Math 130: Final Exam Version A
Spring 2019

- Calculators are NOT allowed.
- No partial credit for multiple choice / no work needs to be shown.
- For short answer questions, you must show work for full and partial credit.
- Sign the honor pledge below after completing the exam.
- Please put all work to be graded on the test itself.

First and last name .... Key ....

PID ....

UNC Email ....

I have not given or received unauthorized help on this exam.

Signature: ....

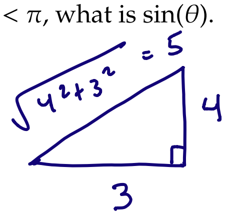
1. (3 points) Is $h(x) = x^2 \sin(x) - 4x$ an even or odd function?

   A. Even
   B. Odd
   C. Neither

$$h(-x) = (-x)^2 \sin(-x) - 4(-x)$$
$$= x^2(-\sin(x)) + 4x$$
$$= -x^2 \sin(x) + 4x$$
$$= -h(x)$$

2. (3 points) If $\tan(\theta) = -\frac{4}{3}$ and $0 < \theta < \pi$, what is $\sin(\theta)$.

   A. $-\frac{4}{5}$
   B. $-\frac{3}{5}$
   C. $\frac{3}{5}$
   D. $\frac{3}{4}$
   E. $\frac{4}{5}$



$|\sin\theta| = \frac{4}{5}$

Since $0 < \theta < \pi$,

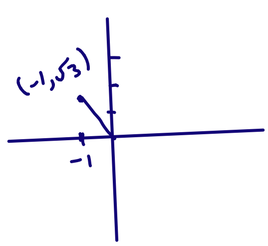
$\sin\theta$ is positive

so $\sin(\theta) = \frac{4}{5}$

3. (3 points) **Select two** polar points that are equivalent to the Cartesian point $\underset{}{\overset{(x,y)}{(-1, \sqrt{3})}}$.

   A. $(2, \frac{\pi}{3})$
   B. $(2, \frac{2\pi}{3})$
   C. $(2, \frac{5\pi}{6})$
   D. $(2, -\frac{5\pi}{6})$
   E. $(-2, \frac{2\pi}{3})$
   F. $(-2, \frac{\pi}{3})$
   G. $(-2, -\frac{\pi}{6})$
   H. $(-2, -\frac{\pi}{3})$

$$r^2 = x^2 + y^2 = (-1)^2 + (\sqrt{3})^2 = 1 + 3 = 4$$
$$r = \pm 2$$
$$\tan\theta = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$
$$\theta = \frac{2\pi}{3} \quad \text{or} \quad \theta = \frac{5\pi}{3}$$



$(2, \frac{2\pi}{3})$ or $(-2, \frac{5\pi}{3})$

or $(-2, -\frac{\pi}{3})$

4. (3 points) Convert the following system of parametric equations to a Cartesian equation.

$$\begin{cases} x(t) = t + 3 \\ y(t) = -2t^2 \end{cases}$$

A. $y = -2x^2$
B. $y = -2x^2 + 6$
C. $y = -2x^2 - 9$
D. $y = -2(x + 3)^2$
E. $y = -2(x - 3)^2$

$x = t + 3 \Rightarrow t = x - 3$
$y = -2t^2 \Rightarrow y = -2(x-3)^2$

5. (5 points) Which of the following equations represents a circle? Select all that apply.

1 pt each correct circling or not circling

A. $3y^2 + 24y - x^2 = 6$
B. $x = 4\sin(t) - 2,\ y = 4\cos(t) + 3$ → b/c $\frac{x+2}{4} = \sin t$ $\quad \frac{y-3}{4} = \cos t$
C. $r = -5$ → same as $r = 5$
D. $r = 7$
E. $r = 3\cos(\theta)$

$\Rightarrow \left(\frac{x+2}{4}\right)^2 + \left(\frac{y-3}{4}\right)^2 = 1$ since $\sin^2 t + \cos^2 t = 1$
$\Rightarrow$ ellipse with same length for major & minor axes $\Rightarrow$ circle

b/c $r^2 = 3r\cos\theta$
$x^2 + y^2 = 3x$
$x^2 - 3x + y^2 = 0$
$x^2 - 3x + (1.5)^2 + y^2 = (1.5)^2$
$(x - 1.5)^2 + y^2 = (1.5)^2$ circle.

