Analysis Project Ideas

ANA 1

Example Text: *Fractal Geometry: Mathematical Foundations and Applications* by Kenneth Falconer

Project: Fractal geometry is concerned with the study of sets which are, in certain respects, small and irregular, but still have a rich geometry. Fractals emerge in many different areas of math, from number theory to dynamical systems to nowhere-differentiable functions. The basic principles of fractal geometry are readily applicable to examples from many different areas of math.

Suggested Prerequisites: Familiarity with the basics of real analysis - up to and including the basic concepts of measure theory- is helpful, but not required.

Keywords: *real analysis, fractal geometry, dimension, analysis, iterated function systems*

ANA 2

Example Text: *Resources for the Study of Real Analysis* by Robert Brabenec

Additional Materials: May occasionally need to reference basic analysis concepts and definitions (limits, derivatives, integrals, etc.) depending on the background of the student.

Project: The story of the development of Calculus and Analysis is fascinating. Newton and Leibniz first discovered Calculus in the mid 16th century; however it was not until the 19th century that mathematicians saw the need for greater rigor and precision in the subject, leading to the formulation of limits, derivatives, and integrals as we know them today. The clean and precise treatments of Calculus and Analysis in typical undergraduate courses, though efficient, can leave the student with a cloudy understanding of the need for certain technical notions. In this project, we will enrich our understanding of and appreciation for Calculus and Analysis by taking a problem based approach with emphasis on the historical context. From this, students will gain an understanding of the historical development of Analysis, and will also engage with some of the key problems that led to the subject taking the form that it has today. In contrast to typical courses, the explicit main priority of this project is not to gain technical fluency with some mathematical concepts, but is instead to help us love, appreciate, and enjoy mathematics more.

Suggested Prerequisites: MATH 231, MATH 232, MATH 381. The project will be adapted based on whether or not the student has taken MATH 521.

Keywords: *calculus, analysis, history of calculus, Euler, Newton, Cauchy, infinite series, Bernoulli family, limits*

ANA 3

Example Text: *Fourier Analysis* by Stein and Shakarchi

Project: Fourier analysis/harmonic analysis is the study of functions by decomposing them with respect to nice bases; classical Fourier analysis decomposes periodic functions into linear combinations of sines and cosines. While this decomposition is useful for solving certain PDEs, Fourier analysis has also proved effective in other domains, including: signal processing/time series data analysis, geometry and topology, and number theory.

In this DRP, we will follow Stein and Shakarchi's "Fourier Analysis" text. Depending on the interests and background of the student, possible goals include: use the Fourier transform to solve PDEs; use the Fourier transform to study finite groups, and/or connect these ideas to signal processing application; explore uses of the Fourier transform in number theory, including Dirichlet's theorem: if q and l are relatively prime positive integers, then there are infinitely many primes of the form $l + kq$ with k an integer).

Suggested Prerequisites: Math 383, Math 521 (preferred but not necessary)

Keywords: *PDEs, spectral theory, Fourier transform, Fourier analysis, harmonic analysis, number theory*

ANA 4

Example Text: *Partial Differential Equations* by Lawrence C. Evans

Additional Materials: Lecture notes from Sigmund Selberg's PDEs course at Johns Hopkins, lecture notes from Jason Metcalfe's Nonlinear Waves Topics Course

Project: Consider ripples caused by dropping a pebble into an infinite pond, which can be mathematically modelled using wave equations. It is well known in physics that the energy associated to this motion must be constant; energy cannot be destroyed. However, as time goes on, the ripples gradually move away from their initial starting point. The energy is dispersed throughout the pond. In this project, we would look at some introductory partial differential equations material on the wave equation. We would define the total energy and prove mathematically that it must be conserved for the homogeneous wave equation. A potential goal for this project would be to explore (and ideally prove) a local energy decay estimate for the wave equation that reflects how the energy would be dispersed throughout the pond.

Suggested Prerequisites: Math 233 and Math 383 are required. 381 and 521 are also highly recommended. (For the local energy estimate proof, our main tool will be integration by parts.)

Keywords: *wave equation, partial differential equation, energy methods, integration by parts*

ANA 5

Example Text: *Differential Equations and Dynamical Systems* by Lawrence Perko

Project: We will study the basics of dynamical systems. Depending on the interest of the student, we can focus more on linear dynamical systems and finding exact solutions, or we can focus on low-dimensional nonlinear systems and qualitative techniques for approximating solutions. It is likely we will do at least some of both. A final presentation could be on the Fundamental Theorem for Linear Systems, or it could be an analysis of a particular applied dynamical system and how approximation techniques give an idea of what solutions look like.

