Math 232 Final Exam Review Answers

Sections covered: 6.1, 6.2, 6.4, 6.5, 7.1, 7.2 (powers of sine and cosine only), 7.3, 7.4, 7.5, 7.8, 10.1, 10.2, 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7, 11.8, 11.9, 11.10, 11.11

Note: There will be some true false and multiple choice concept questions on the exam. I recommend working true false questions and concept checks in the chapter review sections for practice.

Note: The following problems are mostly from the review problems in the textbook.

1. Find the area of the region bounded by the curves:
   (a) \( y = 1 - 2x^2, y = |x| \)
   Answer: 7/12
   (b) \( x + y = 0, x = y^2 + 3y \)
   Answer: 32/3
   (c) \( y = a \sqrt{x}, y = x^2 \)
   Answer: \( \frac{a^2}{3} \)

2. Set up the integral to find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.
   (a) \( x = 0, x = 9 - y^2, \) about \( x = -1 \)
   Answer: \( \int_{-3}^{3} \pi \left( (9 - y^2) - (-1) \right)^2 - [0 - (-1)]^2 \) \( dy = \frac{1656\pi}{3} \)

3. Each integral represents the volume of a solid. Describe the solid.
   (a) \( \int_{0}^{\pi/2} 2\pi \cos^2 x \, dx \)
   Answer: rotate \( R = \{(x, y)|0 \leq x \leq \pi/2, 0 \leq y \leq \sqrt{2} \cos(x)\} \) about the x-axis
   (b) \( \int_{0}^{\pi} \pi(4 - \sin^2 x) \, dx \)
   Answer: rotate \( R = \{(x, y)|0 \leq x \leq \pi, \sin(x) \leq y \leq 2\} \) about the x-axis
   (c) \( \int_{0}^{\pi} \pi(2 - \sin x)^2 \, dx \)
   Answer: rotate \( R = \{(x, y)|0 \leq x \leq \pi, 0 \leq y \leq 2 - \sin(x)\} \) about the x-axis, OR rotate \( R = \{(x, y)|0 \leq x \leq \pi, \sin(x) \leq y \leq 2\} \) around the line \( y = 2 \)

4. The base of a solid is a square with vertices located at \((1, 0), (0, 1), (-1, 0), (0, -1)\). Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.
   Answer: \( 2 \int_{0}^{1} \frac{1}{2} \pi (1 - x)^2 \, dx = \frac{\pi}{3} \)

5. A monument in the shape of a square pyramid has height 20 meters. Its base is a square of side length 5 meters. Find the volume of the monument.
   Answer: \( \int_{0}^{20} \left( \frac{y}{4} \right)^2 \, dx = \frac{500}{3} \, m^3 \)

6. (p. 459 # 5a) Show that the volume of a segment of height \( h \) of a sphere of radius \( r \) is \( V = \frac{1}{3} \pi r^2 (3r - h) \).
   Answer: \( V = \pi h^2 (r - h/3) \)
7. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?
   Answer: 3.2 J

8. A 1600 lb elevator is suspended by a 200 ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?
   Answer: work for elevator = 1600\*30 = 48000, work for bottom 170 feet of cable = 170\*10\*30 = 51000, work for top 30 feet of cable = \(\int_0^3 010x \, dx = 4500\), so total work = 103500 ft-lbs

9. A tank full of water has the shape of a paraboloid of revolution as shown in the figure. That is, its shape is obtained by rotating a parabola about a vertical axis.
   (a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.
   Answer: (a) \(8000\pi/3 = 8378\) ft-lb
   (b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?
   Answer: (b) 2.1 ft

10. A steel tank has the shape of a circular cylinder oriented vertically with diameter 4 m and height 5 m. The tank is currently filled to a level of 3 m with cooking oil that has a density of 920 kg/m\(^3\). Compute the work required to pump the oil out through a 1-m spout at the top of the tank.
    Answer: \(\int_0^3 \pi \cdot 2^2 \cdot 920 \cdot 9.8(6 - y) \, dy = 486,684\pi \) kg m\(^2\)/s\(^2\)

