KEY A

Math 232: Test 1A
Spring 2016
Instructor: Linda Green

- Calculators are allowed.
- For short answer questions, you must show work for full and partial credit. In particular, you must evaluate integrals by hand and show work.
- No partial credit for multiple choice / no work needs to be shown.
- Each multiple choice question is worth 5 points and each short answer question is worth 12 points.
- Give exact values instead of decimal approximations unless otherwise specified.
- Sign the honor pledge below after completing the exam.

First and last name ....

PID ....

UNC Email ....

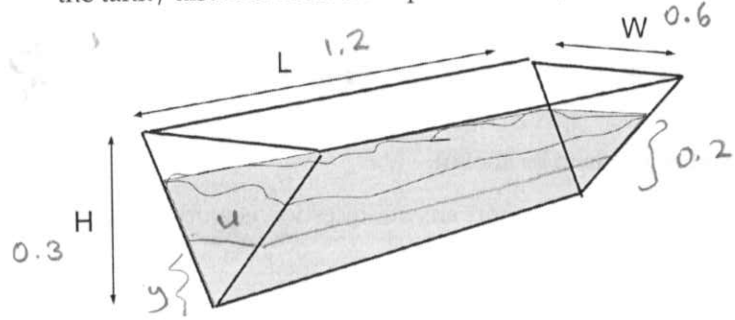
Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature: ....

13 points

1. A tank is shaped like a triangular trough with length 1.2 meters, a width at the top of 0.6 meters, and a height of 0.3 meters. The tank is filled with water to a depth of 0.2 meters. See the figure. SET UP an integral to find the work done to empty the tank by pumping the water up out of the top of the tank. DO NOT simplify or integrate. Use the fact that the acceleration due to gravity is 9.8 m/s^2 and the fact that a cubic meter of water has a mass of 1000 kg.

Please specify: I am using the variable ________ to represent (circle one) height from the bottom of the tank / distance from the top of the tank / distance from the top of the water / other ________



y = height from bottom

distance $= 0.3 - y$

Volume of slice $= u \cdot 1.2\,\Delta y$

$\frac{u}{y} = \frac{0.6}{0.3} \Rightarrow u = 2y$

4 pts (1 pt for 1.2)

$V = 2y \cdot 1.2\,\Delta y$

Force $= V \cdot 1000 \cdot 9.8$

$= 2 \cdot 1.2 \cdot 1000 \cdot 9.8\, y$

$$W = \int_0^{0.2} 2 \cdot 1.2 \cdot 1000 \cdot 9.8\, y(0.3 - y)\, dy$$

4 pts bds

x = distance from top of tank

$d = x$

$V = u \cdot 1.2 \cdot \Delta x$

$\frac{u}{0.3 - x} = \frac{0.6}{0.3}$

$u = 2(0.3 - x)$

$F = 1000 \cdot 9.8 \cdot 1.2 \cdot 2(0.3 - x)$

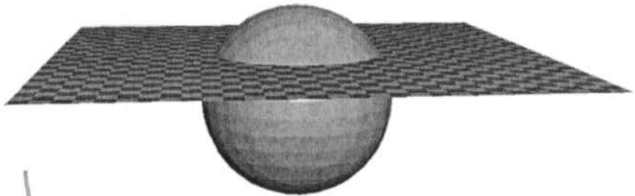
$$W = \int_{0.1}^{0.3} 1000 \cdot 9.8 \cdot 1.2 \cdot 2(0.3 - x)\, x\, dx$$

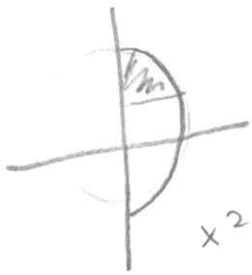
x = distance from top of water

$$W = \int_0^{0.2} 1000 \cdot 9.8 \cdot 1.2 \cdot 2(0.2 - x)(x + 0.1)\, dx$$

12 points

2. A sphere of radius R is cut by a horizontal plane halfway between the equator and the north pole. Use calculus to find the volume of the cap of the sphere that lies above the horizontal plane.