Suggested Prerequisites: 383, 547, (521 also preferred)

Keywords: *dynamical systems, differential equations*

ANA 6

Example Text: *Partial Differential Equations* by Mark S. Gockenbach

Additional Materials: A copy of MATLAB (only if the student is interested in some numerical applications)

Project: The goal of this project is to understand how to generalize the notions of eigenvalues and eigenvectors from matrix algebra (the finite dimensional case) to partial differential operators (the infinite dimensional case). We will learn how to find eigenvalues and "eigenfunctions" for these operators and how to use them in constructing solutions to common boundary value problems that arise in the field of partial differential equations, such as the heat equation and wave equation. This is sometimes referred to as the "spectral method" of solving partial differential equations. A possible final presentation idea could be showing this construction for an interesting example PDE such as the Convection-Diffusion equation, or you could demonstrate some numerical results if that is the direction you prefer.

Suggested Prerequisites: MATH 233 and 383. A good understanding of eigenvalues and eigenvectors of matrices will be very helpful. If the student wants to do numerical work, some coding experience would be strongly recommended.

Keywords: *spectral theorem, partial differential equations, eigenfunctions, fourier series*

ANA 7

Example Text: *A Guide to Distribution Theory and Fourier Transforms* by Robert Strichartz

Project: Is the Dirac-delta function really a function? If it is, can you differentiate it? Can you differentiate a function that isn't continuous? These questions can be answered in the language of distribution theory. Distributions, sometimes called "generalized functions," are not quite functions, but we can still make sense of them in a structured way. We can even differentiate them...in a certain sense. Distribution theory opens up a vast and interesting branch of analysis and has applications to some very central questions in partial differential equations. For example, the fundamental solution of the wave equation is not a function, but it IS a distribution, and this allows us to still compute with it, so long as we understand what kind of object we're working with. Distributions also fit quite nicely into a study of the Fourier transform, and so a reasonable final presentation idea could be a (perhaps informal) proof of the Fourier inversion formula on the space of distributions.

Suggested Prerequisites: MATH 521 is absolutely required. A partial differential equations course is recommended, but not strictly necessary. Familiarity with the Fourier transform would help, but is not required. Be aware that this is a fairly advanced subject.

Keywords: *distribution theory, fourier transform, partial differential equations, weak derivatives*

ANA 8

Example Text: *Introductory Functional Analysis with Applications* by Erwin Kreyszig

Project: In some sense, functional analysis is the extension of linear algebra to the infinite-dimensional setting. Depending on the level of abstraction, this can require increased levels of background analysis needed. Some of this abstraction can obfuscate the direct analogies between the two areas. The goal of this reading is to discuss such infinite dimensional analogues to some of the core material in linear algebra, namely the study of normed spaces, inner product spaces, and linear operators between such spaces. This is appropriate for students with MATH 381 and MATH 547 background, although MATH 521 would be very useful. Some final goals, in increasing order of difficulty, would be the Hahn-Banach theorem, the Baire category theorem and consequences, or spectral theory. One could also study applications, such as the Banach fixed point theorem applied to differential and integral equations, approximation theory, or (for the highly ambitious student) quantum mechanics (this requires the study of almost all of the above topics and unbounded operators, which are somewhat scary).

Suggested Prerequisites: MATH 381, 547 are required, MATH 521 would be useful (although not strictly needed)

Keywords: *analysis, functional analysis, operators*

ANA 9

Example Text: Notes provided.

Project: Semiclassical analysis is a branch of mathematical analysis dedicated to developing techniques for studying partial differential equations motivated by the classical-quantum correspondence. In this reading, we will begin with a brief overview of classical and quantum mechanics, as a means of providing motivation for the subject, demonstrating analogies between the two along the way. From here, we will go through some mathematical preliminaries, namely basic Fourier analysis and stationary phase. Finally, we will discuss the concept of quantization, the mathematical realization of the classical-quantum correspondence. We will develop elementary properties of quantization to demonstrate their encapsulation of this connection. A reasonable goal would be to demonstrate how quantization preserves that the Poisson bracket in classical mechanics corresponds to the commutator in quantum mechanics, with the level of rigor depending on the background of the student. Disclaimer: This is a fairly advanced topic.

Suggested Prerequisites: MATH 383 is required, MATH 521 is very highly recommended

Keywords: *PDE, analysis, fourier*