A

Final Exam for Math 233 (Fall 2017)

NAME Key	
UNC EMAIL	PID
INSTRUCTOR	SECTION

Instructions

- This exam consists of **12 exercises**, each worth **10 points**.
- Write **clear solutions** so that you can get partial points for your reasoning!
- Write your **final answer** in the **BOXES** provided.
- Calculators and other materials are **NOT** allowed.
- The duration of the exam is 3 hours.

Honor pledge

I certify that no unauthorized assistance has been received or given in the completion of this work.

SIGNATURE

Problem 1.

(1) **Find an equation** of the tangent plane to the graph of $f(x,y) = ye^x$ at the point $(0,2)$. *Express the plane in $ax + by + cz = d$ form.*

$f_x = ye^x \qquad f_x(0,2) = 2e^0 = 2$

at $(0,2)$

$f(0,2) = 2$

so $z = 2$

$f_y = e^x \qquad f_y(0,2) = e^0 = 1$

$z - 2 = 2(x-0) + 1(y-2)$

$z - 2 = 2x + y - 2$

$2x + y - z = 0$

(2) **Find a linear approximation** of the function $f(x,y) = ye^x$ at the point $(0,2)$.

$2x + y - z = 0$

$\Rightarrow \quad z = 2x + y$

$L(x,y) = 2x + y$

$\boxed{L(x,y) = 2x + y}$

(3) **Find an approximation** of the value $2.03e^{0.1}$.

$2.03\,e^{0.1} = f(0.1, 2.03)$

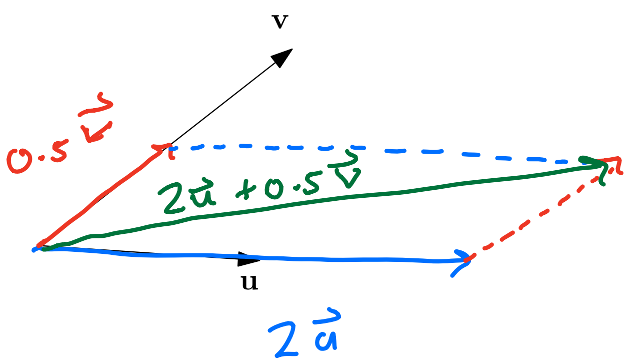
$\approx L(0.1, 2.03) = 2 \cdot 0.1 + 2.03$

$= 2.23$

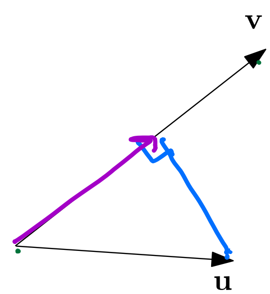
$\boxed{2.23}$

Problem 2.

(a) **Sketch** the vector $2\mathbf{u} + 0.5\mathbf{v}$.



(b) **Sketch** the vector $\text{proj}_{\mathbf{v}}\mathbf{u}$.



(c) Let $\mathbf{u} = \langle 1, 1, 0\rangle$ and $\mathbf{v} = \langle 1, 0, 1\rangle$. **Find the angle** between $\mathbf{u}$ and $\mathbf{v}$.

$$\cos^{-1}\left(\frac{\langle 1,1,0\rangle \cdot \langle 1,0,1\rangle}{\|\langle 1,1,0\rangle\| \; \|\langle 1,0,1\rangle\|}\right)$$

$$= \cos^{-1}\left(\frac{1}{\sqrt{2}\cdot\sqrt{2}}\right)$$

$$= \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3$$

$\boxed{\pi/3}$

(d) Let $\mathbf{u} = \langle 1, 1, 0\rangle$ and $\mathbf{v} = \langle 1, 0, 1\rangle$. **Find the area** of the parallelogram of sides $\mathbf{u}$ and $\mathbf{v}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\cdot 1 - \vec{j}\cdot 1 + \vec{k}(-1)$$

$$= \vec{i} - \vec{j} - \vec{k}$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

$\boxed{\sqrt{3}}$

Problem 3. Find the **absolute minimum** value of the function

$$f(x,y) = x^2 + 2y^2 - x$$

on the region $x^2 + y^2 \leq 4$.

critical points:

$f_x = 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$

$f_y = 4y = 0 \quad y = 0$

boundary: set $y^2 = 4 - x^2$

to get $f(x,y) = x^2 + 2(4 - x^2) - x$

$= x^2 + 8 - 2x^2 - x$

$= -x^2 - x + 8$ on the bdy

let $g(x) = -x^2 - x + 8$

critical pts of g: $g'(x) = -2x - 1 = 0$

$\Rightarrow x = -\frac{1}{2} \Rightarrow g(x) = 8.25$

$\Rightarrow y^2 = 4 - (-\frac{1}{2})^2 = 3.75$

$\Rightarrow y = \pm\sqrt{3.75}$

end pts for bdy: $x = -2$, $x = 2$

x	y	f(x,y)
$\frac{1}{2}$	0	$-\frac{1}{4}$
$-\frac{1}{2}$	$\sqrt{3.75}$	8.25
$-\frac{1}{2}$	$-\sqrt{3.75}$	8.25
-2	0	6
2	0	2

$y = -\frac{1}{4}$

Problem 4. Consider the curve in space parametrized by

$$\mathbf{r}(t) = \langle t^2, t^4, t^3 \rangle, \qquad 0 \le t \le 2.$$

(1) Find parametric equations for the tangent line to the curve at the point $(1, 1, 1)$.

$t = 1$ at pt $(1, 1, 1)$

$\vec{r}\,'(t) = \langle 2t, 4t^3, 3t^2 \rangle$

$\vec{r}\,'(1) = \langle 2, 4, 3 \rangle$

$x = 1 + 2t$

$y = 1 + 4t$

$z = 1 + 3t$

$\boxed{x = 1 + 2t,\ y = 1 + 4t,\ z = 1 + 3t}$

(2) Find the work done by the force $\mathbf{F}(x, y, z) = \langle y, 0, z \rangle$ on a particle that moves along the curve described above.

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{2} \langle y(t), 0, z(t) \rangle \cdot \langle x'(t), y'(t), z'(t) \rangle \, dt$$

$$= \int_{t=0}^{2} \langle t^4, 0, t^3 \rangle \cdot \langle 2t, 4t^3, 3t^2 \rangle \, dt$$

$$= \int_{t=0}^{2} 2t^5 + 3t^5 \, dt = \int_{t=0}^{2} 5t^5 \, dt$$

$$= \frac{5t^6}{6} \Big|_0^2 = \frac{160}{3}$$

$\boxed{\frac{160}{3}}$

Problem 5. Let

$$g(s,t) = f\Big(\overbrace{3s + e^{st}}^{x},\ s^2 \overbrace{+ \sin(t^2 + 2t)}^{y}\Big),$$

where f is an unknown function. Suppose you know that

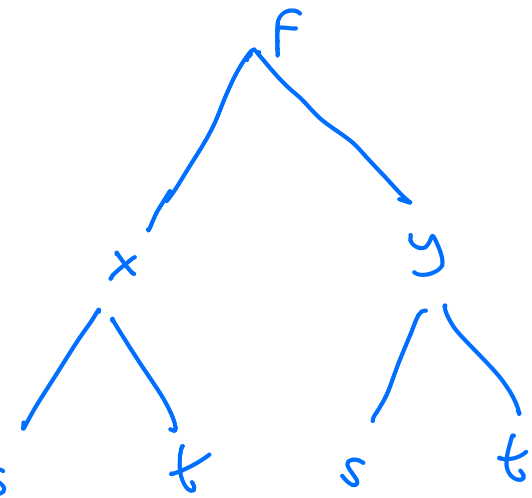
$g(1,0) = 3,$	$f(1,0) = 5,$	$f_x(1,0) = 1,$	$f_y(1,0) = 4,$
$g(4,1) = 0,$	$f(4,1) = 3,$	$f_x(4,1) = -1,$	$f_y(4,1) = 2.$

Find $\frac{\partial g}{\partial s}(1,0)$**.**

when $(s,t)=(1,0)$

$x = 3\cdot 1 + e^{1\cdot 0} = 4$

$y = 1^2 + \sin 0 = 1$



$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\Big|_{(4,1)} \cdot \frac{\partial x}{\partial s}\Big|_{(1,0)} + \frac{\partial f}{\partial y}\Big|_{(4,1)} \cdot \frac{\partial y}{\partial s}\Big|_{(1,0)}$$

$$= f_x(4,1)\cdot (3t + te^{st})\Big|_{(1,0)} + f_y(4,1)\cdot 2s\Big|_{(1,0)}$$

$$= (-1)(3+0) + 2(2)$$

$$= 1$$

$x = 3s + e^{st}$

$\frac{\partial x}{\partial s} = 3t + te^{st}$

$y = s^2 + \sin(t^2+2t)$

$\frac{\partial y}{\partial s} = 2s$

$\frac{\partial g}{\partial s}(1,0) = 1$

Problem 6. Consider the vector field

$$\mathbf{F}(x,y) = \left\langle 2xy^2 + 5\,,\, 2x^2y - 3y^2 \right\rangle.$$

(1) Is $\mathbf{F}$ conservative? If the answer is no, please justify. If the answer is yes find a potential (that is, a function f so that $\nabla f = F$).

