KEY

Math 232: Final Exam A
Spring 2016
Instructor: Linda Green

- Basic or scientific calculators are allowed. Graphing calculators are not allowed. Please list the model of your calculator here.

- There is a formula sheet on the last page. Feel free to tear it off.
- True False / Multiple Choice questions 1 - 26 should go on the scantron. Since you have test version A, please code your scantron sequence number as 111111 (all 1's).
- For short answer questions, you must show work for full and partial credit. All work to be graded needs to go on the test. If you need extra room, please use the formula sheet or blank last page.
- Give exact values instead of decimal approximations unless otherwise specified.
- Sign the honor pledge below after completing the exam.

First and last name .....

PID .....

UNC Email .....

Instructor (circle one): Elizabeth McLaughlin OR Linda Green

Recitation TA (circle one): Carol Sadek, Cass Sherman, Chen Shen, Michael Senter

Honor Pledge: I have neither given nor received unauthorized help on this exam.

Signature: .....

True or False (2 pts each). Note that True means always true, and False means sometimes or always false. n represents a non-negative integer and all a_n and b_n are real numbers.

1. True or False: $\displaystyle\int_{-1}^{1} \frac{1}{x^3}\,dx = -\frac{x^{-2}}{2}\Big|_{-1}^{1} = 0$

   $\displaystyle\int_{-1}^{0} x^{-3}\,dx + \int_{0}^{1} x^{-3}\,dx$

   $= -\frac{1}{2}x^{-2}\Big|_{-1}^{0} + -\frac{x^{-2}}{2}\Big|_{0}^{1}$

   $=$ diverges

   A. True
   B. False

2. True or False: $\displaystyle\int_{1}^{\infty} \sin x\,dx$ diverges.

   A. True
   B. False

3. True or False: If $a_n < a_{n+1} < 0$ for all n, then $\{a_n\}_{n=1}^{\infty}$ converges.

   A. True
   B. False

   monotonic & bdd

4. True or False: If $\{a_n\}_{n=1}^{\infty}$ converges to zero, then $\{a_n b_n\}_{n=1}^{\infty}$ converges to zero for any sequence $\{b_n\}_{n=1}^{\infty}$.

   A. True
   B. False

5. True or False: If $\{a_n\}_{n=1}^{\infty}$ converges , then $\left\{\dfrac{a_n}{n}\right\}_{n=1}^{\infty}$ converges.

   A. True
   B. False

   Squeeze Thm

6. True or False: If $\lim_{n\to\infty} a_n = 0$, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

   A. True
   B. False

(2 pts each) For each of the following series, decide if the series converges or diverges.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$

   A. Converges

   B. Diverges

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln(n+1)$

   A. Converges

   B. Diverges

9. $1 - \frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots$

   A. Converges

   B. Diverges

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$

    A. Converges

    B. Diverges

11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\pi}{4}\right)^n$

    A. Converges

    B. Diverges

12. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+e^n}$

    A. Converges

    B. Diverges

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^{n-1}}{2^{n+1}}$

    A. Converges

    B. Diverges

14. (5 pts) Consider the curve $(x-8)^2 + (y+6)^2 = 100$. This curve can be written in parametric equations as follows.

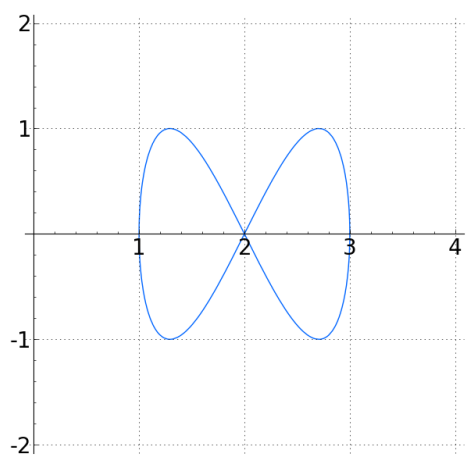
A. $x = t,\ y = \sqrt{100-(t-8)^2} - 6$

B. $x = 10\cos(t) + 8,\ y = 10\sin(t) - 6$

C. $x = 10\cos(t-8),\ y = 10\sin(t+6)$

D. $x = \dfrac{\cos(t)}{10} + 8,\ y = \dfrac{\sin(t)}{10} - 6$

15. (5 pts) Set up an integral to find the arclength of the curve given by equations $x = \cos(t) + 2$, $y = \sin(2t)$ drawn below.



