Math 233: Test 2A
Spring 2014
Instructor: Linda Green

Name: ______________________________

1. Calculators are allowed and a formula sheet ~~as~~ is allowed.
2. You must show work for full and partial credit, except where otherwise noted.
3. Give exact values instead of decimal approximations, except where otherwise noted.
4. Sign the honor pledge after completing the exam.

   I have neither given nor received unauthorized help on this exam.

   Key
   ____

average: $44.4/54 = 82\%$

median $45/54 = 83\%$

8 A's (2 100%'s)

14 B's

9 C's

2 D's

1 F

1. (10 pts) Find the equation of the tangent plane to the surface

$$x^2y = z^2 - 7$$

at the point $(1, 2, 3)$.

$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$

$x^2y - z^2 + 7 = 0$

$f(x, y, z) = x^2y - z^2 + 7$

$f_x = 2xy$ at $(1, 2, 3)$ $f_x = 4$

$f_y = x^2$ $f_y = 1$

$f_z = -2z$ $f_z = -6$

$4(x-1) + 1(y-2) - 6(z-3) = 0$

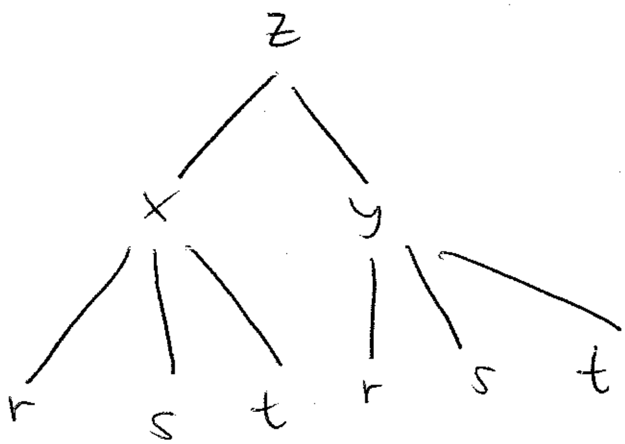
equivalently,

$z - 3 = \frac{2}{3}(x-1) + \frac{1}{6}(y-2)$

equivalently,

$\boxed{4x + y - 6z = -12}$

2. (10 pts) Suppose $z = \frac{x}{y}$, $x = r + s\cos t$, $y = r + s\sin t$. Find $\frac{\partial z}{\partial t}$ when $r = 1$, $s = 2$, and $t = 0$.



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$ 2 pts

4 pts $\quad \frac{\partial z}{\partial x} = \frac{1}{y} \quad \frac{\partial x}{\partial t} = -s\sin t \quad \frac{\partial z}{\partial y} = -\frac{x}{y^2} \quad \frac{\partial y}{\partial t} = s\cos t$

when $r=1$, $s=2$, $t=0$

$x = 1 + 2\cos 0 = 3$, $\quad y = 1 + 2\sin 0 = 1$

$\frac{\partial z}{\partial x} = \frac{1}{1} = 1 \quad \frac{\partial x}{\partial t} = -2\sin 0 = 0 \quad \frac{\partial z}{\partial y} = -\frac{3}{1^2} = -3$

$\frac{\partial y}{\partial t} = 2\cos 0 = 2$

$$\frac{\partial z}{\partial t} = 1 \cdot 0 - 3 \cdot 2 = \boxed{-6}$$

3. (12 pts total) Consider the function $f(x,y) = \ln(3+xy^2)$.

(a) Find the directional derivative of f at the point $(2,3)$ in the direction towards the point $(1,5)$. (Note: you are not given a direction vector, only the first and last point of a direction vector!)

(b) At the point $(2,3)$, in what direction does f increase most rapidly?

(c) What is the maximal rate of increase at the point $(2,3)$?