$$x^2+y^2=R^2$$
$$\Rightarrow x^2=R^2-y^2$$

$$V=\int_{\frac{1}{2}R}^{R} \pi x^2\,dy$$

$$=\int_{R/2}^{R} \pi\left(R^2-y^2\right)dy$$ ——8 points to here

$$=\pi\left[R^2y-\frac{y^3}{3}\right]\Big|_{R/2}^{R}$$

$$=\pi\left[\left(R^3-\frac{R^3}{3}\right)-\left(\frac{R^3}{2}-\frac{R^3}{24}\right)\right]$$

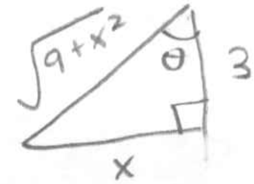
$$=\pi\left[\frac{2}{3}R^3-\frac{11}{24}R^3\right]$$

$$=\frac{5}{24}\pi R^3$$ —— 12 points to here

13 points

13 pt

3. Compute the integral by hand and show work. $\int \frac{\sqrt{9+x^2}}{x^4}\,dx$



$$\int \frac{\sqrt{9+(3\tan\theta)^2}}{(3\tan\theta)^4} \cdot 3\sec^2\theta\, d\theta$$

$\tan\theta = \frac{x}{3}$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

$$= \int \frac{3\sqrt{1+\tan^2\theta} \cdot 3\sec^2\theta\, d\theta}{3^4 \tan^4\theta}$$

$$= \frac{1}{9}\int \frac{\sec\theta \sec^2\theta}{\tan^4\theta}\, d\theta$$

$$= \frac{1}{9}\int \frac{\frac{1}{\cos^3\theta}}{\frac{\sin^4\theta}{\cos^4\theta}}\, d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^4\theta}\, d\theta$$

$u = \sin\theta$

$du = \cos\theta\, d\theta$

$$= \frac{1}{9}\int u^{-4}\, du$$

$$= \frac{1}{9} \frac{u^{-3}}{-3} + C$$

$$= -\frac{1}{27} \cdot \frac{1}{\sin^3\theta} + C$$

$$= -\frac{1}{27} \cdot \frac{1}{\left(\frac{x}{\sqrt{9+x^2}}\right)^3} + C = -\frac{\left(\sqrt{9+x^2}\right)^3}{27x^3}$$

12 points

12 pt

4. Compute the integral by hand and show work. $\int \frac{1}{x(2x-5)}\,dx$

$$\frac{1}{x(2x-5)} = \frac{A}{x} + \frac{B}{2x-5}$$ ✓✓

$$1 = A(2x-5) + Bx$$

$$1 = (2A+B)x - 5A$$

$\Rightarrow -5A = 1 \Rightarrow A = \frac{-1}{5}$ ✓✓

$\Rightarrow 2A + B = 0 \Rightarrow -\frac{2}{5} + B = 0 \Rightarrow B = \frac{2}{5}$ ✓✓

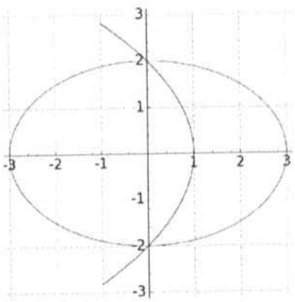
$$\int \frac{1}{x(2x-5)}\,dx = \int \frac{-1/5}{x} + \frac{\frac{2}{5}}{2x-5}\,dx$$

✓✓ ✓ ✓ ✓

$$= -\frac{1}{5}\ln x + \frac{2}{5}\cdot\frac{1}{2}\ln|2x-5| + C$$ ✓

$$= \frac{1}{5}\ln\left|\frac{2x-5}{x}\right| + C$$

5. Which integral represents the area in the first quadrant bounded by the curves $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and $x = 1 - \frac{y^2}{4}$?