$Q_x = 4xy \qquad P_y = 4xy$ conservative

$f_x = 2xy^2 + 5 \Rightarrow f(x,y) = \int 2xy^2 + 5\,dx$

$= x^2y^2 + 5x + g(y)$

$\Rightarrow f_y(x,y) = 2x^2y + g'(y) = 2x^2y - 3y^2$

$\Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = \int -3y^2\,dy$

$= -y^3 + C$

$\Rightarrow f(x,y) = x^2y^2 + 5x - y^3 + C$

can set $C = 0$ or any other constant.

(2) **Evaluate** $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is given by $\mathbf{r}(t) = t^2\mathbf{i} + \sqrt{t}\mathbf{j}$ between $(0,0)$ and $(1,1)$.

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \nabla f \cdot d\vec{r} = f(1,1) - f(0,0)$

$= 5 - 0 = 5$

$\boxed{5}$

(3) **Evaluate** $\int_{C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_2 is the circle $x^2 + y^2 = 4$ travelled in the counter-clockwise direction starting at $(2,0)$.

$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \nabla f \cdot d\vec{r} = 0$

since path is closed

$\boxed{0}$

Problem 7. A bug is crawling on a sheet of metal modelled by the xy-plane. The temperature of the sheet is given by the function

$$T(x, y) = 2x^2y + 3xy + 10.$$

(1) If the bug is currently at the point $(3, 1)$, **find the direction** in which the bug should go to **decrease** temperature most rapidly.

$\nabla T = \langle 4xy + 3y, \quad 2x^2 + 3x \rangle$

$\nabla T(3,1) = \langle 15, \ 27 \rangle$

$-\nabla T(3,1) = \langle -15, -27 \rangle$

$\boxed{\langle -15, -27 \rangle}$

(2) **Find the rate of change** of the temperature at the point $(3, 1)$ in the direction of $\langle -1, 1\rangle$.

$\vec{v} = \langle -1, 1 \rangle$

$\vec{u}$ = unit vector $= \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$\nabla T \circ \vec{u} = \langle 15, 27 \rangle \circ \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

$= \frac{-15}{\sqrt{2}} + \frac{27}{\sqrt{2}} = \frac{12}{\sqrt{2}}$

$\boxed{\frac{12}{\sqrt{2}}}$

(3) Suppose the bug is now at the point $(3, 1)$. **Find an equation** for a curve on which the bug should walk if it wants to stay at exactly the same temperature as it is now.

$T(3,1) = 2 \cdot 3^2 \cdot 1 + 3 \cdot 3 \cdot 1 + 10$

$= 37$

$2x^2y + 3xy + 10 = 37$

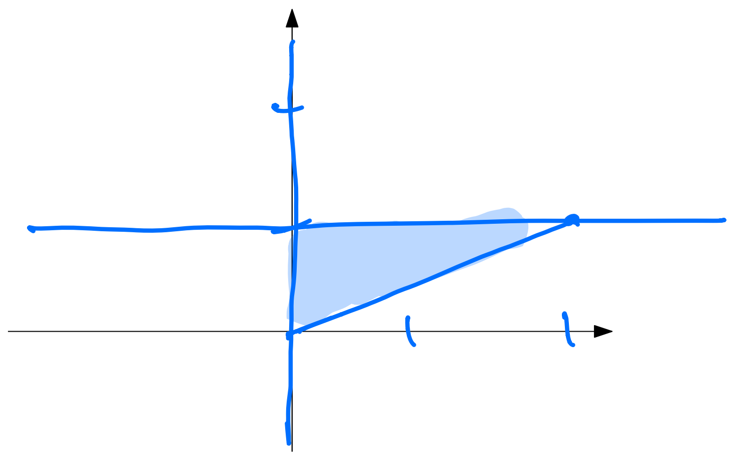
$2x^2y + 3xy = 27$

$\boxed{2x^2y + 3xy = 27}$

Problem 8. A lamina is shaped as the region in $\mathbb{R}^2$ bounded by the curves $x = 0$, $y = x/2$ and $y = 1$.

