BOUNDARY LAYER AND LONG TIME STABILITY FOR MULTI-D VISCOUS SHOCKS

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ABSTRACT. This is an expository paper whose goal is to provide a brief introduction without the full technicalities to the methods used recently in [GMWZ1, GMWZ2] to prove the existence of curved multi-D viscous shocks, to rigorously justify the small viscosity limit, and to prove long time stability of multidimensional planar viscous shocks.

1. INTRODUCTION

We describe a unified approach to the problems of proving long time stability of multi-D planar viscous shocks and of justifying the small viscosity limit for curved multiD viscous shocks. In both cases the main hypothesis is a natural spectral stability hypothesis expressed in terms of Evans functions. The parabolic (hyperbolic + viscosity) problem on the whole space is first reformulated as a doubled boundary problem on a half space with transmission boundary conditions that express regularity across an appropriate interface. This interface (the "viscous front") is initially unknown in the small viscosity problem and has to be constructed. In both problems one has a profile that serves as an approximate solution, and the main difficulty in obtaining an exact solution is to prove good L^2 estimates for the corresponding error problem linearized about the profile. An obstacle to obtaining the L^2 estimate arises because the profile depends on a parameter z that varies from $-\infty$ to $+\infty$, and the algebraic properties of the coefficient matrices thus vary with z. A key step is to conjugate the boundary problem to a new limiting problem in which the z dependence of the coefficients is removed: the coefficients are replaced by their limits as $z \to \pm\infty$. One is now in a position to use Kreiss type symmetrizers to prove the L^2 estimate, but a remaining difficulty is that the limiting problem fails to satisfy the uniform Lopatinski condition: the Lopatinski determinant

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vanishes to first order for zero frequency. By introducing a corresponding degeneracy into the symmetrizers, we obtain singular L^2 estimates that are still strong enough to imply nonlinear stability.

To help out readers not familiar with Kreiss symmetrizers we have included a detailed construction for the case where glancing modes have order at most two. In this case, which in fact includes most of the usual physical examples, many of the linear algebraic subtleties become transparent. The construction of symmetrizers here is nonstandard in two respects. First, Kreiss's hyperbolic construction was based on a matrix perturbation argument with respect to one parameter (called γ here), while our construction for "hyperbolic + viscosity" operators involves two parameters (γ and ρ). Also, as indicated above, our symmetrizers are degenerate in the sense that they become singular as $\rho \to 0$.

2. The two problems

Consider the $m \times m$ hyperbolic system of conservation laws on $\mathbb{R}^{N+1}$

$$\sum_{j=0}^{N} f_j(u)_{x_j} = 0 \tag{2.1}$$

where $f_j : \mathbb{R}^m \to \mathbb{R}^m$ are C^∞ functions with $f_0(u) = u$ and x_0 denotes time. We are given a shock solution $(U^0_\pm, \psi_0)$ to (2.1); this means that U^0_+ (resp. U^0_-) satisfy (2.1) in the classical sense to the right (resp. left) of the shock surface $\mathcal{S}$ defined by $x_N = \psi_0(x')$ (here $x' = (x_0, \ldots, x_{N-1}) = (x_0, x'')$), and in the distribution sense in a neighborhood of $\mathcal{S}$. The piecewise classical solution is a distribution solution near $\mathcal{S}$ if and only if the Rankine-Hugoniot jump condition holds:

$$\sum_{0}^{N-1} [f_j(U^0)]\partial_{x_j}\psi_0 - [f_N(U^0)] = 0 \text{ on } \mathcal{S}. \tag{2.2}$$

In the long time problem (LT) we take $\mathcal{S}$ to be the plane $x_N = sx_0$ and $U^0_\pm$ are constant vectors. In the small viscosity problem (SV) $U^0_\pm$ are generally nonconstant functions of x and the surface $\mathcal{S}$ is curved. The problems are stated in sections 1.2 and 1.3.

2.1. **Hypotheses on the inviscid shock.**

(H1) For states u near $U^0_\pm$, the matrix $\sum_{j=1}^{N} df_j(u)\xi_j$ has simple real eigenvalues for $\xi \in \mathbb{R}^N \setminus 0$ (This can be weakened to: real semisimple eigenvalues with constant multiplicity.)