6. (3 points) The minute hand of a clock is 5 inches long. What distance does the tip of the minute hand travel in 10 minutes?

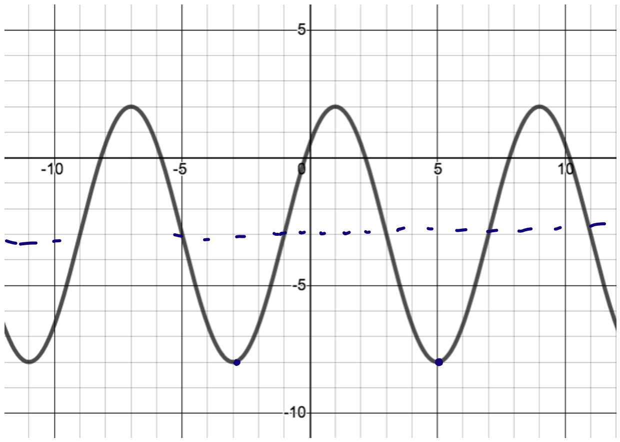
A. $\frac{\pi}{5}$ inches
B. $\frac{5\pi}{3}$ inches
C. $\frac{25\pi}{6}$ inches
D. 10π inches
E. 300 inches

arclength
$S = r \cdot \theta$ ← angle
radius

$\theta = \frac{10}{60} \cdot 2\pi = \frac{2\pi}{6} = \frac{\pi}{3}$

$S = 5 \cdot \frac{\pi}{3} = \frac{5\pi}{3}$

7. (3 points) Find the equation of this sinusoidal function.



A. $y = 3\cos(8x - 1) - 5$

B. $y = 5\cos(8(x - 1)) - 3$

C. $y = 3\cos\left(\frac{\pi}{4}x - 1\right) - 5$

D. $y = 5\cos\left(\frac{\pi}{4}x - 1\right) - 3$

E. $y = 5\cos\left(\frac{\pi}{4}(x - 1)\right) - 3$

period $= 8 \Rightarrow B = \frac{2\pi}{8} = \frac{\pi}{4}$

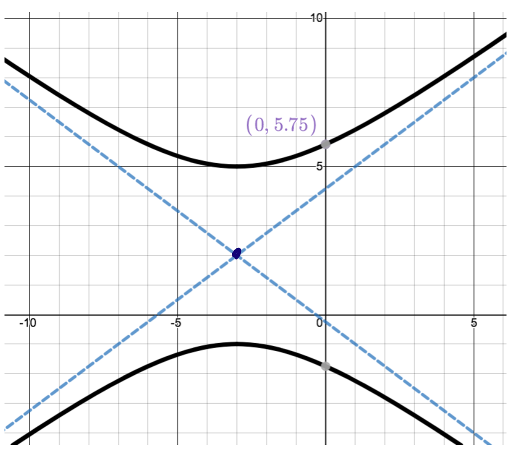
midline $= \frac{2 + (-8)}{2} = -\frac{6}{2} = -3$

amplitude $= 2 - (-3) = 5$

using cosine,
horiz shift is 1 to the right

$y = 5\cos\left(\frac{\pi}{4}(x - 1)\right) - 3$

8. (3 points) Find the equation of the curve graphed below.