Sketch the region that the lamina occupies **(shade it)** on the coordinate plane below and **find the mass** of the lamina if its density is given by

$$\rho(x, y) = x\cos(y^3 - 1) + 2.$$



$$\iint_D \rho(x,y)\, dA$$

$$= \iint_D x\cos(y^3-1) + 2\, dA$$

$$= \int_{y=0}^{1}\int_{x=0}^{2y} x\cos(y^3-1) + 2\, dx\, dy$$

$$= \int_{y=0}^{1} \frac{x^2}{2}\cos(y^3-1) + 2x \Big|_{x=0}^{2y} dy$$

$$= \int_{y=0}^{1} \frac{(2y)^2}{2}\cos(y^3-1) + 4y\, dy = \int_{y=0}^{1} 2y^2\cos(y^3-1)\, dy + 2y^2 \Big|_0^1$$

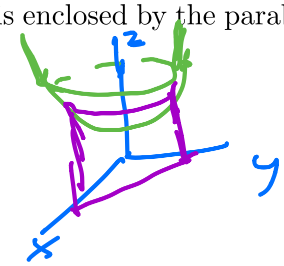
$u = y^3 - 1$ $\quad du = 3y^2\, dy$ $\quad \frac{1}{3}du = y^2\, dy$ $\quad y = 0 \Rightarrow u = -1$ $\quad y = 1 \Rightarrow u = 0$

$$\int_{u=-1}^{0} \frac{2}{3}\cos(u)\, du + 2 = \frac{2}{3}\sin(u)\Big|_{-1}^{0} + 2$$

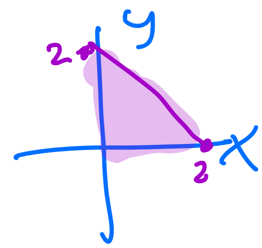
$$= \frac{2}{3}\sin 0 - \frac{2}{3}\sin(-1) + 2 = -\frac{2}{3}\sin(-1) + 2$$

$$\boxed{-\frac{2}{3}\sin(-1) + 2}$$

Problem 9. Find the **volume** of the solid that lies in the **first octant** and is enclosed by the paraboloid $z = 1 + x^2 + y^2$ and the plane $x + y = 2$.



$$\iint_D (1+x^2+y^2)\, dA$$



$$= \int_{y=0}^{2} \int_{x=0}^{2-y} (1+x^2+y^2)\, dx\, dy$$

$$= \int_{y=0}^{2} x + \frac{x^3}{3} + y^2 x \Big|_0^{2-y} dy$$

$$= \int_{y=0}^{2} (2-y) + \frac{(2-y)^3}{3} + y^2(2-y)\; dy$$

$$= \int_{y=0}^{2} 2-y + \frac{8}{3} - 4y + 2y^2 - \frac{y^3}{3} + 2y^2 - y^3\; dy$$

$$= \int_{y=0}^{2} \frac{14}{3} - 5y + 4y^2 - \frac{4}{3}y^3\; dy$$

$$= \frac{14}{3}y - \frac{5}{2}y^2 + \frac{4}{3}y^3 - \frac{1}{3}y^4 \Big|_0^2$$

$$= \frac{28}{3} - 10 + \frac{32}{3} - \frac{16}{3}$$

$$= \frac{14}{3}$$

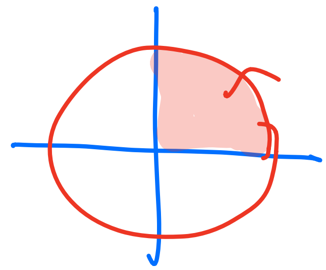
$$\boxed{14/3}$$

Problem 10. Consider the line integral

$$\int_{\partial D} \overset{P}{x^2 y}\, dx - \overset{Q}{y^2 x}\, dy,$$

where D is the region in the first quadrant enclosed between the coordinate axes and the circle $x^2+y^2=4$, and where ∂D is the boundary curve of the region D traversed in counterclockwise direction.

Use Green's Theorem to **compute** this integral.



