Math 232 – Fall 2017 – Thomas – Test 3 Name Key

#1-5 Multiple Choice (11points each)

1. Find a power series representation for $f(x) = \frac{1}{9+x^2}$ and determine its interval of convergence.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$ with interval of convergence $(-3,3)$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^{n+1}}$ with interval of convergence $(-3,3)$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^n}$ with interval of convergence $(-1,1)$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{9^{n+1}}$ with interval of convergence $(-1,1)$

e) $\sum_{n=0}^{\infty} \frac{x^{2n}}{9^{n+1}}$ with interval of convergence $(-3,3)$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{9+x^2} = \frac{1}{9}\left(\frac{1}{1+\frac{x^2}{9}}\right) = \frac{1}{9}\left(\frac{1}{1-\left(\frac{-x^2}{9}\right)}\right)$$

$$= \frac{1}{9}\sum_{n=0}^{\infty}\left(\frac{-x^2}{9}\right)^n = \frac{1}{9}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^n}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{9^{n+1}}}$$

Interval:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{9^{n+1+1}}\cdot\frac{9^{n+1}}{(-1)^n x^{2n}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{9^{n+2}}\cdot\frac{9^{n+1}}{x^{2n}}\right| = \lim_{n\to\infty}\frac{1}{9}|x^2|$$

$$= \frac{1}{9}|x^2| < 1 \qquad |x^2| < 9 \qquad |x| < 3 \qquad \boxed{(-3,3)}$$

2. Find the Maclaurin series for $f(x) = x^5 \sin x^2$.

a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+7}}{(4n+7)(2n+1)!}$

b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$

c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+7}}{(2n+1)!}$

d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+6}}{(2n+1)!}$

e) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+6}}{(2n+6)(2n+1)!}$

using Maclaurin Table:

$$\sin x^2 = \sum_{n=0}^{\infty}\frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$f(x) = x^5 \sin x^2 = x^5\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{4n+7}}{(2n+1)!}}$$

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3. Using series to find the $\lim_{x\to 0}\frac{x^3-3x+3\tan^{-1}(x)}{x^5}$ will simplify to finding which of the following limits?

$\tan^{-1}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}$ by Maclaurin Table

a) $\lim_{x\to 0}\left(\frac{3x^2}{5}-\frac{x^4}{3}+\frac{x^6}{2}-\cdots\right)$

b) $\lim_{x\to 0}\left(\frac{3}{5}-3x+\frac{x^3}{3}-\frac{x^7}{7}+\cdots\right)$

c) $\lim_{x\to 0}\left(\frac{3x^2}{7}-\frac{x^4}{3}+\frac{x^6}{2}-\cdots\right)$

d) $\lim_{x\to 0}\left(\frac{3}{5}-x^2+x^4-\cdots\right)$

e) $\lim_{x\to 0}\left(\frac{3}{5}-\frac{3x^2}{7}+\frac{x^4}{3}-\cdots\right)$

$\lim_{x\to 0}\frac{x^3}{x^5}-\frac{3x}{x^5}+\frac{3\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}}{x^5}=$

$\lim_{x\to 0}\frac{1}{x^2}-\frac{3}{x^4}+3\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-4}}{2n+1}=$

$\lim_{x\to 0}\cancel{\frac{1}{x^2}}-\cancel{\frac{3}{x^4}}+\cancel{\frac{3}{x^4}}-\cancel{\frac{3}{3x^2}}+$

$\boxed{\frac{3}{5}-\frac{3x^2}{7}+\frac{3x^4}{9}+\cdots}$

4. Find the first 3 nonzero terms in the Taylor series for $f(x)=\cos x$ centered at $a=\pi$.

Taylors $\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$

a) $(x-\pi)-\frac{(x-\pi)^3}{3!}+\frac{(x-\pi)^5}{5!}$

b) $(\pi x)-\frac{(\pi x)^3}{3!}+\frac{(\pi x)^5}{5!}$

c) $(x-\pi)+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}$

d) $-1+\frac{(\pi x)^2}{2!}-\frac{(\pi x)^4}{4!}$

e) $-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}$

$f(x)=\cos x$ $\quad f(\pi)=-1$ $\quad n=0$

$f'(x)=-\sin x$ $\quad f'(\pi)=0$ $\quad n=1$

$f''(x)=-\cos x$ $\quad f''(\pi)=1$ $\quad n=2$

$f'''(x)=\sin x$ $\quad f'''(\pi)=0$ $\quad n=3$

$f''''(x)=\cos x$ $\quad f''''(\pi)=-1$ $\quad n=4$

$\sum_{n=0}^{\infty}\frac{f^n(\pi)}{n!}(x-\pi)^n=\frac{-1}{0!}(x-\pi)^0+0+\frac{1}{2!}(x-\pi)^2+0-\frac{1}{4!}(x-\pi$

