Math 232 Final Exam Review Answers

Sections covered: 6.1, 6.2, 6.4, 6.5, 7.1, 7.2 (powers of sine and cosine only), 7.3, 7.4, 7.5, 7.8, 10.1, 10.2, 11.1, 11.2, 11.3, 11.4, 11.5, 11.6, 11.7, 11.8, 11.9, 11.10, 11.11

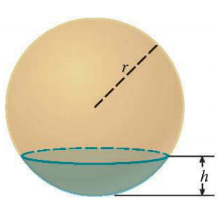
Note: There will be some true false and multiple choice concept questions on the exam. I recommend working true false questions and concept checks in the chapter review sections for practice.

Note: The following problems are mostly from the review problems in the textbook.

1. Find the area of the region bounded by the curves:
   (a) $y = 1 - 2x^2, y = |x|$
   Answer: 7/12
   (b) $x + y = 0, x = y^2 + 3y$
   Answer: 32/3
   (c) $y = a\sqrt{x}, y = x^2$
   Answer: $\frac{a^2}{3}$
2. Set up the integral to find the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.
   (a) $x = 0, x = 9 - y^2$, about $x = -1$
   Answer: $\int_{-3}^{3} \pi\left([(9 - y^2) - (-1)]^2 - [0 - (-1)]^2\right) dy = \frac{1656}{5}\pi$
3. Each integral represents the volume of a solid. Describe the solid.
   (a) $\int_0^{\pi/2} 2\pi \cos^2 x \, dx$
   Answer: rotate $R = \{(x, y)|0 \le x \le \pi/2, 0 \le y \le \sqrt{2}\cos(x)\}$ about the x-axis
   (b) $\int_0^{\pi} \pi(4 - \sin^2 x) \, dx$
   Answer: rotate $R = \{(x, y)|0 \le x \le \pi, \sin(x) \le y \le 2\}$ about the x-axis
   (c) $\int_0^{\pi} \pi(2 - \sin x)^2 \, dx$
   Answer: rotate $R = \{(x, y)|0 \le x \le \pi, 0 \le y \le 2 - \sin(x)\}$ about the x-axis, OR rotate $R = \{(x, y)|0 \le x \le \pi, \sin(x) \le y2\}$ around the line $y = 2$
4. The base of a solid is a square with vertices located at $(1, 0), (0, 1), (-1, 0), (0, -1)$. Each cross-section perpendicular to the x-axis is a semicircle. Find the volume of the solid.

   Answer: $2\int_0^1 \frac{1}{2}\pi(1 - x)^2 \, dx = \pi/3$
5. A monument in the shape of a square pyramid has height 20 meters. Its base is a square of side length 5 meters. Find the volume of the monument.

   Answer: $\int_0^{20} \left(\frac{y}{4}\right)^2 dx = \frac{500}{3}m^3$
6. (p. 459 # 5a) Show that the volume of a segment of height h of a sphere of radius r is $V = \frac{1}{3}\pi h^2(3r - h)$.



   Answer: $V = \pi h^2(r - h/3)$

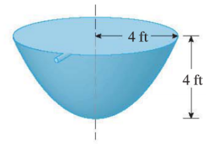
7. A force of 30 N is required to maintain a spring stretched from its natural length of 12 cm to a length of 15 cm. How much work is done in stretching the spring from 12 cm to 20 cm?

   Answer: 3.2 J

8. A 1600 lb elevator is suspended by a 200 ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?

   Answer: work for elevator = 1600*30 = 48000, work for bottom 170 feet of cable = 170*10*30 = 51000, work for top 30 feet of cable = $\int_0^3 010x\,dx = 4500$, so total work = 103500 ft-lbs

9. A tank full of water has the shape of a paraboloid of revolution as shown in the figure. That is, its



   shape is obtained by rotating a parabola about a vertical axis.

   (a) If its height is 4 ft and the radius at the top is 4 ft, find the work required to pump the water out of the tank.

