Math 233, Final Exam, Spring 2018

May 3rd, 2018

Last name . . . .

First name . . . .

UNC E-MAIL (ONYEN) . . . .

- Closed book, closed notes, no calculators.
- Show photo ID when turning in exam.
- Partial credit is important—try all problems.
- Take integrals as far as you can analytically, leaving them as iterated or definite integrals if you must.
- You must show full analytical work to receive full credit, even on the multiple choice problems.
- By putting your name on your paper, you implicitly pledge your adherence to the honor code.

Title page continued.

1. (10 pts) Let $\mathbf{F}(x, y, z) = \langle x\cos(x^2), ze^{y^2}, z^3\rangle$.
Does there exist a function $f(x, y, z)$ defined on all of $\mathbb{R}^3$ with $\mathbf{F} = \nabla f$?
Circle your answer and justify (show your work).

(a) Yes, such a function exists (and your work demonstrates why).
(b) No, such a function does not exist because $\mathbf{F} \neq \mathbf{0}$.
(c) No, such a function does not exist because $\nabla \times \mathbf{F} \neq \mathbf{0}$.
(d) No, such a function does not exist because $\nabla \cdot \mathbf{F} \neq 0$.
(e) Not enough information to determine.

Problem 1 continued.

2. (10 pts) Let $\mathbf{F}(x, y, z) = \langle yz\cos(y^2), xe^y, z^3\rangle$.
Does there exist a vector field $\mathbf{G}$ defined on all of $\mathbb{R}^3$ with $\mathbf{F} = \nabla \times \mathbf{G}$?
Circle your answer and justify.

(a) No, because $\nabla \times \mathbf{F} \neq \mathbf{0}$.
(b) No, because $\nabla \cdot \mathbf{F} \neq 0$.
(c) No, because $\mathbf{F}(0, 0, 0) = \mathbf{0}$.
(d) Yes, such a function exists (and your work demonstrates why).
(e) Not enough information to determine.

Problem 2 continued.

3. (10 pts) Let $f(x, y, z) = x + y^3 + z^2$. Identify the unit vector pointing in the direction in which f *decreases* fastest at the point $(0, 1, 2)$.

Problem 3 continued.

4. (10 pts) Evaluate $\int_0^3 \int_{x^2}^9 xe^{-y^2}\, dy\, dx$ by integrating over the same region with the order of the iterated integrals reversed.

Problem 4 continued.

5. (10 pts) Show that the vector field

$$\mathbf{F}(x, y, z) = \langle yz + y\cos(xy), xz + x\cos(xy), xy + e^z \rangle$$

is conservative and evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C follows a curve along the paraboloid $z = x^2 + y^2$ from $(0, 0, 0)$ to $(1, 1, 2)$.

Problem 5 continued.

6. (20 pts) Consider the function $z = f(x, y) = 4e^{xy} + xy$.

(a) Compute the first partial derivatives of f with respect to x and y.

(b) At the point $(1, 1)$, write the equation for the tangent plane to the surface described by the function.

(c) What is the linear approximation to f at the point $(1, 1)$?

Problem 6 continued.

7. (20 pts) Consider the cone $z^2 = x^2 + y^2$ between $z = 0$ and $z = 1$.

(a) Find the surface area of this cone.

(b) Find the volume of the region above this cone and inside the sphere of radius $\sqrt{2}$ centered at the origin that encloses the cone.

Problem 7 continued.

8. (20 pts) Find and correctly classify all of the local minima, local maxima, and saddles of $f(x, y) = x^2 + y^2 - x^2y$.

Problem 8 continued.

9. (20 pts) Evaluate $\int_C x^4\,dx + (x+\cos y)\,dy$ with C the boundary of the rectangular region defined by $-2 \leq x \leq 2$ and $-3 \leq y \leq 2$, oriented *clockwise.*

Problem 9 continued.

10. (20 pts) Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for $\mathbf{F}(x, y, z) = 2z\mathbf{i} + 3x\mathbf{j} + 4y\mathbf{k}$ where C is the circle of radius 1 in the xy-plane centered at the origin and oriented counterclockwise when viewed from above the xy-plane. (Do not evaluate the line integral; you must evaluate the integral obtained via Stokes' Theorem.)

Problem 10 continued.

11. (20 pts) Consider the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ with $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$ where S is the surface of the cylinder $y^2 + z^2 = 4$ with $0 \leq x \leq 2$. (a) Parametrize this surface and write down (but do not evaluate) the iterated integrals for the surface integral.

(b) Let S' be the closed surface with outward-facing normals obtained by taking the union of the surface S with the planes $x = 0$ and $x = 2$. Use the Divergence Theorem to evaluate the integral $\oiint_{S'} \mathbf{F} \cdot d\mathbf{S}$.

Problem 11 continued.