A. $\frac{(y-2)^2}{9} - \frac{(x+3)^2}{16} = 1$

B. $\frac{(x+3)^2}{16} - \frac{(y-2)^2}{9} = 1$

C. $\frac{(y-2)^2}{9} - \frac{(x+3)^2}{49} = 1$

D. $\frac{(x+3)^2}{49} - \frac{(y-2)^2}{9} = 1$

Center: $(-3, 2)$

$a = 3$

slope of asymptote is $\pm \frac{3}{4}$

$b = 4$

$$\frac{(y-2)^2}{9} - \frac{(x+3)^2}{16} = 1$$

9. (6 points) Evaluate $\cos(\frac{\pi}{6} + \sin^{-1}(-\frac{5}{7}))$ exactly and simplify your answer.

$$\cos\left(\frac{\pi}{6} + \sin^{-1}\left(-\frac{5}{7}\right)\right) = \overset{1}{\cos\frac{\pi}{6}}\ \overset{2}{\cos\left(\sin^{-1}\left(-\frac{5}{7}\right)\right)} - \overset{3}{\sin\left(\frac{\pi}{6}\right)}\ \overset{4}{\sin\left(\sin^{-1}\left(-\frac{5}{7}\right)\right)}$$

1 $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

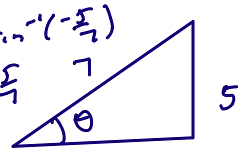
2 $\cos(\underbrace{\sin^{-1}(-\frac{5}{7})}_{\theta})$

3 $\sin(\frac{\pi}{6}) = \frac{1}{2}$

4 $\sin(\sin^{-1}(-\frac{5}{7})) = -\frac{5}{7}$

$\theta = \sin^{-1}(-\frac{5}{7})$

$\sin\theta = -\frac{5}{7}$



$$\sqrt{7^2 - 5^2} = \sqrt{49-25} = \sqrt{24}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{24}}{7}$$

$$\cos\left(\frac{\pi}{6} + \sin^{-1}\left(-\frac{5}{7}\right)\right) = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{24}}{7} - \left(\frac{1}{2}\right)\left(-\frac{5}{7}\right)$$

$$= \frac{\sqrt{72}}{14} + \frac{5}{14} = \frac{\sqrt{72}+5}{14}$$

$$= \boxed{\frac{6\sqrt{2}+5}{14}}$$

Answer: $\boxed{\frac{6\sqrt{2}+5}{14}}$ OR $\frac{\sqrt{72}+5}{14}$ is fine

10. (6 points) Find **all** solutions to the equation: $\cos(2x) + \sin^2(x) = \cos(x)$

$\cos(2x) = \cos^2(x) - \sin^2(x)$

$\cos(2x) + \sin^2(x) = \cos(x)$

$\cos^2(x) - \sin^2(x) + \sin^2(x) = \cos x$ ✓

$\cos^2(x) = \cos(x)$

$\cos^2(x) - \cos(x) = 0$

$\cos(x)\,(\cos(x) - 1) = 0$

$\cos(x) = 0$ ✓ or $\cos(x) - 1 = 0$

$\cos(x) = 1$ ✓

$x = \pi/2 + 2\pi k,\ \frac{3\pi}{2} + 2\pi k$ ✓ $\qquad x = 0 + 2\pi k$ ✓

$\frac{1}{2}$ pt each for $\pi/2$ & $3\pi/2$

-1 if $+2\pi k$ are missing or wrong

$-\frac{1}{2}$ if $+2\pi k$ are partially missing or wrong

$x = \pi/2 + 2\pi k,\ \frac{3\pi}{2} + 2\pi k,\ 0 + 2\pi k$

OR

$x = \pi/2 + \pi k,\ 2\pi k$

Answer: $\boxed{\frac{\pi}{2} + 2\pi k,\ \frac{3\pi}{2} + 2\pi k,\ 2\pi k}$

11. (6 points) Find and simplify the difference quotient for $f(x) = x^2 - 5x + 3$.

