The student will learn basic representation theory of finite groups, working from Isaacs book. We could work on an application to finite unipotent groups, for example: 1) iterative construction of characters via the little groups method, or general semi-direct product method; 2) classifying new families of conjugacy classes and/or coadjoint orbits (in the second case, compute polarizations, dimensions, defining ideals); 3) supercharacter theories. Enumerating the irreducible representations of this group is an "intractable" problem, but finding partial results like these is still useful research and good experience. A lot of work can be done on both projects with just basic combinatorics and not a lot of background knowledge.

Suggested Prerequisites: Abstract algebra courses like 534 or 578, and comfortable with basic combinatorics

Keywords: *finite groups, finite unipotent groups, character theory, representation theory*