Math 233: Test 2A
Fall 2017
Instructor: Linda Green

- Please code your name and PID on your scantron.

- Since you have test version A, please code your scantron PAGE NUMBER as 1.

- Calculators are NOT allowed.

- For short answer questions, you must show work for full and partial credit. Please put all work to be graded on the test, not on scrap paper, since only pages with QR codes will be graded.

- No partial credit for multiple choice / no work needs to be shown.

- Sign the honor pledge below after completing the exam.

First and last name .... *Key* ....

PID ....

UNC Email ....

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature: ....

$\cos(30°) = \frac{\sqrt{3}}{2}$ $\sin(30°) = \frac{1}{2}$

$\cos(45°) = \frac{\sqrt{2}}{2}$ $\sin(45°) = \frac{\sqrt{2}}{2}$

$\cos(60°) = \frac{1}{2}$ $\sin(60°) = \frac{\sqrt{3}}{2}$

$\sqrt{2} \approx 1.41$ $\sqrt{3} \approx 1.73$

1. (2 pts) True or False: The differentiable function $f(x, y)$ must acheive an absolute maximum value in the region $x^2 + y^2 < 1$.

For example $f(x,y) = \frac{1}{1-x^2+y^2}$ has no abs max on $x^2+y^2<1$

   A. True

   B. False *Neither does $z = x^2+y^2$. Note that $\{(x,y)\,|\,x^2+y^2<1\}$ is NOT closed.*

2. (2 pts) True or False: If the differentiable function $f(x, y)$ has a local minimum point at (x_0, y_0, z_0), then $D_{\vec{u}} f(x_0, y_0) = 0$ for any unit vector $\vec{u}$).

(x_0,y_0) must be a critical point

   A. True *Since f is differentiable, f_x & f_y exist so they must be 0.*

   B. False *∴ $D_{\vec{u}} f = \langle f_x, f_y \rangle \cdot \vec{u} = \langle 0,0 \rangle \cdot \vec{u} = 0$ for any unit vector $\vec{u}$.*

The table below contains data about a differentiable function $g(x, y)$ at several points. Use this information to answer the questions below.

(x, y)	$g(x, y)$	$g_x(x, y)$	$g_y(x, y)$	$g_{xx}(x, y)$	$g_{yy}(x, y)$	$g_{xy}(x, y)$
$(-3, 2)$	4	0	0	14	3	4
$(0, 0)$	0	12	0	-2	-6	3
$(1, 4)$	-6	0	0	-5	10	-6

3. (2 pts) The point $(-3, 2)$ is: *$g_x(-3,2)=g_y(-3,2)=0$, so $(-3,2)$ is a critical point*

   A. Not a critical point.

   B. A local minimum point. *$D = g_{xx} g_{yy} - (g_{xy})^2 = 14\cdot3 - 4^2 > 0$*

   C. A local maximum point.   *$g_{xx} > 0$ so local min ⌣*

   D. A saddle point.

   E. None or the above or not enough information to determine.

4. (2 pts) The point $(0, 0)$ is:

   A. Not a critical point.   *$g_x(0,0) \neq 0$*

   B. A local minimum point.

   C. A local maximum point.

   D. A saddle point.

   E. None or the above or not enough information to determine.

5. (2 pts) The point $(1, 4)$ is: *$g_x(1,4)=g_y(1,4)=0$, so $(1,4)$ is a critical point*

   A. Not a critical point.

   B. A local minimum point. *$D = g_{xx} g_{yy} - (g_{xy})^2 = (-5)(10) - (-6)^2$*

   C. A local maximum point.   *$= -50 - 36 < 0$*

   D. A saddle point.

   E. None or the above or not enough information to determine.   *So saddle*

6. (4 pts) Suppose that the water pressure P on the bottom of a lake is given by a function $P(T,z)$ where T is temperature and z is the depth of the water. The temperature is a function of position (x- and y-coordinates) according to the equation $T(x,y) = x^2y$ and the depth of water is given by $z(x,y) = 2xy$. The values of P_T and P_z are given in the table below. (Note: the functions and values are not necessarily realistic, but go with it.)