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^2 - 5(x+h) + 3 - (x^2 - 5x + 3)}{h}$$

(do not need to write this for full credit)

$$= \frac{\cancel{x^2} + 2xh + h^2 - \cancel{5x} - 5h \cancel{+3} - \cancel{x^2} + \cancel{5x} \cancel{-3}}{h}$$

$$= \frac{2xh + h^2 - 5h}{h} = \frac{h(2x+h-5)}{h}.$$

$$= 2x + h - 5$$

Answer: $\boxed{2x + h - 5}$

12. (6 points) Find the exact value of $\sin(15°)$. Simplify your answer.

Method 1:

$15° = 45° - 30°$ ✓✓

$\sin(15°) = \sin(45° - 30°)$

$= \sin 45° \cos 30° - \cos 45° \sin 30°$ ✓✓

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2}$ ✓✓ ($\frac{1}{2}$ pt each trig fn)

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}$

Method 2:

$15° = \frac{30°}{2}$ ✓✓

$\sin(15°) = \sin\left(\frac{30°}{2}\right) = \pm\sqrt{\frac{1 - \cos 30°}{2}}$ ✓✓

$= \pm\sqrt{\frac{1 - \frac{\sqrt{3}}{2}}{2}}$ ✓ ✓

$= \sqrt{\frac{\frac{2}{2} - \frac{\sqrt{3}}{2}}{2}}$

$= \sqrt{\frac{\frac{2-\sqrt{3}}{2}}{\frac{2}{1}}}$

$= \sqrt{\frac{2-\sqrt{3}}{2} \cdot \frac{1}{2}}$

$= \sqrt{\frac{2-\sqrt{3}}{4}}$

Note: these are the same b/c

$\left(\frac{\sqrt{6} - \sqrt{2}}{4}\right)^2 = \frac{(\sqrt{6} - \sqrt{2})(\sqrt{6} - \sqrt{2})}{4^2} = \frac{6 - 2\sqrt{6}\sqrt{2} + 2}{16}$

$= \frac{8 - 2\sqrt{12}}{16}$

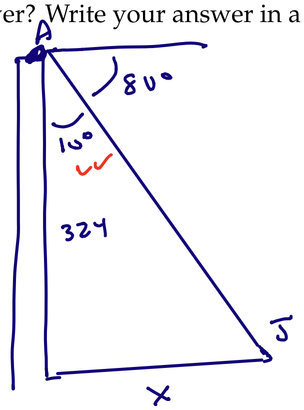
$= \frac{8 - 2 \cdot 2\sqrt{3}}{16}$

$= \frac{4(2 - \sqrt{3})}{16} = \frac{(2 - \sqrt{3})}{4}$

$\therefore \frac{\sqrt{6} - \sqrt{2}}{4} = \sqrt{\frac{2 - \sqrt{3}}{4}}$

Answer: $\boxed{\frac{\sqrt{6} - \sqrt{2}}{4} \left(\text{or } \sqrt{\frac{2 - \sqrt{3}}{4}}\right)}$

13. (6 points) A tiny but horrible alien is standing at the top of the Eiffel Tower (which is 324 meters tall) and he is threatening to destroy the city of Paris! He can see Agent J standing on the ground and the angle of depression from the alien to Agent J is 80°. How far away is Agent J from the base of the Eiffel Tower? Write your answer in a form that could be typed into a calculator to get a numerical answer.

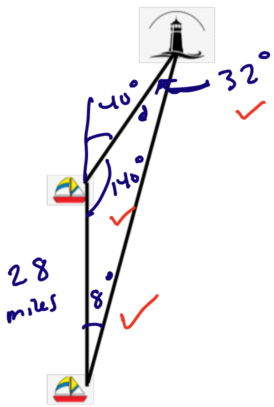


$\tan 10° = \frac{x}{324}$

$x = 324 \tan 10°$ ✓✓ ✓✓

Answer: $324 \tan 10°$

14. (6 points) A boat sailing due north at a rate of 14 mph records the bearing of a lighthouse as $N8°E$. Two hours later, the bearing of he same lighthouse is $N40°E$. Find the distance d from the boat to the lighthouse at the time of the second sighting. Leave your answer in a form that can be typed into a calculator to get a numerical answer.



