Work on as many of the following problems as possible. Turn in all your work.

(1) Consider two bodies of mass $m_1$ and $m_2$, respectively, joined by a spring (Hooke's law, constant $k$), in collinear motion along the $z$-axis of a cartesian frame in $\mathbb{R}^3$. If the position of the first body is at $z = r_1(t)$ and the second is at $z = r_2(t)$, and each body is also subject to an overall central force field $F = -Km/z^4$, where $z \neq 0$, $m_1, m_2$ are the body’s masses and $K$ is a constant, then

(a) Write evolution equations given by Newton’s second law for the functions $r_1$ and $r_2$ of time $t$.

(b) Identify the units of $K$.

(c) Non-dimensionalize the equations of motion; identify non-dimensional parameters; discuss all the possible dominant balances.

(d) Neglecting the central force field, find the solution of these equations corresponding to zero initial velocities, taking $r_1(0) - r_2(0) = h > 0$.

(e) Write an asymptotic expansion for the leading order plus the first correction of the solution assuming $h \ll (r_1(0) + r_2(0))/2$, and define the range of initial conditions that allow the central force to be considered weaker than the spring force.

(f) Sketch the leading order plus the first correction of the solutions and note their time scales of validity.

(2) Consider the integral

$$I(a) = \int_0^{2\pi} \frac{dx}{a + i \cos x} \quad a \in \mathbb{R}$$

(a) Prove that $I(a)$ is real.

(b) Show that $I(a)$ is an odd function of $a$, i.e., $I(-a) = -I(a)$

(c) Continue the integrand in the complex plane $z$, $z = x + iy$, (i.e., $\Re(z) = x$), and use the residue theorem to show that for $a > 0$

$$I(a) = \frac{2\pi}{\sqrt{1 + a^2}}$$

This result seemingly suggests that $I(a)$ is an even function of $a$. How can this be reconciled with the odd property of $I(a)$ proved above? Discuss.

(3) Consider the eigenvalue problem on the real line $x \in \mathbb{R}$

$$\epsilon y'' - (U(x) + \lambda)y = 0, \quad y(x) \to 0, \text{ as } |x| \to \infty.$$ 

with the (square well) potential $U(x) = 1$ for $|x| > 1$ and $U(x) = -2$ for $|x| < 1$

(a) Identify the range of $\lambda$ for bounded eigenfunctions to exist.

(b) Solve for eigenvalues and eigenfunctions exactly.

(c) Study their asymptotic limit as $\epsilon \to 0$.

(d) As $\epsilon \to 0$ compute acceptable solutions directly via WKBJ approach using the two-turning point analysis in this limit. Compare the result with the exact formulae.
(4) Consider the rapidly varying diffusivity:

\[ K(x, y, z; \epsilon) = A + F(x/\epsilon^3) + G(y/\epsilon^2) + H(z/\epsilon) \]

where \( A \) is chosen to guarantee \( K \) is positive, and \( \epsilon \) is a small constant. By applying iterated homogenization, average the following diffusion equation

\[
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( K(x, y, z; \epsilon) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( K(x, y, z; \epsilon) \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( K(x, y, z; \epsilon) \frac{\partial u}{\partial z} \right)
\]

\[ u(x, y, 0) = u_0(x, y, z), \]

by computing a leading order effective equation governing the evolution as \( \epsilon \to 0 \) over the \((x, y)\)-plane, assuming the functions \( F(x) \), \( G(y) \) and \( H(z) \) are mean zero, periodic, and share the same period. Solve the averaged equation in free space.

(5) Consider the following functions:

\[
f(x, t) = \frac{2xt[(1 - t^2)^2 - x^2]}{(1 - t^2)^{3/2}[2t(1 - t^2)^2 + x^2]} \\
g(x, t) = \frac{2xt}{t^2(1 - t^2)^2 + x^2}
\]

(a) Is the function \( g \) asymptotic to \( f \) in an interval, \(-t^\beta < x < t^\delta\), as \( t \to 0^+ \)? If so, for what values of the parameters \( \beta \) and \( \delta \)? Discuss.
(b) Estimate the size of the extrema of \( f \) and their \( x \)-locations and as \( t \to 0^+ \).

(6) Consider the following initial value problem for the time-\( t \) evolution equation in one spatial dimension \( x \in \mathbb{R} \)

\[ T_t + \gamma \left( 2 + \cos t \right) x T_x = \kappa T_{xx}, \quad T(x, 0) = T_0(\alpha x). \]

(a) What are the units of the parameters \( \gamma \), \( \kappa \) and \( \alpha \)?
(b) Non-dimensionalize the equation.
(c) Use the Fourier-transform method with definition

\[ \hat{T}(k, t) \equiv \int_{-\infty}^{\infty} T(x, t)e^{-ikx} \, dx \]

to solve the resulting equation for \( \hat{T} \) by the method of characteristics.
(d) What PDE does \( u \equiv T_x \) solve?
(e) Compare the long-time asymptotics of \( T \) vs. \( u \) assuming that \( T_0 \) is a Heaviside step function.

(7) (a) Explain the difference between pointwise convergence and asymptotic convergence. Illustrate with the particular example of power series.
(b) Define uniform asymptotic convergence.
(c) Are the functions \( f \) and \( g \) in problem (5) uniformly asymptotic to each other as \( t \to 0^+ \) in some \( t \)-dependent interval containing the origin?
(d) Are the derivatives \( \frac{\partial f}{\partial t} \) and \( \frac{\partial g}{\partial t} \) uniformly asymptotic to each other as \( t \to 0^+ \) on some \( t \)-dependent interval containing the origin? If so, estimate the interval size. What about the partial \( x \)-derivatives?