11. Find the average value of the function \(f(t) = t \sin(t^2)\) on the interval [0, 10].
    Answer: \(\frac{1}{20}(1 - \cos 100)\)

12. Integrate by hand:
   (a) \(\int \frac{dt}{2t+3t^2+1}\)
    Answer: \(\ln |2t + 1| - \ln |t + 1| + C\) (partial fractions)
   (b) \(\int_0^{\pi/2} \sin^3 \theta \cos^2 \theta \, d\theta\)
    Answer: \(2/15\) (use trig identities)
   (c) \(\int_1^2 \frac{\sqrt{x^2-1}}{x} \, dx\)
    Answer: \(\sqrt{3} - \pi/3\) (trig substitution)
   (d) \(\int_0^{\pi/6} t \sin 2t \, dt\)
    Answer: \(-\pi/24 + \sqrt{3}/8\) (integration by parts)
   (e) \(\int_1^2 x^5 \ln x \, dx\)
    Answer: \(\frac{32}{3} \ln 2 - \frac{7}{4}\) (integration by parts)
   (f) \(\int e^{2t} \frac{dx}{1+e^t}\)
    Answer: \(\frac{1}{2} \tan^{-1}(e^{2t}) + C\) (u-substitution)
   (g) \(\int \frac{x^2+2}{x^2+2} \, dx\)
    Answer: \(\frac{1}{2}x^2 - 2x + 6 \ln |x + 2| + C\) (partial fractions)
   (h) \(\int e^x \cos x \, dx\)
    Answer: \(\frac{1}{2}e^x(\cos(x) + \sin(x)) + C\)

13. Evaluate the integral or prove that it is divergent.
(a) \( \int_0^4 4 \ln x \, dx \)
Answer: \( 4 \ln 4 - 8 \)

(b) \( \int_0^\infty \frac{\ln x}{x^3} \, dx \)
Answer: diverges because \( \int_0^4 \frac{\ln x}{x^3} \, dx \) diverges

(c) \( \int_0^1 \frac{1}{2x^3} \, dx \)
Answer: diverges because \( \int_0^{2/3} \frac{1}{2x} \left[ -\frac{1}{4} \ln |2 - 3x| \right]^x_0 = \infty \)

(d) \( \int_1^\infty \frac{2 + \sin x}{\sqrt{x}} \, dx \)
Answer: diverges by comparing to \( \frac{1}{\sqrt{x}} \)

14. Determine if the integral converges or diverges and prove your answer.

(a) \( \int_1^\infty \frac{1}{\sqrt{1+3t}} \, dx \)
Answer: converges by comparing to \( \int_0^\infty \frac{1}{x^2} \, dx \)

(b) \( \int_1^\infty \frac{x + 1}{\sqrt{x^4 - x}} \, dx \)
Answer: diverges by comparing to \( \int_1^\infty \frac{1}{x} \, dx \)

(c) \( \int_0^1 \frac{\sec^2 x}{x \sqrt{x}} \, dx \)
Answer: diverges by comparing to \( \int_0^1 \frac{1}{\sqrt{x}} \, dx \)

15. Find the length of the curve \( y = \frac{x^4}{16} + \frac{1}{2x^2}, \ 1 \leq x \leq 2 \)
Answer: \( \frac{21}{16} \)

16. Find parametric equations for the following curves:

(a) The curve \( y = \sqrt{x} \) Answer: \( x = t, \ y = \sqrt{t} \) for \( t \geq 0 \), or \( y = t, \ x = t^2 \) for \( t \geq 0 \).

(b) The curve \( (x - 3)^2 + (y - 5)^2 = 36 \) Answer: \( x = 6 \cos(t) + 5, \ y = 6 \sin(t) + 3, \ 0 \leq t < 2\pi \)

(c) The line segment between the points \((-2, 5)\) and \((3, 7)\). Answer: \( x = -2 + 5t, \ y = 5 + 2t, \ 0 \leq t \leq 1 \)

17. Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

(a) \( x = 2 \cos \theta, \ y = 1 + \sin \theta \)
Answer: \( x^2 + (y - 1)^2 = 1 \)

(b) \( x = t^2 + 4t, \ y = 2 - t, -4 \leq t \leq 1 \)
Answer: \( x = 12 - 8y + y^2, \ 1 \leq y \leq 6 \)

18. Use the graphs of \( x = f(t) \) and \( y = g(t) \) to sketch the parametric curve \( x = f(t), \ y = g(t) \). Indicate with arrows the direction in which the curve is traced as \( t \) increases.