$$\frac{d}{\sin 8°} = \frac{28}{\sin 32°}$$

$$d = \frac{28 \sin 8°}{\sin 32°}$$

1 pt 28 miles

1 pt each angle of Δ (3 pts total)

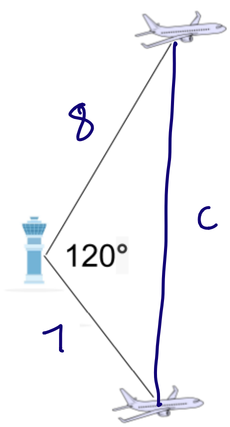
1 pt idea of Law of Sines

1 pt correct use of Law of Sines

-1 if compute wrong distance - from 1st sighting point not 2nd

Answer: $28 \frac{\sin 8°}{\sin 32°}$

15. (6 points) In tracking the relative location of two aircraft, a controller determines that the distance from the station to the first aircraft is 8 miles and the distance to the second aircraft is 7 miles. If the angle between the controller's sight lines to the two aircraft is 120°, how far apart are the two planes? **Simplify your answer.**



$c^2 = a^2 + b^2 - 2ab\cos 120°$ ✓✓ don't need to write this for full credit

$c^2 = 8^2 + 7^2 - 2 \cdot 8 \cdot 7 \cos 120°$ ✓✓

$c^2 = 64 + 49 - 2 \cdot 56 \cdot (-\frac{1}{2})$ 1 pt evaluating $\cos(120°) = -\frac{1}{2}$

$c^2 = 64 + 49 + 56 = 169$

$c = \sqrt{169} = 13$ 1 pt final answer

Answer: 13

16. (6 points) Prove the identity $\dfrac{\cos(x)}{1-\sin(x)} = \sec(x) + \tan(x)$

$$\frac{\cos(x)}{1-\sin(x)} = \frac{\cos(x)}{1-\sin(x)} \cdot \frac{(1+\sin x)}{(1+\sin x)}$$

2 pts multiplying by conj or something equivalent

$$= \frac{\cos(x)\,(1+\sin x)}{1-\sin^2 x}$$

2 pts using Pythag id to get from $1-\sin^2(x)$ to $\cos^2(x)$

$$= \frac{\cos(x)\,(1+\sin x)}{\cos^2(x)}$$

$$= \frac{1+\sin(x)}{\cos(x)}$$

$$= \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}$$

2 pts translating b/wn $\sec(x)+\tan(x)$ & $\dfrac{1}{\cos(x)} + \dfrac{\sin(x)}{\cos(x)}$

$$= \sec(x) + \tan(x) \quad \blacksquare$$

17. (6 points) A window above a doorway is constructed in the shape of the top half on an ellipse, as shown in the figure. The window is 2 feet tall at its highest point and 8 feet wide at the bottom. Find the height of the window 1 foot from the center of the base.

$b = 2$

$a = 4$

$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$

$$\frac{x^2}{16} + \frac{y^2}{4} = 1$$ 3 pts eqn (-1 if use $a=8$ instead of 4)

when $x = 1$ solve for y 1 pt setting $x = 1$

$$\frac{1^2}{16} + \frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 1 - \frac{1}{16} \Rightarrow \frac{y^2}{4} = \frac{15}{16}$$

$$\Rightarrow y^2 = \frac{15}{16} \cdot 4 \Rightarrow y^2 = \frac{15}{4} \Rightarrow y = \frac{\sqrt{15}}{2}$$

2 pts solving for y

Equation: $\boxed{\sqrt{15}/2}$

18. (8 points) For the function $g(x) = \frac{3-5x}{2x+7}$.