(H2) The inviscid shock is a Lax shock. This means that the normal matrices

$$A_N(U^0_\pm, d\psi_0) \equiv df_N(U^0_\pm) - \sum_{j=0}^{N-1} \partial_j \psi_0 df_j(U^0_\pm)$$

are invertible, and that if we let k (resp. l) be the number of positive (resp. negative) eigenvalues of $A_N(U^0_+, d\psi_0)$ (resp. $A_N(U^0_-, d\psi_0)$), then $k + l = m - 1$. (Here we evaluate at $x = (x', 0)$).

Remark 2.1. (H1) implies that eigenvalues λ of $-i \sum_{j=1}^N df_j(u)\xi_j - |\xi|^2$ satisfy $\Re\lambda = -|\xi|^2$ for states u near $U^0_\pm$.

Consider also a corresponding system of viscous conservation laws

$$\sum_{j=0}^{N} f^j(u)_{x_j} = \epsilon \triangle u, \tag{2.3}$$

where

$$\triangle u = \sum_{j=1}^{N} \partial^2_{x_j} u.$$

In (LT) we take $\epsilon = 1$ while in (SV) we have $\epsilon \in (0, 1]$. We assume we're given a smooth function $\mathcal{U}^0(x', z)$ satisfying the travelling wave equation

$$\begin{aligned} &C^0(x')\partial_z \mathcal{U}^0 = \mathbb{G}(\mathcal{U}^0, d\psi_0) - \mathbb{G}(U^0_-, d\psi_0) \text{ where} \\ &\mathbb{G}(u, d\phi) \equiv f_N(u) - \sum_{0}^{N-1} f_j(u)\partial_j\phi \text{ and } C^0(x') = 1 + |\nabla_{x''}\psi_0|^2, \end{aligned} \tag{2.4}$$

and connecting the endstates $U^0_\pm(x', 0)$:

$$\lim_{z\to\pm\infty} \mathcal{U}^0(x', z) = U^0_\pm(x', 0). \tag{2.5}$$

Note that in (LT) $C^0 = 1$, $\mathcal{U}^0$ is independent of x', and (2.4) reduces to

$$\partial_z \mathcal{U}^0 = (f_N(\mathcal{U}^0) - s\mathcal{U}^0) - (f_N(U^0_-) - sU^0_-). \tag{2.6}$$

$\mathcal{U}^0$) is variously referred to as a *connection*, a *profile*, and a *viscous shock.*

An important point for entire argument is that the profile decays exponentially to its endstates. Indeed, since the shock is noncharacteristic, the center manifold of the travelling wave ODE is trivial at the rest points $U^0_\pm$.

Observe that in (LT) $\mathcal{U}^0(x_N - sx_0)$ is an exact solution of (2.3), while in (SV) the function $\mathcal{U}^0(x', \frac{x_N-\psi_0}{\epsilon})$ is an approximate solution in the sense that after plugging into (2.3), the coefficient of $\frac{1}{\epsilon}$ vanishes. In (LT) after redefining x_N, f_N as $\tilde{x}_N = x_N - sx_0$ and $\tilde{f}_N(u) = f_N(u) - su$, we can and will henceforth assume $s = 0$.

2.2. **The long time stability problem (LT).** We wish to understand the stability of the profile $\mathcal{U}^0$ under multidimensional perturbations. Let $\mathcal{A}$ denote some set of admissible perturbations to be specified later.

Definition 2.1. For $v_0 \in \mathcal{A}$ let $u(x)$ be the solution to the system (2.3) (with $\epsilon = 1$) with initial data at $x_0 = 0$ given by

$$u_0(x'', x_N) = \mathcal{U}^0(x_N) + \delta v_0(x'', x_N). \tag{2.7}$$

We say that $\mathcal{U}^0$ is *nonlinearly stable* with respect to perturbations in $\mathcal{A}$ if there exists a $\delta_0 > 0$ (depending on $|v_0|_{\mathcal{A}}$) such that for $\delta \leq \delta_0$, the solution $u(x)$ exists for all time and

$$|u(x) - \mathcal{U}^0(x_N)|_{L^\infty(x)} \to 0 \text{ as } x_0 \to \infty. \tag{2.8}$$

Assuming that the profile $\mathcal{U}^0$ satisfies a spectral stability (Evans) assumption (H3) described below, the problem is to show that the profile is nonlinearly stable with respect to as large a set of perturbations as possible. Results on this problem in 1D include [Go, KK, ZH]. This problem is studied in multiD in [Z] by explicit construction and estimation of Green's functions. In this paper we describe the approach to (LT) taken in [GMWZ1].