Find the rate at which the pressure P is changing with respect to x when $x = 1$ and $y = 2$..

$P_T(T,z)$

z \ T	0	1	2	3	4
0	8	6	3	2	2
1	5	2	0	-2	-3
2	4	2	-1	-3	-4
3	1	3	2	0	-2
4	-2	-1	1	1	-1

$P_z(T,z)$

z \ T	0	1	2	3	4
0	-14	-12	-10	-7	-4
1	-10	-7	-5	-2	2
2	-6	-3	1	4	8
3	-1	3	6	11	15
4	-6	-3	1	4	8

A. -20

B. -16

C. 16

D. 20

E. 24
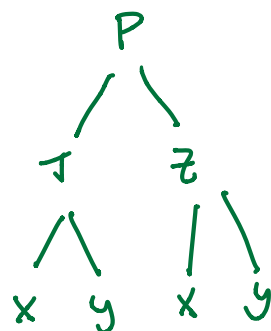
$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial T} \cdot \frac{\partial T}{\partial x} + \frac{\partial P}{\partial z} \cdot \frac{\partial z}{\partial x}$$

when $(x,y) = (1,2)$
$(T,z) = (2,4)$

$= P_T(2,4) \cdot 2xy|_{(1,2)} + P_z(2,4) \, 2y|_{(1,2)}$

$= -4 \cdot 4 + 8(4) = 16$

7. (4 pts) Find the directional derivative of $f(x,y) = x^2y$ at the point $(2,3)$ in the direction of the vector $\vec{i} + 2\vec{j}$

A. 16

B. 20

C. $4\sqrt{5}$

D. $\frac{16\sqrt{5}}{5}$

E. $20\sqrt{5}$

$f_x = 2xy$ $f_y = x^2$

$f_x(2,3) = 12$ $f_y(2,3) = 4$

$\vec{u} = \frac{\langle 1,2 \rangle}{\sqrt{1^2+2^2}} = \langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \rangle$

$D_{\vec{u}} f(2,3) = 12 \cdot \frac{1}{\sqrt{5}} + 4 \cdot \frac{2}{\sqrt{5}} = \frac{20}{\sqrt{5}}$

$= \frac{20}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{20\sqrt{5}}{5} = 4\sqrt{5}$

8. (4 pts) Is the function $f(x,y)$ defined below continuous at $(0,0)$?

$$f(x) = \begin{cases} \dfrac{x^2y^2}{x^4 + y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

x-axis
$x=t \quad y=0$
$\lim_{t \to 0} \frac{t^2 \cdot 0^2}{t^4 + 0^4} = \lim_{t \to 0} 0 = 0$

A. Yes, because it is a quotient of continuous functions. No, b/c denom is 0 at $(0,0)$ & defined piecewise

y-axis
$x=0 \quad y=t$
$\lim_{t \to 0} \frac{0^2 t^2}{0^4 + t^4} = 0$

B. Yes, because the limit of $f(x,y)$ is 0 as (x,y) approaches $(0,0)$ along the x-axis and the limit of $f(x,y)$ is 0 as (x,y) approaches $(0,0)$ along the y-axis. No, this doesn't guarantee limit exists

C. No, because the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the x-axis is different from the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the line $y = x$.

D. No, because the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the line $y = x$ is different from the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the line $y = -x$. No limits are both $\frac{1}{2}$
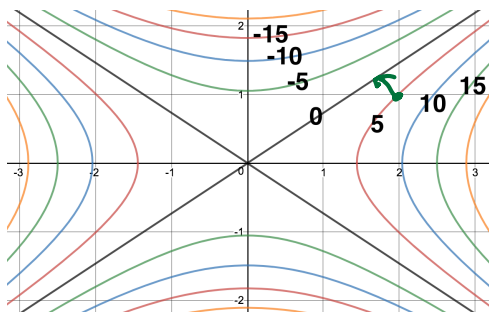
$y=x$
$x=t \quad y=t$
$\lim_{t \to 0} \frac{t^2 \cdot t^2}{t^4 + t^4} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2}$

$y=-x$
$x=t \quad y=-t$
$\lim_{t \to 0} \frac{t^2(-t)^2}{t^4 + (-t)^4} = \lim_{t \to 0} \frac{t^4}{t^4 + t^4} = \frac{1}{2}$