(d) Find a unit ~~tangent~~ vector $\vec{w}$ for which $D_{\vec{w}}f(2,3) = 0$.

a) $\nabla f = \langle f_x, f_y \rangle = \left\langle \frac{y^2}{3+xy^2}, \frac{2xy}{3+xy^2} \right\rangle$ (2 pts)

$\nabla f(2,3) = \left\langle \frac{9}{21}, \frac{12}{21} \right\rangle = \left\langle \frac{3}{7}, \frac{4}{7} \right\rangle$

direction vector: $\langle 1-2, 5-3 \rangle = \langle -1, 2 \rangle$

unit direction vector: $\vec{u} = \frac{\langle -1, 2 \rangle}{\|\langle -1, 2 \rangle\|} = \left\langle \frac{-1}{\sqrt5}, \frac{2}{\sqrt5} \right\rangle$ (2 pts)

$D_{\vec{u}} f = \nabla f \circ \vec{u}$

$= \left\langle \frac{3}{7}, \frac{4}{7} \right\rangle \circ \left\langle \frac{-1}{\sqrt5}, \frac{2}{\sqrt5} \right\rangle = \frac{5}{7\sqrt5} = \boxed{\frac{\sqrt5}{7}}$ (2 pts)

b) In the direction of ∇f: $\left\langle \frac{3}{7}, \frac{4}{7} \right\rangle$

as a unit vector: $\frac{\langle \frac{3}{7}, \frac{4}{7} \rangle}{\|\langle \frac{3}{7}, \frac{4}{7} \rangle\|} = \boxed{\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle}$ (2 pts)

c) $\|\nabla f\| = \left\|\left\langle \frac{3}{7}, \frac{4}{7} \right\rangle\right\| = \boxed{\frac{5}{7}}$ (2 pts)

d) perpendicular to ∇f

Since unit vector in direction of ∇f is $\left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$,

unit vector perp to this is $\boxed{\left\langle \frac{-4}{5}, \frac{3}{5} \right\rangle \text{ or } \left\langle \frac{4}{5}, \frac{-3}{5} \right\rangle}$ (2 pts)

Note: don't need to do any algebra to figure this out:

If $\langle a,b \rangle$ is a vector, then $\langle -b, a \rangle$ and $\langle b, -a \rangle$ are perpendicular to it, since $\langle a,b \rangle \circ \langle -b, a \rangle = 0$ etc.

If $\langle a,b \rangle$ is a unit vector, so is $\langle -b, a \rangle$

4. (12 pts) True or false. True means always true. False means sometimes or always false. NO WORK NEEDED.

Assume that f is defined on all of $\mathbb{R}^2$ with continuous first and second partial derivatives.

(a) **T** **F** If $(5,7)$ is a critical point of $f(x,y)$ and $f_{xx}(5,7) > 0$ and $f_{yy}(5,7) < 0$, then f has a saddle point at $(5,7)$.

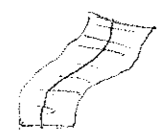
b/c $D = \underbrace{f_{xx} f_{yy}}_{<0} - \underbrace{(f_{xy})^2}_{<0}$ is negative

so saddle

(b) **T** **F** If $\nabla f(1,2) = \vec{0}$, then f has either a local minimum, a local maximum, or a saddle point at $(1,2)$.

b/c $f(x,y) = x^3$ has $\nabla f(0,0) = \vec{0}$

but no local max, min, or saddle



(c) **T** **F** If $\nabla f(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$, then $P_y(x,y) = Q_x(x,y)$.

b/c $P(x,y) = f_x(x,y)$ $\quad Q(x,y) = f_y(x,y)$

so $P_y = f_{xy} = f_{yx} = Q_x$

by Clairaut's Thm

(d) **T** **F** If $\lim_{x\to 0} f(x,0) = 1$ and $\lim_{y\to 0} f(0,y) = 1$, then $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

For example, $f(x,y) = \dfrac{x^2+y^2}{x^2+2xy+y^2}$ $\quad \lim_{x\to 0} f(x,0) = 1$ $\quad \lim_{y\to 0} f(0,y) = 1$