$P = x^2 y \qquad Q = -y^2 x$

$Q_x - P_y = -y^2 - x^2$

$$\iint_D Q_x - P_y \, dA$$

$$= \iint_D -y^2 - x^2 \, dA = \int_{\theta=0}^{\pi/2}\int_{r=0}^{2} -r^2 \, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} d\theta \int_{r=0}^{2} -r^3 \, dr$$

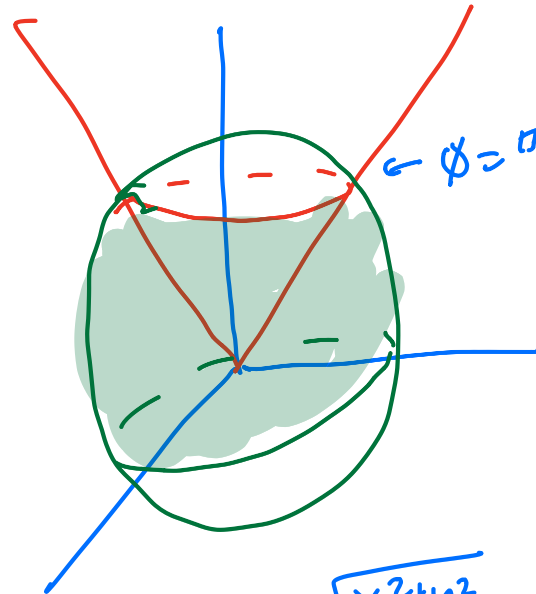
$$= \theta\Big|_0^{\pi/2} \left(-\frac{r^4}{4}\right)\Big|_0^2$$

$$= \pi/2\,(-4) = -2\pi$$

$$\boxed{-2\pi}$$

Problem 11. Find the **volume** of the solid that lies within the sphere $x^2 + y^2 + z^2 = 1$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

use spherical coordinates



$$\int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_{\theta=0}^{2\pi} d\theta \int_{\phi=\pi/4}^{\pi/2} \sin\phi \, d\phi \int_{\rho=0}^{1} \rho^2 \, d\rho$$

$$= \theta \Big|_0^{2\pi} \left(-\cos\phi\right)\Big|_{\pi/4}^{\pi/2} \frac{\rho^3}{3}\Big|_0^1$$

$$= 2\pi \left(0 + \frac{\sqrt{2}}{2}\right) \cdot \frac{1}{3}$$

$$= \frac{2\pi}{3}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi\sqrt{2}}{3}$$

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1$$

$$\Rightarrow \phi = \pi/4$$

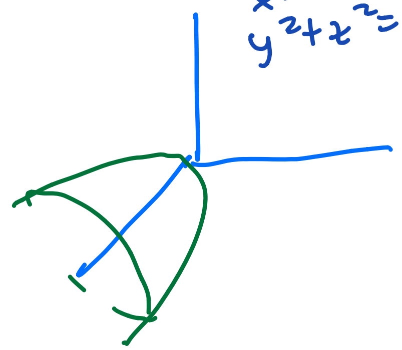
$$0 \le \rho \le 1$$
$$\pi/4 \le \phi \le \pi/2$$
$$0 \le \theta \le 2\pi$$

$$\boxed{\frac{\pi\sqrt{2}}{3}}$$

Problem 12. Find the **surface area** of the part of the paraboloid $x = y^2 + z^2$ between the planes $x = 0$ and $x = 4$.

$\vec{r}(u,v) = \langle u^2+v^2, u, v \rangle$

$x=4 \Rightarrow y^2+z^2=4$



$\vec{r}_u = \langle 2u, 1, 0 \rangle$

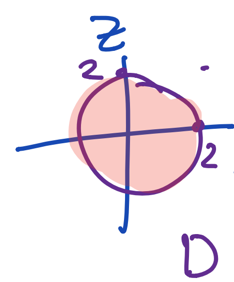
$\vec{r}_v = \langle 2v, 0, 1 \rangle$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 1 & 0 \\ 2v & 0 & 1 \end{vmatrix} = \vec{i}\cdot 1 - \vec{j}\cdot 2u + \vec{k}(-2v)$$

$$= \langle 1, -2u, -2v \rangle$$

$$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{1+4u^2+4v^2}$$

$$\iint_D \sqrt{1+4u^2+4v^2}\, dA$$



$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$u = 1+4r^2$

$du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

$r=0 \Rightarrow u=1$

$r=2 \Rightarrow u=17$

$$= \int_{\theta=0}^{2\pi} d\theta \int_{u=1}^{17} \sqrt{u}\cdot\frac{1}{8}\,du$$

$$= 2\pi\; u^{3/2}\cdot\frac{2}{3}\cdot\frac{1}{8}\Big|_1^{17}$$

$$= \frac{\pi}{6}\left(17\sqrt{17}-1\right)$$

$$\boxed{\frac{\pi}{6}\left(17\sqrt{17}-1\right)}$$