(a) Find $g^{-1}(x)$

$y = \frac{3-5x}{2x+7}$

$x = \frac{3-5y}{2y+7}$ 1 pt

$x(2y+7) = 3-5y$

$2xy+7x = 3-5y$ 1 pt

$2xy+5y = 3-7x$ 1 pt

$y(2x+5) = 3-7x$

$y = \frac{3-7x}{2x+5}$ 1 pt

$g^{-1}(x) = \boxed{\frac{3-7x}{2x+5}}$

(b) Find the domain and range of of $g(x)$ and $g^{-1}(x)$. Write your answers in interval notation.

domain of $g(x)$

$2x+7 \neq 0$

$2x \neq -7$

$x \neq -7/2$

Domain of $g(x)$

$\boxed{(-\infty,-\frac{7}{2})\cup(-\frac{7}{2},\infty)}$ 1 pt

Range of $g(x)$

$\boxed{(-\infty,-\frac{5}{2})\cup(-\frac{5}{2},\infty)}$

1 pt
full credit for wrong answer that agrees with domain of $g^{-1}(x)$

domain of $g^{-1}(x)$

$2x+5 \neq 0$

$2x \neq -5$

$x \neq -\frac{5}{2}$

Domain of $g^{-1}(x)$

$\boxed{(-\infty,-\frac{5}{2})\cup(-\frac{5}{2},\infty)}$ 1 pt

Range of $g^{-1}(x)$

$\boxed{(-\infty,-\frac{7}{2})\cup(-\frac{7}{2},\infty)}$ 1 pt

full credit for wrong answer that agrees with domain of $g(x)$

19. (6 points) Find the vertex, focus, and directrix for the parabola given by the equation $x^2+6x+8y-23=0$

$$x^2+6x=-8y+23$$

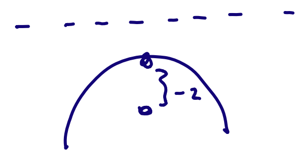
$$x^2+6x+9=-8y+23+9$$

$$(x+3)^2=-8y+32$$

$$(x+3)^2=-8(y-4)$$

✓ ✓ ✓

$p=-2$



Vertex: $(-3,4)$ 1 pt

Focus: $(-3,2)$ 1 pt

Directrix: $y=6$ 1 pt

20. (6 points) True or False and justify your answer.

(a) $\sin(3\theta) = 3\sin\theta\cos\theta$

A. True B. False ✓

Justification

For $\theta = \frac{\pi}{3}$

$\sin(3\cdot\frac{\pi}{3}) = \sin(\pi) = 0$ ✓✓

$3\sin\frac{\pi}{3}\cos\frac{\pi}{3} = 3\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{3\sqrt{3}}{4} \neq 0$

(b) $\cos^{-1}(\cos(193°)) = 193°$

A. True B. False ✓

Justification

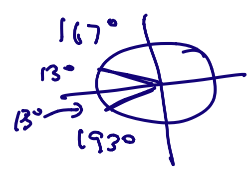
$\cos^{-1}$ always gives an angle btwn $0°$ & $180°$

but $193°$ is not between $0°$ & $180°$ ✓✓

In fact, $\cos^{-1}(\cos 193°) = 167°$

b/c $167°$ is btween $0°$ & $180°$

and has the same cosine as $193°$



FORMULAS

$\cos^2(\theta) + \sin^2(\theta) = 1$ $\qquad$ $\tan^2(\theta) + 1 = \sec^2(\theta)$ $\qquad$ $\cot^2(\theta) + 1 = \csc^2(\theta)$

$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ $\qquad$ $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$ $\qquad$ $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ $\qquad$ $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$ $\qquad$ $\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$

$c^2 = a^2 + b^2 - 2ab\cos(C)$ $\qquad$ $\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$