   (b) After 4000 ft-lb of work has been done, what is the depth of the water remaining in the tank?

   Answer: (a) $8000\pi/3 = 8378$ ft-lb, (b) 2.1 ft

10. A steel tank has the shape of a circular cylinder oriented vertically with diameter 4 m and height 5 m. The tank is currently filled to a level of 3 m with cooking oil that has a density of 920 kg/m^3. Compute the work required to pump the oil out through a 1-m spout at the top of the tank.

    Answer: $\int_0^3 \pi \cdot 2^2 \cdot 920 \cdot 9.8(6-y)\,dy = 486,684\pi \;\; kg\, m^2/s^2$

11. Find the average value of the function $f(t) = t\sin(t^2)$ on the interval $[0, 10]$.

    Answer: $\frac{1}{20}(1 - \cos 100)$

12. Integrate by hand:

    (a) $\int \frac{dt}{2t^2+3t+1}$
    Answer: $\ln|2t+1| - \ln|t+1| + C$ (partial fractions)

    (b) $\int_0^{\pi/2} \sin^3\theta\cos^2\theta\,d\theta$
    Answer: 2/15 (use trig identities)

    (c) $\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx$
    Answer: $\sqrt{3} - \pi/3$ (trig substitution)

    (d) $\int_0^{\pi/6} t\sin 2t\,dt$
    Answer: $-\pi/24 + \sqrt{3}/8$ (integration by parts)

    (e) $\int_1^2 x^5 \ln x\,dx$
    Answer: $\frac{32}{3}\ln 2 - \frac{7}{4}$ (integration by parts)

    (f) $\int \frac{e^{2x}}{1+e^{4x}}\,dx$
    Answer: $\frac{1}{2}\tan^{-1}(e^{2x}) + C$ (u-substitution)

    (g) $\int \frac{x^2+2}{x+2}\,dx$
    Answer: $\frac{1}{2}x^2 - 2x + 6\ln|x+2| + C$ (partial fractions)

    (h) $\int e^x\cos x\,dx$
    Answer: $\frac{1}{2}e^x(\cos(x) + \sin(x)) + C$

13. Evaluate the integral or prove that it is divergent.

(a) $\int_0^4 \frac{\ln x}{\sqrt{x}}\, dx$

Answer: $4 \ln 4 - 8$

(b) $\int_0^\infty \frac{\ln x}{x^4}\, dx$

Answer: diverges because $\int_0^1 \frac{\ln x}{x^4}\, dx$ diverges

(c) $\int_0^1 \frac{1}{2-3x}\, dx$

Answer: diverges because $\int_0^{2/3} = \lim_{t \to \frac{2}{3}^-} [-\frac{1}{3} \ln|2-3x|]_0^t = \infty$

(d) $\int_1^\infty \frac{2+\sin x}{\sqrt{x}}\, dx$

Answer: diverges by comparing to $\frac{1}{\sqrt{x}}$

14. Determine if the integral converges or diverges and prove your answer.

(a) $\int_1^\infty \frac{1}{\sqrt{1+x^4}}\, dx$

Answer: converges by comparing to $\int_0^\infty \frac{1}{x^2}\, dx$

(b) $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\, dx$

Answer: diverges by comparing to $\int_1^\infty \frac{1}{x}\, dx$

(c) $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\, dx$

Answer: diverges by comparing to $\int_0^1 \frac{1}{x^{3/2}}\, dx$

15. Find the length of the curve $y = \frac{x^4}{16} + \frac{1}{2x^2}, 1 \le x \le 2$

Answer: 21/16

16. Find parametric equations for the following curves:

(a) The curve $y = \sqrt{x}$ Answer: $x = t, y = \sqrt{t}$ for $t \ge 0$, or $y = t, x = t^2$ for $t \ge 0$.

