

Math 232: Test 3A
Spring 2016
Instructor: Linda Green

- Calculators are allowed.
- For short answer questions, you must show work for full and partial credit.
- No partial credit for multiple choice / no work needs to be shown.
- Since you have test version A, please code your scantron sequence number as 11111 (all 1's).
- Give exact values instead of decimal approximations unless otherwise specified.
- Sign the honor pledge below after completing the exam.

First and last name *Key*

PID

UNC Email

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature:

True or False. (2 points each) Recall that true means always true, and false means sometimes or always false. n represents non-negative integers and the coefficients c_n are real numbers that could be positive, negative, or 0.

1. True or False: If $f(x) = \sum_{n=0}^{\infty} 3x^n$, then $f^{(n)}(0) = 3$ for all $n \geq 0$.

- A. True
 B. False

$$\frac{f^{(n)}(0)}{n!} = c_n = 3 \Rightarrow f^{(n)}(0) = 3 \cdot n!$$

2. True or False: Suppose the series $\sum_{n=0}^{\infty} a_n x^n$ and the series $\sum_{n=0}^{\infty} c_n x^n$ converge to the same values for all x . Then $a_n = c_n$ for all n .

- A. True
 B. False

The Taylor series for a function is unique.

3. True or False: If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges for all x -values, then the power

series $\sum_{n=1}^{\infty} n \cdot c_n x^{n-1}$ also converges for all x -values.

- A. True
 B. False

$$\sum_{n=1}^{\infty} n \cdot c_n x^{n-1} \text{ is } \frac{d}{dx} \left(\sum_{n=0}^{\infty} c_n x^n \right)$$

the radius of convergence for both is the same.

4. True or False: If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is 9, then the radius of convergence of $f(x) = \sum_{n=0}^{\infty} c_n x^{2n}$ is 3.

- A. True
 B. False

$\sum_{n=0}^{\infty} c_n x^n$ converges when $|x| < 9$
 the second series is obtained by plugging x^2 into the first, so it converges when $|x^2| < 9 \Rightarrow |x| < 3$.

5. True or False: If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is the Taylor series for $f(x) = 3x^2 - 5x + 6$, then $c_n = 0$ for all $n > 2$.

- A. True
 B. False

The Taylor series of a polynomial is the polynomial itself.

6. True or False: If $f(x) = \sum_{n=0}^{\infty} c_n 4^n$ converges, then $f(x) = \sum_{n=0}^{\infty} c_n (-4)^n$ also converges.

- A. True
- B. False

The interval of convergence could be $(-4, 4]$ for $\sum_{n=0}^{\infty} c_n x^n$

7. True or False: If $f(x) = \sum_{n=0}^{\infty} c_n (-5)^n$ converges, then $f(x) = \sum_{n=0}^{\infty} c_n 3^n$ also converges.

- A. True
- B. False

The radius of convergence must be at least 5 for $\sum_{n=0}^{\infty} c_n x^n$

8. True or False: If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)! 2^n}$ and $g(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n! 2^{n-1}}$, then $f(x) = g(x)$ for all values of x .

- A. True
- B. False

$k = n-1$
 $n = 1 \Rightarrow k = 0$
 $n = k+1$
 $\sum_{k=0}^{\infty} \frac{x^k}{(k+1)! 2^k}$

9. (5 pts) Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(-2)^n (x-3)^n}{n^3}$

- A. $\frac{1}{2}$
- B. 1
- C. 2
- D. 3
- E. $\frac{7}{2}$

$\lim_{n \rightarrow \infty} \frac{(-2)^{n+1} (x-3)^{n+1}}{(n+1)^3} \cdot \frac{(n)^3}{(-2)^n (x-3)^n}$
 $= \lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1}}{(-2)^n} \cdot \frac{(x-3)^{n+1}}{(x-3)^n} \right| \left(\frac{n}{n+1} \right)^3$
 $= \lim_{n \rightarrow \infty} 2 |x-3| = 2 |x-3|$
 set $2|x-3| < 1 \Rightarrow |x-3| < \frac{1}{2}$

10. (5 pts) The series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-7)^n}{4^n \sqrt{n}}$ has a radius of convergence of 4. What is its interval of convergence?

- A. (3, 11)
- B. [3, 11]
- C. [3, 11)
- D. [3, 11]

$x = 3$
 $\sum_{n=1}^{\infty} \frac{(-1)^n (3-7)^n}{4^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (-4)^n}{4^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n (4)^n}{4^n \sqrt{n}}$
 $= \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-test
 $\sum_{n=1}^{\infty} \frac{(-1)^n (11-7)^n}{4^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{4^n \sqrt{n}}$ conv by Alt series test

11. (5 pts) The sum of the series $\sqrt{3} - \frac{3\sqrt{3}}{3} + \frac{3^2\sqrt{3}}{5} - \frac{3^3\sqrt{3}}{7} + \dots$ is equal to which of the following expressions?

