Math 232: Final Exam Version A
Spring 2015
Instructor: Linda Green

KEY

Name: ______________________________

1. Calculators are allowed.
2. You must show work for full and partial credit unless otherwise noted. In particular, you must evaluate integrals by hand and show work.
3. Give exact values instead of decimal approximations.
4. Sign the honor pledge below after completing the exam.

HONOR PLEDGE: (Sign if true; otherwise, leave blank.)
I have neither given nor received unauthorized help on this exam.

______________________________(Signature)

1. (10 points each) Pick TWO integrals to compute. Show work for credit.

   (a) $\int \cos^3(x)\sin^2(x)\,dx$

   (b) $\int \frac{x+3}{3x^2+4x+1}\,dx$

   (c) $\int \sqrt{4-x^2}\,dx$

a) $\int \cos^3(x)\sin^2(x)\,dx = \int \cos^2(x)\sin^2(x)\cos(x)\,dx$

$u = \sin(x)$

$du = \cos(x)\,dx$

$= \int (1-\sin^2(x))\sin^2(x)\cos(x)\,dx$

$= \int (1-u^2)\,u^2\,du = \int u^2 - u^4\,du$

$= \frac{u^3}{3} - \frac{u^5}{5} + C = \boxed{\frac{\sin^3(x)}{3} - \frac{\sin^5(x)}{5} + C}$

---

b) $\int \frac{x+3}{3x^2+4x+1}\,dx$

$\frac{x+3}{3x^2+4x+1} = \frac{A}{3x+1} + \frac{B}{x+1}$

$x+3 = A(x+1) + B(3x+1)$

$x=-1 \Rightarrow 2 = B(-2) \Rightarrow B=-1$

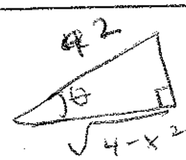
$x=-\frac{1}{3} \Rightarrow \frac{8}{3} = A(\frac{2}{3}) \Rightarrow A=4$

$\int \frac{4}{3x+1} + \frac{-1}{x+1}\,dx$

$= \boxed{\frac{4}{3}\ln|3x+1| - \ln|x+1| + C}$

---

c) $\int \sqrt{4-x^2}\,dx$



$\sin\theta = \frac{x}{2}$ $\quad x = 2\sin\theta$

$dx = 2\cos\theta\,d\theta$

$= \int \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\,d\theta$

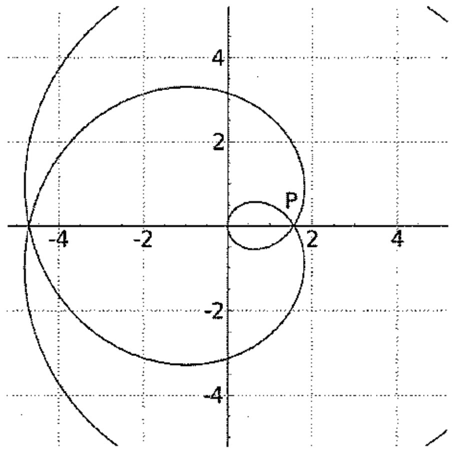
$= \int 2\sqrt{1-\sin^2\theta}\cdot 2\cos\theta\,d\theta = 4\int \sqrt{\cos^2\theta}\,\cos\theta\,d\theta$

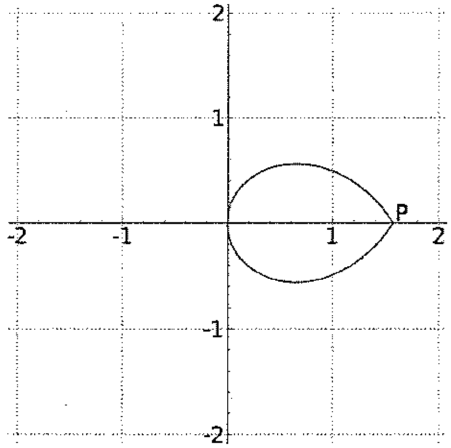
$= 4\int \cos^2\theta\,d\theta = 4\int \frac{1}{2} + \frac{\cos 2\theta}{2}\,d\theta = 2\int 1+\cos 2\theta\,d\theta$

$= 2\left[\theta + \frac{\sin 2\theta}{2}\right] + C = 2\left[\arcsin(\tfrac{x}{2}) + \frac{2\sin\theta\cos\theta}{2}\right] + C$

$= 2\left[\arcsin(\tfrac{x}{2}) + \frac{x}{2}\cdot\frac{\sqrt{4-x^2}}{2}\right] + C = \boxed{2\arcsin(\tfrac{x}{2}) + \frac{x\sqrt{4-x^2}}{2} + C}$

2. Consider the curve given parametrically by $x = t\sin(t)$, $y = t\cos(t)$. See the graph below and the close-up of the "teardrop" at right.