(b) The curve $(x-3)^2 + (y-5)^2 = 36$ Answer: $x = 6\cos(t) + 5, y = 6\sin(t) + 3, 0 \le t < 2\pi$

(c) The line segment between the points $(-2, 5)$ and $(3, 7)$. Answer: $x = -2 + 5t, y = 5 + 2t, 0 \le t \le 1$

17. Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

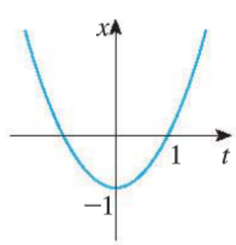
(a) $x = 2\cos\theta, y = 1 + \sin\theta$

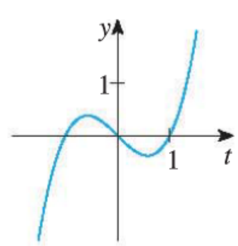
Answer: $\frac{x^2}{4} + (y-1)^2 = 1$

(b) $x = t^2 + 4t, y = 2 - t, -4 \le t \le 1$

Answer: $x = 12 - 8y + y^2, 1 \le y \le 6$

18. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t), y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.





Answer:

19. Find the length of the curve $x = 3t^2, y = 2t^3$ between the origin and the point $(12, 16)$ on the x-y plane.

Answer: $2(5\sqrt{5} - 1)$

20. Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

(a) $a_n = \frac{2+n^3}{1+2n^3}$
Answer: 1/2

(b) $a_n = \cos(n\pi/2)$
Answer: diverges

(c) $a_n = \frac{n \sin n}{n^2+1}$
Answer: 0

(d) $a_n = \frac{\ln n}{\sqrt{n}}$
Answer: 0

(e) $\left\{(1 + 3/n)^{4n}\right\}$
Answer: e^{12}

(f) $\left\{\frac{(-10)^n}{n!}\right\}$
Answer: 0

(g) $a_n = \frac{(-1)^n 3^n}{2^{2n}}$
Answer: 0

Answer: All a_n are bounded above by 2. To see this, note that $a_1 < 2$. For any n, if it is true that $a_n < 2$, then $a_{n+1} = \frac{1}{3}(a_n + 4) < \frac{1}{3}(2 + 4) = 2$, so $a_{n+1} < 2$. So by induction, for all n, $a_n < 2$. Also, the a_n's are increasing. To see this, note that since $a_n < 2$, we have $a_{n+1} - a_n = \frac{1}{3}(a_n+4) - a_n = -\frac{2}{3}a_n + \frac{4}{3} > -\frac{4}{3}a_n + \frac{4}{3} = 0$, so the a_n's are increasing. An increasing sequence that is bounded above has to converge.

21. A series $\sum_{n=1}^{\infty} a_n$ has partial sums $s_n = 2 - (\frac{1}{3})^n$. Decide whether the series converges or diverges. Justify your answer. If it converges, find the sum.

    Answer: Since the partial sums converge to $2 - 0 = 2$, the series converges to 2 by definition.

22. Determine whether the series converges or diverges. Justify your answer. State the convergence test and check that any necessary conditions apply.

    (a) $\sum_{n=1}^{\infty} \frac{1}{n+3^n}$
    Answer: converges by limit comparison with $\frac{1}{3^n}$

    (b) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$
    Answer: diverges because the terms don't go to zero

    (c) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$
    Answer: converges by the alternating series test

    (d) $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$
    Answer: converges by the ratio test

    (e) $\sum_{n=2}^{\infty} \frac{2n^2-3n+6}{n^3-1}$
    Answer: diverges by the limit comparison test

    (f) $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$
    Answer: converges using the p-test and the geometric series test

    (g) $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$
    Answer: converges by comparison test with $\frac{1}{k^2}$

    (h) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$
    Answer: converges by the ratio test

    (i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
    Answer: converges by the ratio test

(j) $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$
Answer: The absolute valued series converges by comparison to $\frac{1}{2^n}$. Absolutely convergent implies convergent.