Answer:

19. Find the length of the curve \( x = 3t^2, \ y = 2t^3 \) between the origin and the point \((12, 16)\) on the x-y plane.
Answer: \( 2\sqrt{5} \left( \sqrt{5} - 1 \right) \)

20. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.
(a) \( a_n = \frac{2 + n^2}{1 + 2n^3} \)
Answer: 1/2

(b) \( a_n = \cos(n\pi/2) \)
Answer: diverges

(c) \( a_n = \frac{\sin n}{n^2 + 1} \)
Answer: 0

(d) \( a_n = \frac{\ln n}{\sqrt{n}} \)
Answer: 0

(e) \( \left\{ (1 + 3/n)^{4n} \right\} \)
Answer: \( e^{12} \)

(f) \( \left\{ \frac{(-1)^n}{n!} \right\} \)
Answer: 0

(g) \( a_n = \frac{(-1)^n3^n}{2^n} \)
Answer: 0

Answer: All \( a_n \) are bounded above by 2. To see this, note that \( a_1 < 2. \) For any \( n, \) if it is true that \( a_n < 2, \) then \( a_{n+1} = \frac{1}{3}(a_n + 4) < \frac{1}{3}(2 + 4) = 2, \) so \( a_{n+1} < 2. \) So by induction, for all \( n, a_n < 2. \) Also, the \( a_n \)'s are increasing. To see this, note that since \( a_n < 2, \) we have \( a_{n+1} - a_n = \frac{1}{3}(a_n + 4) - a_n = -\frac{2}{3}a_n + \frac{4}{3} > -\frac{2}{3}a_n + \frac{4}{3} = 0, \) so the \( a_n \)'s are increasing. An increasing sequence that is bounded above has to converge.

21. A series \( \sum_{n=1}^{\infty} a_n \) has partial sums \( s_n = 2 - \left( \frac{1}{3} \right)^n. \) Decide whether the series converges or diverges. Justify your answer. If it converges, find the sum.
Answer: Since the partial sums converge to \( 2 - 0 = 2, \) the series converges to 2 by definition.

22. Determine whether the series converges or diverges. Justify your answer. State the convergence test and check that any necessary conditions apply.

(a) \( \sum_{n=1}^{\infty} \frac{1}{n + 3^n} \)
Answer: converges by limit comparison with \( \frac{1}{3^n} \)

(b) \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2} \)
Answer: diverges because the terms don’t go to zero

(c) \( \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2} \)
Answer: converges by the alternating series test

(d) \( \sum_{n=1}^{\infty} \frac{n^22^{n-1}}{(-5)^n} \)
Answer: converges by the ratio test

(e) \( \sum_{n=2}^{\infty} \frac{2n^2-3n+6}{n!} \)
Answer: diverges by the limit comparison test

(f) \( \sum_{n=1}^{\infty} \left( \frac{1}{n} + \frac{1}{3^n} \right) \)
Answer: converges using the p-test and the geometric series test

(g) \( \sum_{k=1}^{\infty} \frac{2^k}{k^{k+1}} \)
Answer: converges by comparison test with \( \frac{1}{k^2} \)

(h) \( \sum_{n=1}^{\infty} \frac{3^n2^n}{n!} \)
Answer: converges by the ratio test

(i) \( \sum_{n=1}^{\infty} \frac{n!}{n^n} \)
Answer: converges by the ratio test
(j) $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2n}$
Answer: The absolute valued series converges by comparison to $\frac{1}{2^n}$. Absolutely convergent implies convergent.