(a) (2 points) Find the exact x and y coordinates of the point P of self-intersection shown on the graph.

$(x, y) = \boxed{\left(\pi/2,\ 0\right)}$

(b) (6 points) SET UP the integral to compute the area inside the innermost teardrop shape. Be sure to include bounds of integration.

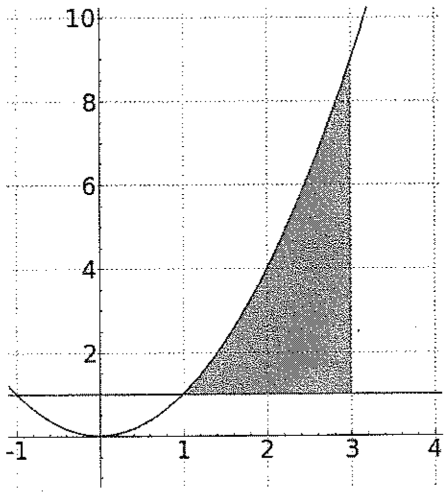
$$2\int_0^{\pi/2} y\,dx = 2\int_0^{\pi/2} (t\cos t)(t\cos t + \sin t)\,dt$$

(c) (6 points) SET UP the integral for the arclength of the boundary of the innermost teardrop shape. Be sure to include bounds of integration.

$$\int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= 2\int_0^{\pi/2} \sqrt{(t\cos t + \sin t)^2 + (-t\sin t + \cos t)^2}\,dt$$

3. Consider the region bounded by $y = x^2$, $y = 1$, and $x = 3$.



(a) (6 points) SET UP the integral to compute the volume of the solid formed by rotating this region around the horizontal line $y = 1$.

$$\boxed{\int_{x=1}^{x=3} \pi \left(x^2 - 1\right)^2 \, dx}$$

(b) (6 points) SET UP the integral to compute the volume of the solid formed by rotating the region around the y-axis.

$$\int_{y=1}^{y=9} \pi \left(r_{outer}\right)^2 - \pi \left(r_{inner}\right)^2 \, dy \qquad x = \sqrt{y}$$

$$\int_1^9 \pi 9 - \pi \left(\sqrt{y}\right)^2 \, dy$$

$$\boxed{\pi \int_1^9 9 - y \, dy}$$

✓E for 'squared' in the right places

4. (12 points) Evaluate the integral

$$\int_0^\infty e^{-x}\sin(x)\,dx$$

$u = \sin x \qquad dv = e^{-x}\,dx$

$du = \cos x\,dx \qquad v = -e^{-x}$

$$\int e^{-x}\sin x\,dx = -\sin x\, e^{-x} - \int -e^{-x}\cos x\,dx$$

$$= -\sin x\, e^{-x} + \int e^{-x}\cos x\,dx$$

$u = \cos x \qquad dv = e^{-x}\,dx$

$du = -\sin x\,dx \qquad v = -e^{-x}$

$$= -\sin x\, e^{-x} + \left[-e^{-x}\cos x - \int -e^{-x}(-\sin x)\,dx\right]$$

So

$$\underbrace{\int e^{-x}\sin x\,dx}_{A} = -\sin x\, e^{-x} - \cos x\, e^{-x} - \underbrace{\int e^{-x}\sin x\,dx}_{A}$$

$$\Rightarrow \quad 2A = -\sin x\, e^{-x} - \cos x\, e^{-x}$$

$$\Rightarrow \quad A = -\tfrac{1}{2}\sin x\, e^{-x} - \tfrac{1}{2}\cos x\, e^{-x}$$