9. (4 pts) Use the graph of the level curves of $f(x,y)$ below to estimate $D_{\vec{u}}f(2,1)$ where $\vec{u} = <-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}>$.
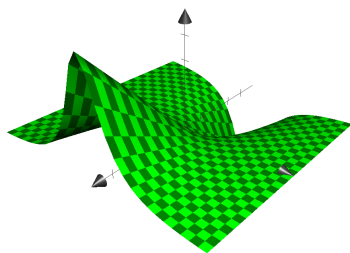Pick the closest answer.

A. -10

B. -5

C. 0

D. 5

E. 10

$\frac{rise}{run} \approx \frac{-5}{\frac{1}{2}} = -10$

Alternatives $f_x \approx \frac{5}{\frac{1}{2}} = 10$

$f_y \approx \frac{-5}{\frac{1}{2}} = -10$

$D_{<-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}>}f \approx -\frac{\sqrt{2}}{2} \cdot 10 + \frac{\sqrt{2}}{2}(-10)$

$\approx -\sqrt{2} \cdot 10 \approx -14$

10. (4 pts) Match the graph with an equation.

$z = $ constant
intersect w/ horizontal plane $\rightarrow$ get parabolas

$x = $ constant $\rightsquigarrow$ get curves like that:

$y = $ constant $\rightsquigarrow$ get gentle curves, possibly lines

A. $z = xy$ No, b/c setting $z = c \Rightarrow y = \frac{c}{x}$ hyperbolas

B. $z = \frac{x}{y^2 + 1}$

C. $z = x^2 - y^2$ No b/c $z = c \Rightarrow c = x^2 - y^2$ hyperbolas

D. $z = \sqrt{x^2 + y^2 + 1}$ No b/c $z = c \Rightarrow c = \sqrt{x^2 + y^2 + 1} \Rightarrow x^2 + y^2 = c^2 - 1$ circles

E. $z = -x^2 + y$ $z = c$ does yield parabolas, but $y = c \Rightarrow z = -x^2 + c$
parabolas

$z = c$
$\Rightarrow$
$x = cy^2 + c$
parabolas

$y = c \Rightarrow$
$z = \frac{x}{c^2 + 1}$ lines

$x = c \Rightarrow$
$z = \frac{c}{y^2 + 1}$ fosters this

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11. (8 pts) The surface area of a torus (donut shape) with an inner radius r and an outer radius $R > r$ is given by $S = \pi^2(R^2 - r^2)$. Use the differential to estimate the change in surface area when R increases by from 5 to 5.25 and r decreases from 3 to 2.75.
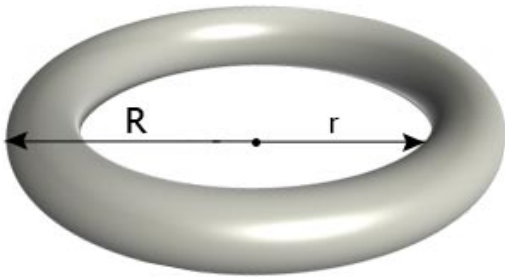
$$S = \pi^2 R^2 - \pi^2 r^2$$

$$dS = S_R \, dR - S_r \, dr$$

$$dS = 2\pi^2 R \, dR - 2\pi^2 r \, dr$$

$$dS = 2\pi^2 \cdot 5 \cdot 0.25 + 2\pi^2 \cdot 3 \cdot 0.25$$

$$= \frac{10}{4}\pi^2 + \frac{6}{4}\pi^2 = 4\pi^2 \checkmark\checkmark$$

$R = 5 \quad dR = 0.25 \checkmark$

$r = 3 \quad dr = -0.25$

-1 if use 0.25 for dr

Estimate of change: $\boxed{4\pi^2}$

12. (16pts) Find the absolute maximum and minimum value of $f(x, y) = x^2 + y^2 - 2x + 2y + 5$ on the region $R = \{(x, y) \mid x^2 + y^2 \le 4\}$. Show work to justify your answer using techniques from class.