A. $\sqrt{3} \arctan(3)$

B. $\arctan(\sqrt{3})$

C. $\sqrt{3} \sin(3)$

D. $\sin(\sqrt{3})$

E. $\ln(1 + \sqrt{3})$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\text{so } \arctan(\sqrt{3}) = \sqrt{3} - \frac{(\sqrt{3})^3}{3} + \frac{(\sqrt{3})^5}{5} - \frac{(\sqrt{3})^7}{7} + \dots$$

which is equivalent to the given series.

12. (10 pts) Use Taylor series to find the limit:

$$\lim_{x \rightarrow 0} \frac{xe^x - x}{\ln(1 + 5x^2)}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

$$xe^x - x = \left(x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \dots \right) - x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\ln(1+5x^2) = 5x^2 - \frac{5^2 x^4}{2} + \frac{5^3 x^6}{3} - \frac{5^4 x^8}{4} + \dots$$

$$\lim_{x \rightarrow 0} \frac{xe^x - x}{\ln(1+5x^2)} = \frac{x^2 + \frac{x^3}{2} + \frac{x^4}{6} + \dots}{5x^2 - \frac{5^2 x^4}{2} + \frac{5^3 x^6}{3} - \dots}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \left(1 + \frac{x}{2} + \frac{x^2}{6} + \dots \right)}{x^2 \left(5 - \frac{5^2 x^2}{2} + \frac{5^3 x^4}{3} - \dots \right)}$$

$$= \boxed{\frac{1}{5}}$$

13. (14 pts) In order to approximate $e^{x/2}$ on the interval $[0.2, 1.4]$, Anne wants to use a degree 3 Taylor polynomial centered at 0.8, but Wendy wants to use a degree 6 Taylor polynomial centered at $x = 0$, instead. Whose estimate will be more accurate? Use remainder estimates to give a quantitative justification for your answer.

Anne

$$|R_3(x)| \leq \frac{M_1}{4!} |x - 0.8|^4$$

$$f^{(4)}(x) \leq M_1$$

$$f(x) = e^{x/2} \Rightarrow f^{(4)}(x) = \left(\frac{1}{2}\right)^4 e^{x/2}$$

which is increasing so it has max value on $(0.2, 1.4)$ at $x = 1.4$

use $M_1 = \left(\frac{1}{2}\right)^4 e^{1.4/2}$

$$|x - 0.8|^4 \leq 0.6^4 \text{ for } x \text{ in } [0.2, 1.4]$$

so

$$|R_3(x)| \leq \frac{\left(\frac{1}{2}\right)^4 e^{1.4/2} (0.6)^4}{4!}$$

$$= 0.00067964$$

2 pts for remainder formula if little else is correct

Wendy

$$|R_6(x)| \leq \frac{M_2}{7!} |x|^7$$

$$f^{(7)}(x) \leq M_2$$

$$f^{(7)}(x) = \left(\frac{1}{2}\right)^7 e^{x/2}$$

use $M_2 = \left(\frac{1}{2}\right)^7 e^{1.4/2}$

$$|R_6(x)| \leq \frac{\left(\frac{1}{2}\right)^7 e^{1.4/2} (1.4)^7}{7!}$$

$$= 0.0000329$$

lower number

Whose estimate will be more accurate?

Anne

Wendy

14. (10 pts)

(a) Find the Maclaurin series for $x^5 \cos(5x^3)$.

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \checkmark \checkmark$$

$$\cos(5x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n (5x^3)^{2n}}{(2n)!} \quad \checkmark \checkmark$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n}}{(2n)!}$$

$$x^5 \cos(5x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+5}}{(2n)!} \quad \checkmark \checkmark$$

(can get 6 points even if don't simplify)

(b) Find the Maclaurin series for $\int x^5 \cos(5x^3) dx$.

$$\int x^5 \cos(5x^3) dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+5}}{(2n)!} dx$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5^{2n} x^{6n+6}}{(6n+6)(2n)!} + C \quad \checkmark$$

-1 if start at $n=1$ not $n=0$ or if this is to is missing

✓✓✓
can give 1 pt

1 pt for + C

7 for correctly simplifying above in order to do the integration even if actual integration is wrong or missing.

15. (10 pts) Find the Taylor polynomial of degree 3 for the function $f(x) = \sqrt{3-x}$ centered around $a = 2$.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$(3-x)^{1/2}$	1 ✓
1	$\frac{1}{2}(3-x)^{-1/2}(-1)$	$-\frac{1}{2}$ ✓
2	$-\frac{1}{2} \cdot \frac{1}{2}(3-x)^{-3/2}(-1)(-1)$	$-\frac{1}{4}$ ✓
3	$\frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}(3-x)^{-5/2}(-1)$	$-\frac{3}{8}$ ✓

$$T_3(x) = 1 - \frac{1}{2}(x-2) - \frac{1}{4} \frac{(x-2)^2}{2!} - \frac{3}{8} \frac{(x-2)^3}{3!}$$

do not need to simplify for full credit.

$$= 1 - \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3$$

2 pts correct center

2 pts correct factorials in denominator

2 pts correct exponents and correct number of terms

-1 if give T_2 instead of T_3 or omit the constant term

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad R = \infty$$

$$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \quad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \dots \quad R = 1$$