$\boxed{x^2 = 9 - \frac{9y^2}{4} \Rightarrow x = \sqrt{9 - \frac{9y^2}{4}}}$

$\int_0^2 \sqrt{9 - \frac{9y^2}{4}} - (1 - \frac{y^2}{4})\, dy$

A. $\int_1^3 \sqrt{4 - \frac{4x^2}{9}} - \sqrt{4 - 4x}\, dx$

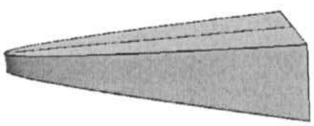
B. $\int_1^3 (4 - \frac{4x^2}{9}) - (4 - 4x)\, dx$

C. $\int_0^2 \sqrt{9 - \frac{9y^2}{4}} - (1 - \frac{y^2}{4})\, dy$

D. $\int_0^2 (1 - \frac{y^2}{4}) - \sqrt{9 - \frac{9y^2}{4}}\, dy$

E. $\int_0^2 (9 - \frac{9y^2}{4}) - (1 - \frac{y^2}{4})^2\, dy$

6. The surface of a pool is in the shape of the parabola $x = y^2$ for $-3 \le y \le 3$, where x and y are measured in feet. For any point (x, y) on the surface of the pool, the depth at that point in feet is given by $D(x, y) = 1 + \frac{1}{3}x$. What is the volume of water in the pool?



cross-sections are rectangles

$\int_{x=0}^{x=9} \underbrace{2y}_{\text{width}} \underbrace{(1 + \tfrac{1}{3}x)}_{\text{depth}}\, dx$

$\boxed{x = y^2 \Rightarrow y = \sqrt{x}}$

$\int_0^9 2\sqrt{x}(1 + \tfrac{1}{3}x)\, dx$

$\int 2x^{1/2} + \frac{2}{3}x^{3/2}\, dx$

A. $\int_0^3 2x(1 + \frac{1}{3}x)dx$

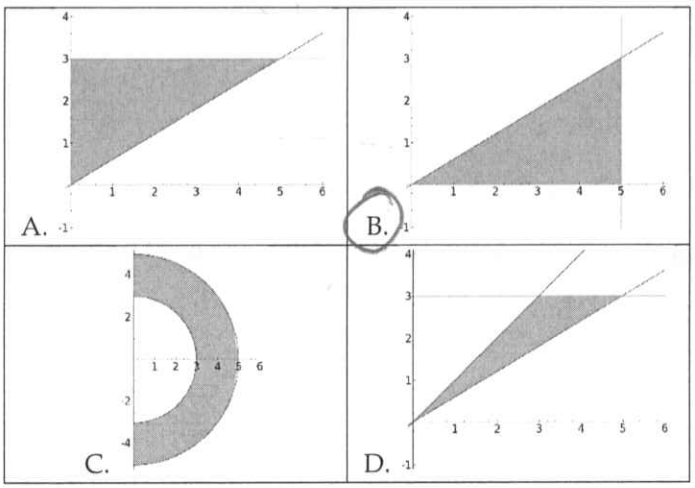
B. $\int_0^9 (2x^{1/2} + \frac{2}{3}x^{3/2})dx$

C. $\int_0^9 2\sqrt{x}\, dx$

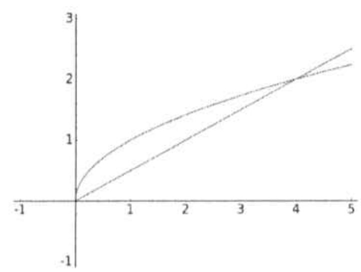
D. $\int_{-3}^3 (4x^2 + \frac{4}{3}x^4)\, dx$

E. $\int_0^9 (\frac{2}{3}x^{3/2})\, dx$

7. A shape is formed by taking a cylinder of radius 5 cm and height 3 cm and removing a cone of radius 5 cm and height 3 cm. Which of the following regions could be rotated around an axis to form the shape?