$f_x = 2x - 2$ ✓ $f_y = 2y + 2$ ✓

$2x - 2 = 0$ $2y + 2 = 0$

$x = 1$ ✓ $y = -1$ ✓

boundary: $x^2 + y^2 = 4$

$x = 2\cos t$ $y = 2\sin t$ ✓✓

$g(t) = f(2\cos t, 2\sin t)$

$\quad = 4\cos^2 t + 4\sin^2 t - 4\cos t + 4\sin t + 5$

$\quad = 4 - 4\cos t + 4\sin t + 5$

$\quad = 9 - 4\cos t + 4\sin t$ ✓

$g'(t) = 4\sin t + 4\cos t = 0 \Rightarrow \sin t = -\cos t$ ✓✓

$\Rightarrow t = 3\pi/4$ ✓ or $7\pi/4$ ✓

$\Rightarrow (x, y) = \left(-2\frac{\sqrt{2}}{2}, 2\frac{\sqrt{2}}{2}\right)$ or $(x, y) = \left(2\frac{\sqrt{2}}{2}, -2\frac{\sqrt{2}}{2}\right)$

end pts: $0, 2\pi \Rightarrow x = 2, y = 0$

No points taken off for omitting end pts of $t = 0, t = 2\pi$ since it is not actually required to check them for periodic fns like this

x	y	f
1	-1	③ ✓
$-\sqrt{2}$	$\sqrt{2}$	$9 + 4\sqrt{2}$ ✓
$\sqrt{2}$	$-\sqrt{2}$	$9 - 4\sqrt{2}$
2	0	5 ✓

Since $\sqrt{2} \approx 1.4$

$4\sqrt{2} \approx 5.6$

so $9 - 4\sqrt{2} \approx 3.4$

Alternate parametrizations are possible (and may require checking endpts)

MAXimum value: $9 + 4\sqrt{2}$ ✓   Minimum value: 3 ✓

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13. (9 pts) Lisette is observing the density of the crowd, measured in people/m^2. If she walks east, the density of people **increases** at the rate of 2 people/m^2 per meter that she walks. If she walks north, the density of people **decreases** at the rate of 3 people/m^2 per meter that she walks. Assume that east is the direction $\vec{i}$ and north is the direction $\vec{j}$.

(a) In what direction should she walk if she wants the density of people to **decrease** the most rapidly? Give your answer as a vector in the form $a\vec{i} + b\vec{j}$ for some numbers a an b. It does not have to be a unit vector.

$$\boxed{\langle -2, 3 \rangle}$$

✓✓✓

+1 if mention $-\nabla f$
or opposite

+1 for $\langle 2, -3 \rangle$

$\nabla f = \langle 2, -3 \rangle$

$-\nabla f = \langle -2, 3 \rangle$

(b) In what direction could she walk if wants the density to change as little as possible? Give your answer as a vector in the form $a\vec{i} + b\vec{j}$.

$$\boxed{\langle 3, 2 \rangle}$$

+1 for $\perp$ to ∇f

Need a vector $\perp$ to ∇f

$\langle 3, 2 \rangle$ or $\langle -3, -2 \rangle$ works

(c) In part (a), at what rate is the density decreasing in this direction?

$$\boxed{\sqrt{13}}$$

+1 for $\|\nabla f\|$

$\|\nabla f\| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

14. (11 pts) Find the tangent plane to the surface $z^2 x + \dfrac{y}{z} = 8$ at the point $(1, 8, 2)$.

This is a level surface for $g(x, y, z) = z^2 x + \dfrac{y}{z}$

$g_x = z^2 \quad\quad g_y = \dfrac{1}{z} \quad\quad g_z = 2zx - \dfrac{y}{z^2}$

$g_x(1,8,2) = 4 \quad g_y(1,8,2) = \dfrac{1}{2} \quad g_z(1,8,2) = 2 \cdot 2 - \dfrac{8}{4}$
$$= 4 - 2 = 2$$

$4(x-1) + \dfrac{1}{2}(y-8) + 2(z-2) = 0$

$4x + \dfrac{1}{2}y + 2z = 12$

$8x + y + 4z = 24$

Tangent plane equation: $\boxed{8x + y + 4z = 24}$

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