So $$\int_0^\infty e^{-x}\sin x\,dx = -\tfrac{1}{2}\sin x\, e^{-x} - \tfrac{1}{2}\cos x\, e^{-x}\Big|_0^\infty$$

$$= \lim_{t\to\infty} -\tfrac{1}{2}\sin x\, e^{-x} - \tfrac{1}{2}\cos x\, e^{-x}\Big|_0^t$$

$$= \lim_{t\to\infty}\left(-\tfrac{1}{2}\sin t\, e^{-t} - \tfrac{1}{2}\cos t\, e^{-t}\right) - \left(-\tfrac{1}{2}\cancel{\sin 0}^{\,0}\, e^{-0} - \tfrac{1}{2}\cos 0\, e^{-0}\right)$$

$$= \lim_{t\to\infty} \frac{-\frac{1}{2}\sin t}{e^t} - \frac{\frac{1}{2}\cos t}{e^t} + \frac{1}{2}$$

Since $-1 \le \sin t \le 1$

$\frac{-1}{e^t} \le \frac{\sin t}{e^t} \le \frac{1}{e^t}$ and since $\lim_{t\to\infty}\frac{1}{e^t} = 0 = \lim_{t\to\infty}\frac{-1}{e^t}$

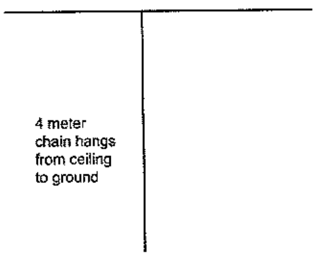
by squeeze thm $\lim_{t\to\infty}\frac{\sin t}{e^t} = 0 \Rightarrow \lim_{t\to\infty}\frac{-\frac{1}{2}\sin t}{e^t} = 0$

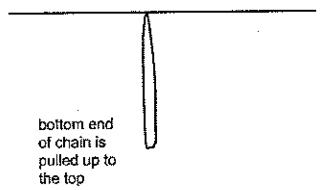
Similarly, $\lim_{t\to\infty}\frac{-\frac{1}{2}\cos t}{e^t} = 0 \Rightarrow$ final answer is $\boxed{\frac{1}{2}}$

5. (8 pts) SET UP the integral for ONE of the following two problems.

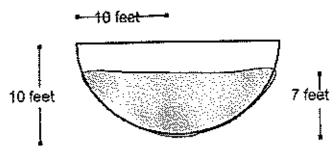
PROBLEM A) A 20 kg chain that is 4 meters long hangs from the ceiling of a 4 meter high room, so that the bottom is just touching the ground. How much work is required to lift the lower end of the chain up to the ceiling so that it is level with the upper end, as in the figure? See the figure. (Acceleration due to gravity is $g = 9.8 m/s^2$.)

Hint: if a small chunk of chain from the bottom half of the chain starts out x meters from the floor, it will end up x meters from the ceiling. How far does it have to travel to get there?

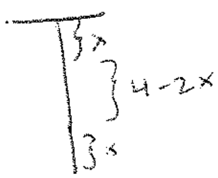




PROBLEM B) A hemispherical tank of water of radius 10 feet is filled with water to a depth of 7 feet. How much work will it take to empty the tank by pumping water out of the top? (One cubic foot of water weighs 62.5 pounds.)



A)



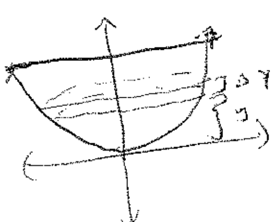
$F = \frac{20}{4}\ kg/m \cdot \Delta x\ m \cdot 9.8\ m/s^2 = 5 \cdot 9.8\, \Delta x$ Newtons

$d = 4 - 2x$

✓ nothing extraneous

$$\boxed{\int_0^2 5 \cdot 9.8\,(4-2x)\,dx} = 196 \text{ J}$$

B)



$d = 10 - y$

$(y-10)^2 + x^2 = 10^2$

$x^2 = 10^2 - (y-10)^2$

$F = \Delta y\, \pi r^2 \cdot 62.5$

$= \Delta y\, \pi x^2 \cdot 62.5$

$= \pi(100 - (y-10)^2) \cdot 62.5\, \Delta y$

$$\text{Work} = \int_0^7 \pi\left(100 - (y-10)^2\right) \cdot 62.5\,(10-y)\,dy$$

.. (6 points) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{3^n(x-5)^n}{n!}$. Show work for credit.