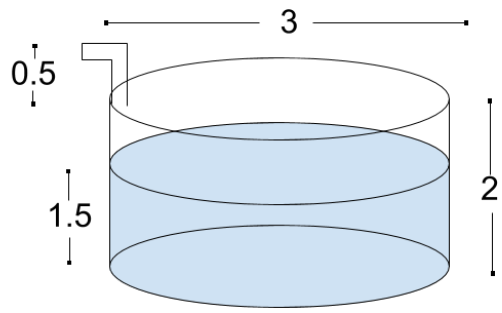
A. $2\displaystyle\int_1^3 \sqrt{\sin^2(t) + 4\cos^2(2t)}\,dt$

B. $2\displaystyle\int_0^\pi \sqrt{\sin^2(t) + 4\cos^2(2t)}\,dt$

C. $2\displaystyle\int_1^3 \sqrt{\cos^2(t) + 4\cos(t) + 4 + \sin^2(2t)}\,dt$

D. $2\displaystyle\int_0^\pi \sqrt{\cos^2(t) + 4\cos(t) + 4) + \sin^2(2t)}\,dt$

16. (5 points) A tank is shaped like a cylinder with a height of 2 feet and a diameter at the top of 3 feet. The tank is filled with water to a depth of 1.5 feet. SET UP an integral to find the work done to empty the tank by pumping the water up out of the top of the tank, through a tube that extends 0.5 feet above the top of the tank. Use the fact that one cubic foot of water weighs 62.5 lbs. Let y represent the distance from the bottom of the tank.



A. $\int_0^2 62.5\pi(1.5)^2\,dy$

B. $\int_0^{1.5} 62.5\pi(1.5)^2(2-y)\,dy$

C. $\int_0^{1.5} 62.5\pi(1.5)^2(2.5-y)\,dy$

D. $\int_0^2 62.5\pi(1.5)^2(y)\,dy$

E. $\int_0^{1.5} 62.5\pi(1.5)^2(y+0.5)\,dy$

17. (5 pts) The base of a solid is the triangular region bounded by the lines $y = 2x$, $y = -2x$, and $y = 6$. Cross-sections perpendicular to the y-axis are squares. Which expression represents the volume of the solid?

A. $\int_0^3 4x^2\,dx$

B. $\int_0^3 \frac{x^2}{4}\,dx$

C. $\int_0^6 4y^2\,dy$

D. $\int_0^6 \frac{y^2}{4}\,dy$

E. $\int_0^6 y^2\,dy$

18. (5 pts) $\displaystyle\int_1^\infty \frac{1+e^{-x}}{x}\,dx$

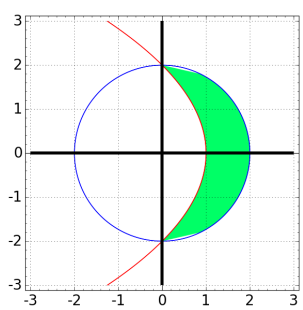
A. converges because $0 \le \dfrac{1+e^{-x}}{x} \le \dfrac{2}{x}$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ converges

B. diverges because $0 \le \dfrac{1+e^{-x}}{x} \le \dfrac{2}{x}$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges

C. converges because $\dfrac{1}{x} \le \dfrac{1+e^{-x}}{x}$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ converges

D. diverges because $\dfrac{1}{x} \le \dfrac{1+e^{-x}}{x}$ and $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges

19. (5 pts) Which expression gives the area of the shaded region between the curves $x^2 + y^2 = 4$ and $x = 1 - \dfrac{y^2}{4}$?



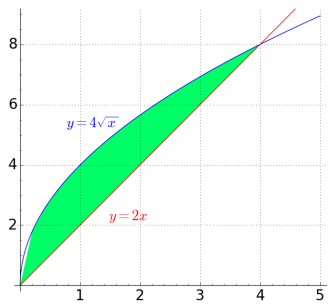
A. $\displaystyle 2\int_1^2 \sqrt{4-x^2} - \sqrt{4-4x}\,dx$

B. $\displaystyle 2\int_1^2 \sqrt{4-4x} - \sqrt{4-x^2}\,dx$

C. $\displaystyle 2\int_0^2 \sqrt{4-y^2} - 1 + \frac{y^2}{4}\,dy$

D. $\displaystyle 2\int_0^2 1 - \frac{y^2}{4} - \sqrt{4-y^2}\,dy$

20. (5 pts) The shaded region between the curves $y = 4\sqrt{x}$ and $y = 2x$ is rotated around the line $y = 8$. Which expression gives the volume of the resulting solid?