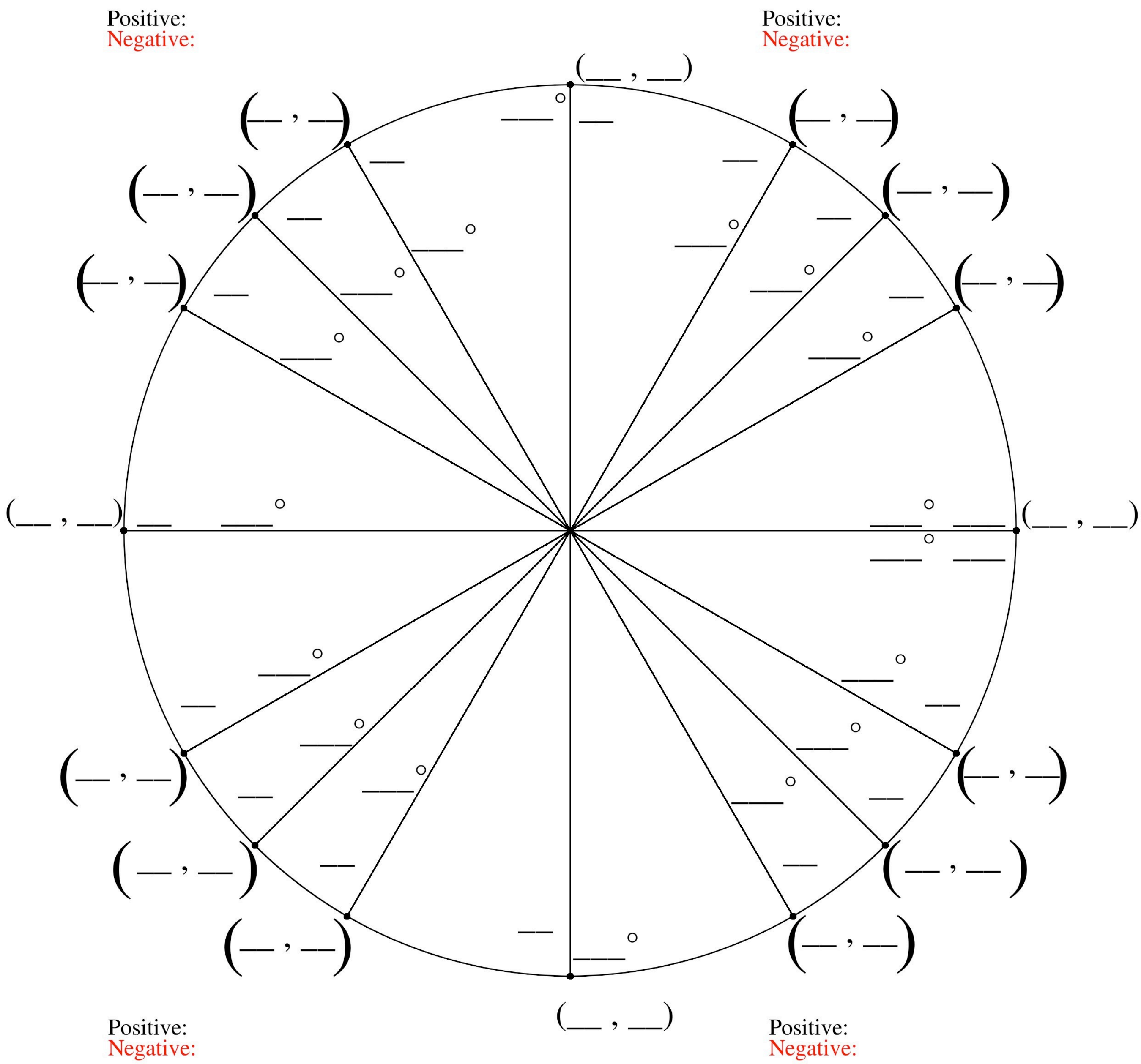
$4p(y-k) = (x-h)^2$ $\qquad$ $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ $\qquad$ $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

$4p(x-h) = (y-k)^2$ $\qquad$ $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $\qquad$ $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

$c^2 = a^2 - b^2$ $\qquad$ $c^2 = a^2 + b^2$

$x = r\cos(\theta)$ $\qquad$ $y = r\sin(\theta)$

$r^2 = x^2 + y^2$ $\qquad$ $\tan(\theta) = \frac{y}{x}$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