$R = \boxed{\infty}$ ✓

$$\lim_{n\to\infty} \left| \frac{\frac{3^{n+1}(x-5)^{n+1}}{(n+1)!}}{\frac{3^n(x-5)^n}{n!}} \right| \quad ✓✓$$

$$= \lim_{n\to\infty} \left| 3(x-5) \frac{n!}{(n+1)!} \right| \quad ✓$$

$$= \lim_{n\to\infty} \left| \frac{3(x-5)}{n+1} \right| \quad ✓$$

7. (6 points) The value and the first 5 derivatives of $f(x) = x^x$ at $x = 1$ are give below. Give the 4th degree Taylor polynomial for $f(x)$ centered at 1. (That is, the first 5 terms of the Taylor series.)

$f(1) = 1$
$f'(1) = 1$
$f''(1) = 2$
$f^{(3)}(1) = 3$
$f^{(4)}(1) = 8$
$f^{(5)}(1) = 10$

$$T_4(x) = f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f^{(3)}(1)}{3!}(x-1)^3 + \frac{f^4(1)(x-1)^4}{4!}$$

$$= 1 + (x-1) + (x-1)^2 + \tfrac{1}{2}(x-1)^3 + \tfrac{1}{3}(x-4)^4$$

2 pts $(x-1)^n$ — subtract 1 pt if centered at 0

1 pt $n!$ — 2 pt substitute in #'s

1 pt $f^{(n)}(1)$ — -1 pt incorrect # terms & adding extraneous

formula only $\to$ 3 pts

8. (10 points)

(a) Find a Taylor series representation for $\int e^{-\frac{x^2}{2}}\,dx$. Write your answer in sigma notation.

(b) Use the first three terms of your Taylor series to estimate $\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-\frac{x^2}{2}}\,dx$. (This represents the area under the standard normal curve between -1 and 1 standard deviations.)

a) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ✓✓

$e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$

✓✓

$\int_{-1}^{1} e^{-x^2/2}\,dx = \int_{-1}^{1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}\,dx$

b) $= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^n n!(2n+1)}\Bigg|_{-1}^{1}$ ✓✓

$= \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2^n n!(2n+1)} - \sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2^n n!(2n+1)}$ ✓✓

$= 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)}$

$\frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{-x^2/2}\,dx = \frac{2}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!(2n+1)}$

1st 3 terms:

$\frac{2}{\sqrt{2\pi}}\left(\frac{1}{1} - \frac{1}{2\cdot 1\cdot 3} + \frac{1}{4\cdot 2\cdot 5}\right) = \frac{103}{120}\frac{\sqrt{2}}{\sqrt{\pi}}$ ✓✓

≈ 0.685

or $\frac{103}{60\sqrt{2\pi}}$

9. (4 points each) Find the exact sum of the series:

(a) $-\frac{3}{5}+\frac{12}{25}-\frac{48}{125}+\frac{192}{625}-\cdots$

geometric: $a = -\frac{3}{5}$ $r = -\frac{4}{5}$

$$\text{sum} = \frac{-\frac{3}{5}}{1-(-\frac{4}{5})} = \boxed{-\frac{1}{3}}$$

(b) $\frac{1}{2}-\frac{1}{2^3\cdot 3}+\frac{1}{2^5\cdot 5}-\frac{1}{2^7\cdot 7}+\cdots$

arctan($\frac{1}{2}$)

~~2 pts decimal~~

~~2 pts exact answer~~

~~1 pt arctan~~

2 pts arctan, 2 pts $\frac{1}{2}$ OR 2 pts decimal answer

10. (2 points each, no work needs to be shown) True or False. Remember that True means always true, and False means sometimes or always false. Be careful to distinguish between SEQUENCES and SERIES. The a_n represent real numbers, and could be positive, negative, or zero.

(a) **True** **False** Every elementary function has an elementary antiderivative. (Recall that an elementary function is a function that can be built out of the usual functions like polynomials, trig functions, and exponential and log functions using addition, subtraction, multiplication, division, and composition. Recall that an antiderivative is the same thing as an indefinite integral.)

(b) **True** **False** If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} |a_n|$ also converges.

(c) **True** **False** If $\sum_{n=1}^{\infty} a_n$ converges, then $\{a_n\}_{n=1}^{\infty}$ also converges.

(d) **True** **False** $\sum_{k=0}^{\infty} \frac{1}{(k+1)^2}$ and $\sum_{k=2}^{\infty} \frac{1}{(k-1)^2}$ both converge to the exact same number.

2 pts 1 pt

11. (4 points each) For each series, circle CONVERGES or DIVERGES, circle the correct justification, and fill in the blanks. If more than one justification applies, just circle one justification that represents the first step in your argument. You DO NOT have to complete the problem or show work.