A. $\pi \int_0^4 16x - 4x^2 \, dx$

B. $\pi \int_0^4 (2x - 4\sqrt{x} - 8)^2 \, dx$

C. $\pi \int_0^4 (4\sqrt{x} - 2x)^2 \, dx$

D. $\pi \int_0^4 (8 - 4\sqrt{x})^2 - (8 - 2x)^2 \, dx$

E. $\pi \int_0^4 (8 - 2x)^2 - (8 - 4\sqrt{x})^2 \, dx$

21. (5 pts) $\int \sqrt{x} \ln x \, dx = \frac{2}{3}x^{3/2} \ln x - A$, where A is:

A. $\frac{1}{2}x^{-1/2} - x \ln x + x + C$

B. $\frac{1}{3}x^{-1/2} + C$

C. $\frac{4}{9}x^{3/2} - x \ln x + x + C$

D. $\frac{4}{9}x^{3/2} + C$

E. $x \ln x - x + C$

22. (5 pts) Find the radius of convergence for the power series $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{5^n \sqrt[3]{n}}$

A. 0

B. 1

C. 2

D. 5

E. 7

23. (5 pts) The power series $\sum_{n=1}^{\infty} \frac{2^n (x-1)^n}{n+3}$ has a radius of convergence of $\frac{1}{2}$. Find the interval of convergence.

A. $\left[\frac{1}{2}, \frac{3}{2}\right]$

B. $\left[\frac{1}{2}, \frac{3}{2}\right)$

C. $\left(\frac{1}{2}, \frac{3}{2}\right]$

D. $\left(\frac{1}{2}, \frac{3}{2}\right)$

24. (5 pts) Suppose $f(x)$ has a power series given by $f(x) = \sum_{n=0}^{\infty} \frac{3(x-13)^n}{n+1}$

What is $f^{(n)}(13)$, the nth derivative of $f(x)$ at $x = 13$?

A. 3

B. $\frac{3}{n+1}$

C. $\frac{3}{(n+1)!}$

D. $\frac{3n!}{n+1}$

E. $3(n+1)!$

25. (5 pts) For a series $a_1 + a_2 + a_3 + \cdots$, suppose that the first partial sum is $S_1 = 7 - \frac{1}{4}$, the second partial sum is $S_2 = 7 - \frac{1}{5}$, the third partial sum is $S_3 = 7 - \frac{1}{6}$, the fourth partial sum is $S_4 = 7 - \frac{1}{7}$ and so on. Which one of the following can we conclude?

A. The series $a_1 + a_2 + a_3 + \cdots$ diverges by the Divergence Test.

B. The series $a_1 + a_2 + a_3 + \cdots$ diverges because the harmonic series diverges.

C. The series $a_1 + a_2 + a_3 + \cdots$ converges, and the sum of the series is 7.

D. The series $a_1 + a_2 + a_3 + \cdots$ converges, but not enough information is given to determine the sum of the series.

E. Not enough information is given to determine convergence or divergence.

26. (5 pts) Find the sum of the series. $\displaystyle\sum_{n=2}^{\infty} \frac{(-2)^n}{3^{n-1}}$

A. $\frac{3}{5}$

B. $\frac{4}{5}$

C. 3

D. 4

E. The series diverges.

For each series, circle CONVERGES or DIVERGES, circle the correct justification, and fill in the appropriate blank. If more than one justification applies, just circle one justification that represents the first step in your argument. You DO NOT have to complete the problem or show work.

27. (6 pts) $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{(n-1)\sqrt{n+4}}$ CONVERGES DIVERGES

A. Divergence Test, where limit of terms is $\boxed{}$

B. Comparison Test (ordinary or limit), comparing series with $\sum b_n$ where $b_n =$ $\boxed{\frac{1}{\sqrt{n}}}$

C. Integral Test, using function $f(x) =$ $\boxed{}$

D. Alternating Series Test

E. Ratio Test, where the limit of ratio is $\boxed{}$

28. (6 pts) $\sum_{n=1}^{\infty} \frac{10^{n+1}}{n!}$ CONVERGES DIVERGES

A. Divergence Test, where limit of terms is ☐

B. Comparison Test (ordinary or limit), comparing series with $\sum b_n$ where $b_n =$ ☐

C. Integral Test, using function $f(x) =$ ☐

D. Alternating Series Test

E. Ratio Test, where the limit of ratio is ☐ 0

29. (6 pts) $\sum_{n=1}^{\infty} \frac{1}{e^{1/n}}$ CONVERGES DIVERGES

A. Divergence Test, where limit of terms is ☐ 1

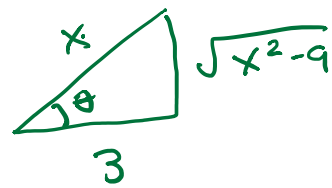
B. Comparison Test (ordinary or limit), comparing series with $\sum b_n$ where $b_n =$ ☐

C. Integral Test, using function $f(x) =$ ☐

D. Alternating Series Test

E. Ratio Test, where the limit of ratio is ☐

30. (10 pts) Using an appropriate substitution, transform $\int \frac{dx}{x^2\sqrt{x^2-9}}$ into $\frac{1}{9}\int \cos u\, du$. Do not integrate.