$\boxed{=-1+\frac{(x-\pi)^2}{2!}-\frac{(x-\pi)^4}{4!}}$

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5. Find the sum of the series $1+2+\frac{4}{2!}+\frac{8}{3!}+\frac{16}{4!}+\cdots$ $= \sum_{n=0}^{\infty} \frac{2^n}{n!}$

a) e^2

b) -1

c) $\ln 3$

d) 2^k

e) $\sin 2$

$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ where $x=2$

$f(x) = e^x$ from Maclaurin Table

where $x=2$

$f(2) = \boxed{e^2}$

#6-7 True/False (5 points each)

6. The Radius of Convergence for $\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n$ is $R=\frac{1}{2}$. TRUE FALSE

$\lim_{n\to\infty}\left|\frac{x^{n+1}}{2^{n+1}}\cdot\frac{2^n}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{2} = \frac{|x|}{2} < 1$

$|x| < 2$

$\boxed{R=2}$

7. If a series $\sum_{n=0}^{\infty} a_n$ is convergent, then the series $\sum_{n=0}^{\infty} a_n$ is also absolutely convergent.

TRUE FALSE

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#8-10 Free Response/Partial Credit(#8-9 worth 13 points each, #10 worth 9 points)

8. Write the function as a power series $f(x) = \ln(x^2 + 2)$.

$f(x) = \ln(x^2+2)$

$f'(x) = \dfrac{2x}{x^2+2} = 2x\left(\dfrac{1}{2+x^2}\right) = \dfrac{2x}{2}\left(\dfrac{1}{1+\frac{x^2}{2}}\right) = x\left(\dfrac{1}{1-\left(-\frac{x^2}{2}\right)}\right)$

$= x\sum_{n=0}^{\infty}\left(\dfrac{-x^2}{2}\right)^n = x\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{2^n} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{2^n}$

$f(x) = \int f'(x)\,dx = \int \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{2^n}\,dx = \boxed{\sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+2}}{(2n+2)(2^n)} + C}$

for fun: $f(0) = \ln(0^2+2) = \ln(2)$; plug zero into Σ for x;
$f(0) = C$; therefore $C = \ln(2)$

9. Use a power series to approximate the definite integral $\int_0^{\frac{1}{10}} e^{-x^2}\,dx$ with $|error| \le \frac{1}{300}$.
(Helpful information: $.1^2 = \frac{1}{100}$ $.1^3 = \frac{1}{1000}$ $.1^4 = \frac{1}{10000}$, etc)

Using Maclaurin Table:

$e^{-x^2} = \sum_{n=0}^{\infty}\dfrac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{n!}$

$\int_0^{\frac{1}{10}} e^{-x^2}\,dx = \int_0^{\frac{1}{10}}\dfrac{(-1)^n x^{2n}}{n!} = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)\,n!}\Bigg]_0^{\frac{1}{10}} = \left[\dfrac{x}{1} - \dfrac{x^3}{3} + \dfrac{x^5}{5\cdot 3!} + \cdots\right]$

$= \dfrac{1}{10} - \dfrac{\left(\frac{1}{10}\right)^3}{3} + \dfrac{\left(\frac{1}{10}\right)^5}{5\cdot 2!} + \cdots$

$= \dfrac{1}{10} - \dfrac{1}{3000} + \dfrac{1}{1000000} + \cdots$

alternating series so $|R_n| \le b_{n+1} \le \frac{1}{300}$

$b_0 = \frac{1}{10}$ not satisfy $\le \frac{1}{300}$

$b_1 = \frac{1}{3000}$ satisifies $\le \frac{1}{300} \Rightarrow R_0 \Rightarrow S_n = S_0 = b_0 = \boxed{\frac{1}{10}}$

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10. Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$. Must prove all statements.

Alternating Series → yes

limit = 0 → yes

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$$

decreasing → yes

$f(x) = n^{-\frac{1}{3}}$

$f'(x) = -\frac{1}{3} n^{-\frac{4}{3}} = \frac{-1}{3\sqrt[3]{n^4}} < 0$

$-1 < 0$ always

$\therefore$ f(n) is decreasing for all n

By alternating series Test,

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$ converges