(k) $\sum_{n=1}^{\infty} \frac{1.3\cdots(2n-1)}{2.5\cdots(3n-1)}$
Answer: converges by the Ratio Test, or by comparison to $\left(\frac{2}{3}\right)^n$

(l) $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$
Answer: diverges by the integral test

23. Determine whether the series is conditionally convergent, absolutely convergent, or divergent. Justify your answer.
   (a) $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$
   Answer: diverges because terms don’t go to zero
   (b) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$
   Answer: converges conditionally

24. Find the sum of the series.
   (a) $\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n}$
   Answer: $\frac{1}{11}$
   (b) $\sum_{n=4}^{\infty} \frac{3}{n^3 - 4}$
   Answer: $\frac{77}{80}$
   (c) $\sum_{n=0}^{\infty} \frac{(-1)^n n^n}{3^n (2n)!}$
   Answer: $\cos\left(\frac{\sqrt{3}}{3}\right)$
   (d) $1 - e + \frac{e^2}{2} - \frac{e^3}{3} + \frac{e^4}{4} - \cdots$
   Answer: $e - e^{\frac{1}{2}}$
   (e) $\sum_{n=1}^{\infty} \frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots$
   Answer: $\arctan\left(\frac{1}{2}\right)$

25. Express $10.1335 = 10.135353535353535$ as a ratio of integers.
   Answer: $\frac{10034}{990}$

26. Find the partial sum $s_3$ for the series and estimate the error in using it as an approximation for the sum of the series.
   (a) $\sum_{n=1}^{\infty} \frac{1}{n^6}$
   Answer: $s_3 \approx 1.016997, R_3 < 1/(5 \cdot 3^5) \approx 0.000823$
   (b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n^3 + n}$
   Answer: $s_3 \approx 0.22083, R_3 < 1/52 = 0.01923$

27. For each of the above two sequences, how many terms are needed to approximate the sum to within 0.001?
   Answer: 3 and 18

28. Find the radius of convergence and the interval of convergence of the series.
   (a) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^{2/3}}$
   Answer: $R = 5, I = [-5, 5]$
(b) \[ \sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!} \]
Answer: \( R = \infty, I = (-\infty, \infty) \)

(c) \[ \sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}} \]
Answer: \( R = 1/2, I = [5/2, 7/2) \)

29. Find the Taylor series of \( f(x) = \sin x \) at \( a = \pi/6 \)
Answer: \[ \frac{\pi}{6} + \sum_{n=0}^{\infty} \frac{(x-\pi/6)^n}{n!} \] \( R = 1/2, I = (-1/2, 1/2) \)

30. Find the Maclaurin series for \( f \) and its radius of convergence.

(a) \( f(x) = \frac{x}{1+x} \)
Answer: \( \sum_{n=0}^{\infty} (-1)^n x^{n+1} \) with \( R = 1 \)

(b) \( f(x) = \ln(4-x) \)
Answer: \( \ln 4 - \sum_{n=1}^{\infty} \frac{x^n}{n4^n}, R = 4 \)

(c) \( f(x) = xe^{2x} \)
Answer: \( \sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}, R = \infty \)

(d) \( f(x) = 10^x \)
Answer: \( \sum_{n=0}^{\infty} \frac{(\ln 10)^n x^n}{n!}, R = \infty \)

(e) \( f(x) = \sin(x^4) \)
Answer: \( \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1.5.9... (4n-3)}{2^{4n}n! 16^n} x^n, R = 16 \)

(f) \( f(x) = 6x^3 - 4x^2 + 2x + 7 \)
Answer: \( 6x^3 - 4x^2 + 2x + 7 \)

31. Use series to approximate \( \int_0^1 x \arctan(x^4) \, dx \) correct to two decimal places.
Answer: \( 0.16 - 1/42 + 1/120 - \cdots \approx 0.15 \)

32. Use series to evaluate the following limit: \( \lim_{x \to 0} \frac{\sin(x-x)}{x^3} \)
Answer: \(-1/6\)

33. Use a degree 3 Taylor polynomial, centered at \( a = 1 \), to approximate \( f(x) = \ln(1+2x) \).
Use Taylor’s inequality to estimate the accuracy of the approximation when \( 0.5 \leq x \leq 1.5 \).
Answer: \( |R_3| < \frac{1}{64} \) using Taylor’s Inequality, or \( |R_3| < \frac{1}{324} \) using the Alternating Series Estimate for Remainders.