(k) $\sum_{n=1}^{\infty} \frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{2\cdot5\cdot8\cdot\dots\cdot(3n-1)}$
Answer: converges by the Ratio Test, or by comparison to $\left(\frac{2}{3}\right)^n$

(l) $\sum_{n=1}^{\infty} \frac{1}{n \ln n}$
Answer: diverges by the integral test

23. Determine whether the series is conditionally convergent, absolutely convergent, or divergent. Justify your answer.

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n \sqrt{n}}{\ln n}$
Answer: diverges because terms don't go to zero

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n \ln n}{\sqrt{n}}$
Answer: converges conditionally

24. Find the sum of the series.

(a) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{2^{3n}}$
Answer: 1/11

(b) $\sum_{n=4}^{\infty} \frac{3}{n^2-4}$
Answer: 77/80

(c) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{3^{2n}(2n)!}$
Answer: $\cos(\frac{\sqrt{\pi}}{3})$

(d) $1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \cdots$
Answer: e^{-e}

(e) $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \cdots$
Answer: $\arctan(\frac{1}{2})$

25. Express $10.1\overline{35} = 10.135353535353535$ as a ratio of integers.

Answer: 10034/990

26. Find the partial sum s_3 for the series and estimate the error in using it as an approximation for the sum of the series.

(a) $\sum_{n=1}^{\infty} 1/n^6$
Answer: $s_3 \approx 1.016997, R_3 < 1/(5 * 3^5) \approx 0.000823$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n^2+n}$
Answer: $s_3 \approx 0.22083, R_3 < 1/52 = 0.01923$

27. For each of the above two sequences, how many terms are needed to approximate the sum to within 0.001?

Answer: 3 and 18

28. Find the radius of convergence and the interval of convergence of the series.

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n^2 5^n}$
Answer: $R = 5, I = [-5, 5]$

(b) $\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$
Answer: $R = \infty, I = (-\infty, \infty)$

(c) $\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$
Answer: $R = 1/2, I = [5/2, 7/2)$

29. Find the Taylor series of $f(x) = \sin x$ at $a = \pi/6$

Answer: $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}(x-\frac{\pi}{6})^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)!}(x-\frac{\pi}{6})^{2n+1}$

30. Find the Maclaurin series for f and its radius of convergence.

(a) $f(x) = \frac{x^2}{1+x}$
Answer: $\sum_{n=0}^{\infty}(-1)^n x^{n+2}$ with $R = 1$

(b) $f(x) = \ln(4-x)$
Answer: $\ln 4 - \sum_{n=1}^{\infty} \frac{x^n}{n4^n}, R = 4$

(c) $f(x) = xe^{2x}$
Answer: $\sum_{n=0}^{\infty} \frac{2^n x^{n+1}}{n!}, R = \infty$

(d) $f(x) = 10^x$
Answer: $\sum_{n=0}^{\infty} \frac{(\ln 10)^n x^n}{n!}, R = \infty$

(e) $f(x) = \sin(x^4)$
Answer: $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1\cdot 5\cdot 9\cdots(4n-3)}{2\cdot 4^n\cdot n!\cdot 16^n} x^n, R = 16$

(f) $f(x) = 6x^3 - 4x^2 + 2x + 7$
Answer: $6x^3 - 4x^2 + 2x + 7$

31. Use series to approximate $\int_0^1 x \arctan(x^4)\, dx$ correct to two decimal places.
Answer: $1/6 - 1/42 + 1/120 - \cdots \approx 0.15$

32. Use series to evaluate the following limit: $\lim_{x\to 0} \frac{\sin x - x}{x^3}$
Answer: $-\frac{1}{6}$

33. Use a degree 3 Taylor polynomial, centered at $a = 1$, to approximate $f(x) = \ln(1+2x)$.
Use Taylor's inequality to estimate the accuracy of the approximation when $0.5 \le x \le 1.5$.

Answer: $|R_3| < \dfrac{1}{64}$ using Taylor's Inequality, or $|R_3| < \dfrac{1}{324}$ using the Alternating Series Estimate for Remainders.