Note $\lim_{(x,y)\to(0,0)}$ along line $y=x$ is $\lim_{t\to 0} \dfrac{t^2+t^2}{t^2+2t\cdot t+t^2} = \dfrac{1}{2}$

so lim $\nexists$ DNE

(e) **T** **F** $\displaystyle\int_2^4 \int_1^6 x^2 e^{xy^3}\, dy\, dx = 0$

b/c $x^2 e^{xy^3}$ is always positive on $[2,4]\times[1,6]$

(f) **T** **F** If $\vec{r}(t)$ is a reparametrization of $\vec{q}(u)$ such that $\vec{r}(t) = \vec{q}(2t)$, then the curvature of $\vec{r}$ at $t = 0$ is 4 times the curvature of $\vec{q}$ at $u = 0$.

b/c curvature doesn't depend on parametrization

CHOOSE YOUR ADVENTURE: Pick one of the following two problems. Please specify which problem you want graded.

5. (10 pts) Suppose you want to find the point(s) on the surface $3xyz = 1$ that is / are closest to the origin, using the method of Lagrange multipliers. SET UP the four equations you would need to solve. (You DO NOT need to solve the equations or finish the problem!) Please indicate the 4 equations clearly by numbering them ① ② ③ ④

OR

6. (10 pts) Find the absolute maximum and absolute minimum of the function $f(x,y) = 3x^2 - 2y^2 - 4y$ on the region bounded by the curves $y = x^2$ and $y = 3$.

(5) $d^2 = f(x,y,z) = x^2+y^2+z^2$ (1 pt) $\quad g(x,y,z) = 3xyz$ (1 pt)

$\nabla f = \langle 2x, 2y, 2z \rangle$ (2 pts) $\quad \nabla g = \langle 3yz, 3xz, 3xy \rangle$ (2 pts)

$\nabla f = \lambda \nabla g$ (2 pts) $\Rightarrow$

① $2x = \lambda 3yz$

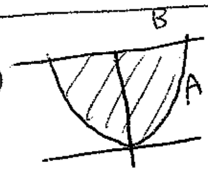
② $2y = \lambda 3xz$

③ $2z = \lambda 3xy$ (2 pts)

④ $3xyz = 1$

$f(x,y) = 3x^2 - 2y^2 - 4y$

(6)



critical pts:

$f_x = 6x \qquad f_y = -4y - 4$

$0 = -4y - 4 \Rightarrow y = -1$

(2 pts) $0 = 6x \Rightarrow x = 0$

but $(0,-1)$ is not in the region!

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bdry A: $y = x^2 \rightsquigarrow (t, t^2) \quad g(t) = f(t,t^2) = 3t^2 - 2t^4 - 4t^2$

$= -2t^4 - t^2$

$g'(t) = -8t^3 - 2t = 0 \Rightarrow t = 0$ or $\cancel{t^2 = -\frac{1}{4}}$

$-2t(4t^2+1)$

(2 pts) $\boxed{f(0,0) = 0}$ $\quad t = 0 \Rightarrow (x,y) = (0,0)$

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bdry B: $y = 3$ parametrize by $(t, 3)$

$h(t) = f(t,3) = 3t^2 - 18 - 12 = 3t^2 - 30$

$h'(t) = -6t = 0 \Rightarrow t = 0 \Rightarrow (x,y) = (0,3)$

(2 pts) $\boxed{f(0,3) = -30}$

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check end pts: $(-\sqrt{3}, 3)$, $(\sqrt{3}, 3)$ (2 pts)

$f(-\sqrt{3}, 3) = f(\sqrt{3}, 3) = -21$

$\boxed{\begin{array}{l} \text{max: } 0 \text{ at } (0,0) \\ \text{min: } -30 \text{ at } (0,3) \end{array}}$ (2 pts)