8. Find the volume of the solid obtained by rotating the region bounded by $y = \sqrt{x}$ and $y = x/2$ about the x-axis.



$$V = \pi \int_{x=0}^{x=4} r_{outer}^2 - r_{inner}^2 \, dx$$

$$= \pi \int_0^4 (\sqrt{x})^2 - \left(\frac{x}{2}\right)^2 dx$$

$$= \pi \int_0^4 x - \frac{x^2}{4}$$

A. $\pi \int_0^4 (\sqrt{x} - \frac{x}{2})^2 \, dx$

B. $\pi \int_0^4 (x - \frac{x^2}{4}) \, dx$

C. $\pi \int_0^4 \frac{x^2}{4} - x \, dx$

D. $\pi \int_0^4 2x - x^2 \, dx$

E. $\pi \int_0^4 \sqrt{x} - \frac{x}{2} \, dx$

9. $\int x^2 \sin(x)\,dx - \int 2x\cos(x)\,dx =$

A. $-x^2\cos x - \int 4x\cos x\,dx$

B. $x^2\cos x - \int 4x\cos x\,dx$

C. $x^2\sin x + C$

D. $-x^2\cos x + C$

E. $-\frac{1}{3}x^3\cos x - x^2\sin x + C$

$u = x^2 \quad dv = \sin x\,dx$

$du = 2x\,dx \quad v = -\cos x$

$\int x^2 \sin x\,dx = -x^2\cos x - \int -\cos x \; 2x\,dx$

$\Rightarrow \int x^2 \sin x\,dx - \int 2x\cos x\,dx = -x^2\cos x + C$

10. Suppose $f(x)$ is a continuous function and $\int_3^7 f(x)\,dx = 4$. Which of the following must be true?

A. $f(x) = 1$ for at least one x in $(3,7)$.

B. $f(x) = 2$ for at least one x in $(3,7)$.

C. $f(x) = 4$ for at least one x in $(3,7)$.

D. $f(x) = 6$ for at least one x in $(3,7)$.

E. None of these must be true.

$f_{ave} = \dfrac{\int_3^7 f(x)\,dx}{7-3} = \dfrac{4}{4} = 1$

So MVT for Integrals says $f(c) = f_{ave}$ for some c in $(3,7)$

11. Which technique of integration is most appropriate to compute $\int e^x \sin x\,dx$?

A. integration by parts

B. trig identities

C. partial fractions

D. trig substitution

E. u-substitution alone

$u = \sin x \quad dv = e^x\,dx$ (or vice versa)

see class notes for a similar problem

12. Which technique of integration is most appropriate to compute $\int \sin^3(x)\sqrt{\cos(x)}\,dx$?

A. integration by parts

B. trig identities

C. partial fractions

D. trig substitution

E. u-substitution alone

convert $\sin^2(x)$ to $1-\cos^2(x)$

$\int \sin^2(x)\sqrt{\cos(x)}\;\sin(x)\,dx$

$= \int (1-\cos^2(x))\sqrt{\cos x}\;\sin x\,dx$

$u = \cos x \quad du = -\sin x\,dx$

$-du = \sin x\,dx$

$= \int (1-u^2)(u)^{1/2}(-du)$

$= -\int u^{1/2} - u^{5/2}\,du$ etc.

13. Which technique of integration is most appropriate to compute $\int \frac{1}{x \ln x}\, dx$?

    A. integration by parts
    B. trig identities
    C. partial fractions
    D. trig substitution
    E. u-substitution alone

$u = \ln x \quad du = \frac{1}{x}\,dx$

$\int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|\ln(x)| + C$

14. Which technique of integration is most appropriate to compute $\int \arctan x \, dx$?

    A. integration by parts
    B. trig identities
    C. partial fractions
    D. trig substitution
    E. u-substitution alone

$u = \arctan x \quad dv = dx$

See class notes.