

## Analysis Project Ideas

### ANA 1

**Spring 2021 Mentor:** Wesley Hamilton

**Text:** *Fourier Analysis* by Stein and Shakarchi

**Project:** Fourier analysis/harmonic analysis is the study of functions by decomposing them with respect to nice bases; classical Fourier analysis decomposes periodic functions into linear combinations of sines and cosines. While this decomposition is useful for solving certain PDEs, Fourier analysis has also proved effective in other domains, including: signal processing/time series data analysis, geometry and topology, and number theory.

In this DRP, we will follow Stein and Shakarchi's "Fourier Analysis" text. Depending on the interests and background of the student, possible goals include: use the Fourier transform to solve PDEs; use the Fourier transform to study finite groups, and/or connect these ideas to signal processing application; explore uses of the Fourier transform in number theory, including Dirichlet's theorem: if  $q$  and  $l$  are relatively prime positive integers, then there are infinitely many primes of the form  $l + kq$  with  $k$  an integer).

**Suggested Prerequisites:** Math 383, Math 521 (preferred but not necessary)

**Keywords:** *PDEs, spectral theory, Fourier transform, Fourier analysis, harmonic analysis, number theory*

### ANA 2

**Text:** *Fractal Geometry: Mathematical Foundations and Applications* by Kenneth Falconer

**Project:** Fractals occur in many places, from dynamics to number theory to non-differentiable functions just some plain cool-looking pictures. This project will involve an introduction to the tools of fractal geometry, especially concepts of fractal dimension and measure. Once the basics are established, specific topics can be narrowed in on, including but not limited to number theory, fractal groups, graphs of real functions, fractals arising from dynamical systems, and random fractals.

**Suggested Prerequisites:** A familiarity with basic real analysis (MATH 521) will be beneficial, but not necessary.

**Keywords:** *real analysis, fractal geometry*

### ANA 3

**Text:** *Differential Equations and Dynamical Systems* by Lawrence Perko

**Additional Materials:** Chris Jones' lecture notes on dynamical systems

**Project:** We will study the basics of dynamical systems. Depending on the interest of the student, we can focus more on linear dynamical systems and finding exact solutions, or we can focus on low-dimensional nonlinear systems and qualitative techniques for approximating solutions. It is likely we will do at least some of both. A final presentation could be on the Fundamental Theorem for Linear Systems, or it could be an analysis of a particular applied dynamical system and how approximation techniques give an idea of what solutions look like.

**Suggested Prerequisites:** 383, 547, (521 also preferred)

**Keywords:** *dynamical systems, differential equations*

### ANA 4

**Text:** *Partial Differential Equations* by Mark S. Gockenbach

**Additional Materials:** A copy of MATLAB (only if the student is interested in some numerical applications)

**Project:** The goal of this project is to understand how to generalize the notions of eigenvalues and eigenvectors from matrix algebra (the finite dimensional case) to partial differential operators (the infinite dimensional case). We will learn how to find eigenvalues and "eigenfunctions" for these operators and how to use them in constructing solutions to common boundary value problems that arise in the field of partial differential equations, such as the heat equation and wave equation. This is sometimes referred to as the "spectral method" of solving partial differential equations. A possible final presentation idea could be showing this construction for an interesting example PDE such as the Convection-Diffusion equation, or you could demonstrate some numerical results if that is the direction you prefer.

**Suggested Prerequisites:** MATH 233 and 383. A good understanding of eigenvalues and eigenvectors of matrices will be very helpful. If the student wants to do numerical work, some coding experience would be strongly recommended.

**Keywords:** *spectral theorem, partial differential equations, eigenfunctions, fourier series*

### ANA 5

**Text:** *A Guide to Distribution Theory and Fourier Transforms* by Robert Strichartz

**Project:** Is the Dirac-delta function really a function? If it is, can you differentiate it? Can you differentiate a function that isn't continuous? These questions can be answered in the language of distribution theory. Distributions, sometimes called "generalized functions," are not quite functions, but we can still make sense of them in a structured way. We can even differentiate them...in a certain sense. Distribution theory opens up a vast and interesting branch of analysis and has applications to some very central questions in partial differential equations. For example, the fundamental solution of the wave equation is not a function, but it IS a distribution, and this allows us to still compute with it, so long as we understand what kind of object we're working with. Distributions also fit quite nicely into a study of the Fourier transform, and so a reasonable final presentation idea could be a (perhaps informal) proof of the Fourier inversion formula on the space of distributions.