1 pt

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(3n+7)}{\ln(5n+1)}$ CONVERGES DIVERGES

i. Geometric Series Test, where $r = \boxed{\quad}$
ii. P-Test, with $p = \boxed{\quad}$
iii. Divergence Test, where limit of terms is $\boxed{1}$
iv. Ordinary Comparison Test, comparing series with $\sum b_n$ where $b_n = \boxed{\quad}$
v. Limit Comparison Test, comparing series with $\sum b_n$ where $b_n = \boxed{\quad}$
vi. Integral Test, using function $f(x) = \boxed{\quad}$
vii. Alternating Series Test
viii. Ratio Test
ix. Root Test

(b) $\displaystyle\sum_{n=1}^{\infty} ne^{-2n}$ CONVERGES DIVERGES

i. Geometric Series Test, where $r = \boxed{\quad}$
ii. P-Test, with $p = \boxed{\quad}$
iii. Divergence Test, where limit of terms is $\boxed{\quad}$
iv. Ordinary Comparison Test, comparing series with $\sum b_n$ where $b_n = \boxed{\quad}$
v. Limit Comparison Test, comparing series with $\sum b_n$ where $b_n = \boxed{\quad}$
vi. Integral Test, using function $f(x) = \boxed{\quad}$
vii. Alternating Series Test
viii. Ratio Test
ix. Root Test

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{5n+7}{3n^3-2n^2+1}$ CONVERGES DIVERGES

i. Geometric Series Test, where $r = \boxed{\quad}$
ii. P-Test, with $p = \boxed{\quad}$
iii. Divergence Test, where limit of terms is $\boxed{\quad}$
iv. Ordinary Comparison Test, comparing series with $\sum b_n$ where $b_n = \boxed{\quad}$
v. Limit Comparison Test, comparing series with $\sum b_n$ where $b_n = \boxed{\frac{1}{n^2}}$
vi. Integral Test, using function $f(x) = \boxed{\quad}$
vii. Alternating Series Test
viii. Ratio Test
ix. Root Test

(d) $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ CONVERGES DIVERGES

i. Geometric Series Test, where $r =$ $\boxed{}$

ii. P-Test, with $p =$ $\boxed{}$

iii. Divergence Test, where limit of terms is $\boxed{}$

iv. Ordinary Comparison Test, comparing series with $\sum b_n$ where $b_n =$ $\boxed{}$

v. Limit Comparison Test, comparing series with $\sum b_n$ where $b_n =$ $\boxed{}$

vi. Integral Test, using function $f(x) =$ $\boxed{\frac{1}{x \ln x}}$

vii. Alternating Series Test

viii. Ratio Test

ix. Root Test

12. (5 points) Prove that $\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ converges to a finite value. You DO NOT need to find the value.

$$\int_0^\infty \frac{1}{\sqrt{x}+x^2}\,dx = \int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx + \int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx \quad \checkmark$$

$\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$: compare to $\frac{1}{\sqrt{x}}$ since $\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges by p-test $p = \frac{1}{2} < 1$ ✓

and $\frac{1}{\sqrt{x}+x^2} < \frac{1}{\sqrt{x}}$ ✓

$\int_0^1 \frac{1}{\sqrt{x}+x^2}\,dx$ also converges

$\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ compare to $\frac{1}{x^2}$ since $\int_1^\infty \frac{1}{x^2}\,dx$ converges by p-test $p = 2 > 1$ ✓

and $\frac{1}{\sqrt{x}+x^2} < \frac{1}{x^2}$ ✓

$\int_1^\infty \frac{1}{\sqrt{x}+x^2}\,dx$ converges also.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

TRIG FORMULAS

Pythagorean identities

$\cos^2(\theta) + \sin^2(\theta) = 1$

$\tan^2(\theta) + 1 = \sec^2(\theta)$

$\cot^2(\theta) + 1 = \csc^2(\theta)$

Even / Odd

$\cos(-t) = \cos(t)$

$\sin(-t) = -\sin(t)$

Angle Sum and Angle Difference

$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$

Double Angle Formulas

$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$

$\cos(2\theta) = 1 - 2\sin^2(\theta)$

$\cos(2\theta) = 2\cos^2(\theta) - 1$

Half Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Phase Shift

$\sin(t) = \cos(t - \frac{\pi}{2})$

$\cos(t) = \sin(t + \frac{\pi}{2})$