$\cos\theta = \frac{3}{x}$

$x = \frac{3}{\cos\theta}$

optio

$x = 3\sec\theta$ ✓✓

$dx = 3\sec\theta\tan\theta\, d\theta$ ✓✓

$$\int \frac{dx}{x^2\sqrt{x^2-9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2-9}}$$ ✓✓

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{9\sec^2\theta\sqrt{9\tan^2\theta}}$$ ✓

$$= \int \frac{3\sec\theta\cancel{\tan\theta}\, d\theta}{27\sec^2\theta\cancel{\tan\theta}}$$ ✓

$$= \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int \cos\theta\, d\theta$$ ✓✓

$$= \frac{1}{9}\int u\, du$$

31. (10 pts) Using an appropriate substitution, transform $\int \sin^8 x \cos^3 x \, dx$ into $\int (u^8 - u^{10}) \, du$. Do not integrate.

$$\int \sin^8 x \cos^3 x \, dx = \int \sin^8 x \cos^2 x \cos x \, dx$$

$u = \sin x$ $du = \cos x \, dx$

$$= \int \sin^8 x \, (1 - \sin^2 x) \cos x \, dx$$

$$= \int u^8 (1 - u^2) \, du$$

$$= \int u^8 - u^{10} \, du$$

32. (10 pts) Find a Taylor series for $f(x) = 3e^{-4x}$ centered at $a = 5$. Write your answer in summation notation.

n	$f^{(n)}(x)$	$f^{(n)}(5)$
0	$3e^{-4x}$	$3e^{-20}$
1	$3(-4)e^{-4x}$	$3(-4)e^{-20}$
2	$3(-4)^2 e^{-4x}$	$3(-4)^2 e^{-20}$
3	$3(-4)^3 e^{-4x}$	$3(-4)^3 e^{-20}$
n	$3(-4)^n e^{-4x}$ ✓✓	$3(-4)^n e^{-20}$ ✓✓

$$T(x) = \sum_{n=0}^{\infty} \frac{3(-4)^n e^{-20}}{n!} (x-5)^n$$ ✓✓ ✓✓

2 pts deriv

2 pts plugging in 5 to deriv

2 pts center at 5

2 pts $n!$

2 pts x^n term and $\sum_{n=0}^{\infty}$

Answer: $$\boxed{T(x) = \sum_{n=0}^{\infty} \frac{3(-4)^n e^{-20}}{n!} (x-5)^n}$$ ✓✓ ✓✓

33. (11 pts)

(a) Write down the Maclaurin series for $\ln\left(1+\frac{x^2}{3}\right)$. Write your answer in summation notation.

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln\left(1+\frac{x^2}{3}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\left(\frac{x^2}{3}\right)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{3^n \cdot n}$$

Answer: $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{3^n \cdot n}}$

(b) Write down the Maclaurin series for $\int \ln\left(1+\frac{x^2}{3}\right) dx$. Write your answer in summation notation.

$$\int \ln\left(1+\frac{x^2}{3}\right) dx = \int \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{3^n \cdot n}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{3^n \cdot n(2n+1)} + C$$

-0.5 for missing $+C$

Answer: $\boxed{\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{3^n \cdot n(2n+1)} + C}$

(c) Use the first three terms for this series to approximate $\int_0^1 \ln\left(1+\frac{x^2}{3}\right) dx$. Your answer can be an exact number or a decimal rounded to 4 decimal places.

$$\int_0^1 \ln\left(1+\frac{x^2}{3}\right) dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n+1}}{3^n \cdot n(2n+1)} \Bigg|_0^1$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 1^{2n+1}}{3^n \cdot n(2n+1)} - 0 \approx \frac{1}{3 \cdot 1 \cdot 3} - \frac{1}{9 \cdot 2 \cdot 5} + \frac{1}{27 \cdot 3 \cdot 7}$$

$$= \frac{577}{5670} \approx 0.1018$$

Answer: $\boxed{\frac{577}{5670} \text{ or } 0.1018}$

TAYLOR SERIES FORMULAS

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

TRIG FORMULAS

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$\cot^2(\theta) + 1 = \csc^2(\theta)$

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$\cos(2\theta) = 1 - 2\sin^2(\theta)$

$\cos(2\theta) = 2\cos^2(\theta) - 1$

$\sin^2(\theta) = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$