**Suggested Prerequisites:** MATH 521 is absolutely required. A partial differential equations course is recommended, but not strictly necessary. Familiarity with the Fourier transform would help, but is not required. Be aware that this is a fairly advanced subject.

**Keywords:** *distribution theory, fourier transform, partial differential equations, weak derivatives*

### ANA 6

**Text:** *Resources for the Study of Real Analysis* by Robert Brabenc

**Additional Materials:** May occasionally need to reference basic analysis concepts and definitions (limits, derivatives, integrals, etc.) depending on the background of the student.

**Project:** The story of the development of Calculus and Analysis is fascinating. Newton and Leibniz first discovered Calculus in the mid 16th century; however it was not until the 19th century that mathematicians saw the need for greater rigor and precision in the subject, leading to the formulation of limits, derivatives, and integrals as we know them today. The clean and precise treatments of Calculus and Analysis in typical undergraduate courses, though efficient, can leave the student with a cloudy understanding of the need for certain technical notions. In this project, we will enrich our understanding of and appreciation for Calculus and Analysis by taking a problem based approach with emphasis on the historical context. From this, students will gain an understanding of the historical development of Analysis, and will also engage with some of the key problems that led to the subject taking the form that it has today. In contrast to typical courses, the explicit main priority of this project is not to gain technical fluency with some mathematical concepts, but is instead to help us love, appreciate, and enjoy mathematics more.

**Suggested Prerequisites:** MATH 231, MATH 232, MATH 381. The project will be adapted based on whether or not the student has taken MATH 521.

**Keywords:** *calculus, analysis, history of calculus, Euler, Newton, Cauchy, infinite series, Bernoulli family, limits*

### ANA 7

**Text:** Notes provided.

**Project:** Semiclassical analysis is a branch of mathematical analysis dedicated to developing techniques for studying partial differential equations motivated by the classical-quantum correspondence. In this reading, we will begin with a brief overview of classical and quantum mechanics, as a means of providing motivation for the subject, demonstrating analogies between the two along the way. From here, we will go through some mathematical preliminaries, namely basic Fourier analysis and stationary phase. Finally, we will discuss the concept of quantization, the mathematical realization of the classical-quantum correspondence. We will develop elementary properties of quantization to demonstrate their encapsulation of this connection. A reasonable goal would be to demonstrate how quantization preserves that the Poisson bracket in classical mechanics corresponds to the commutator in quantum mechanics, with the level of rigor depending on the background of the student. Disclaimer: This is a fairly advanced topic.

**Suggested Prerequisites:** MATH 383 is required, MATH 521 is very highly recommended

**Keywords:** *PDE, analysis, fourier*

### ANA 8

**Text:** *Introductory Functional Analysis with Applications* by Erwin Kreyszig

**Project:** In some sense, functional analysis is the extension of linear algebra to the infinite-dimensional setting. Depending on the level of abstraction, this can require increased levels of background analysis needed. Some of this abstraction can obfuscate the direct analogies between the two areas. The goal of this reading is to discuss such infinite dimensional analogues to some of the core material in linear algebra, namely the study of normed spaces, inner product spaces, and linear operators between such spaces. This is appropriate for students with MATH 381 and MATH 547 background, although MATH 521 would be very useful. Some final goals, in increasing order of difficulty, would be the Hahn-Banach theorem, the Baire category theorem and consequences, or spectral theory. One could also study applications, such as the Banach fixed point theorem applied to differential and integral equations, approximation theory, or (for the highly ambitious student) quantum mechanics (this requires the study of almost all of the above topics and unbounded operators, which are somewhat scary).

**Suggested Prerequisites:** MATH 381, 547 are required, MATH 521 would be useful (although not strictly needed)

**Keywords:** *analysis, functional analysis, operators*