2.3. **The small viscosity problem (SV).** Under the same Evans assumption as above, the problem is to show that exact solutions u^ϵ (viscous shocks) to the parabolic problem (2.3) exist on a fixed time interval $[0, T_0]$ independent of ϵ and converge in some appropriate sense (e.g., L^2_{loc}) to the original inviscid shock $U^0_\pm$ as $\epsilon \to 0$.

In fact the only approach to (SV) that we know of involves doing much more than this. We construct arbitrarily high order approximate solutions $\tilde{u}^\epsilon$ to (2.3) in which the inviscid shock appears in the leading term. We then prove the existence of nearby exact solutions u^ϵ such that $u^\epsilon - \tilde{u}^\epsilon = O(\epsilon^M)$ in L^∞ on $[0, T_0]$ for some large M. Since it is (or will be) obvious that the approximate solutions converge to the inviscid shock in L^2_{loc}, the same is therefore clear for the u^ϵ.

Results on this problem in 1D include [GX, R, Y]. Here we describe the approach to (SV) taken in [GMWZ2].

3. THE DOUBLED FORWARD ERROR PROBLEMS

3.1. **The long time error problem.** From now on we set $A_j = df_j$. In (LT) we look for $u(x)$ solving (2.3),(2.7) of the form

$$u(x) = \mathcal{U}^0(x_N) + \delta v(x), \tag{3.1}$$

and then to obtain a problem with zero initial data (a forward problem) we look for v as a sum

$$v(x) = w(x) + e^{-x_0} v_0(x). \tag{3.2}$$

A short computation shows that the problem satisfied by w is

$$\begin{aligned} &\sum_{j=0}^{N} (A_j(\mathcal{U}^0)w)_{x_j} + \delta div_{x'',x_N}(B(x)w) + \delta div_{x'',x_N}(H(x,w)) = \\ &\qquad \triangle w + e^{-x_0} v_0 + div_{x'',x_N}(\mathcal{F}(x)) \\ &w|_{x_0=0} = 0, \end{aligned} \tag{3.3}$$

where B and $\mathcal{F}$ are smooth functions of $(e^{-x_0}v_0, e^{-x_0}\nabla_{x'',x_N}v_0, \mathcal{U}^0)$ and

$$|h(x,w)| \leq C|w|^2. \tag{3.4}$$

The main difficulty in solving (3.3) is to obtain good energy estimates for the corresponding linear problem:

$$\begin{aligned} &\sum_{j=0}^{N} (A_j(\mathcal{U}^0)w)_{x_j} - \triangle w = f \\ &w|_{x_0=0} = 0. \end{aligned} \tag{3.5}$$

Let $\xi \in \mathbb{R}^{N+1}$ denote the dual variable to x, extend w and f in (3.5) by zero in $x_0 < 0$, and Fourier-Laplace transform in (x_0, x'') to get the *eigenvalue equation*

$$\hat{w}_{x_N x_N} - (A_N(\mathcal{U}^0)\hat{w})_{x_N} - M(x_N, \lambda, \xi'')\hat{w} = \hat{f}(x_N, \lambda, \xi'') \tag{3.6}$$

where $\lambda = i\xi_0 + \gamma$ with $\gamma \geq 0$ and

$$M(x_N, \lambda, \xi'') = \sum_{j=1}^{N-1} A_j(\mathcal{U}^0) i\xi_j + \lambda I + |\xi''|^2 I. \tag{3.7}$$

To put this in a form convenient for symmetrizer arguments we first rewrite (3.6) as a $2m \times 2m$ first order system on $\mathbb{R}$

$$\begin{pmatrix} \hat{w} \\ \hat{z} \end{pmatrix}_{x_N} = \begin{pmatrix} A_N(\mathcal{U}^0) & I \\ M(x_N, \lambda, \xi'') & 0 \end{pmatrix} \begin{pmatrix} \hat{w} \\ \hat{z} \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{f}(x_N, \lambda, \xi'') \end{pmatrix}, \tag{3.8}$$

or

$$\mathcal{U}_{x_N} = \mathcal{G}\mathcal{U} + \mathcal{F} \tag{3.9}$$

for short.

Finally, we rewrite (3.8) as an equivalent "doubled" $4m \times 4m$ boundary problem on $x_N \geq 0$. Given a function $h(x_N)$ defined on the whole real line, for $x_N \geq 0$ set

$$h_\pm(x_N) = h(\pm x_N),$$

and observe that $\mathcal{U}$ satisfies (3.8) on $\mathbb{R}$ if and only if $U \equiv (\mathcal{U}_+, \mathcal{U}_-)$ satisfies the problem on $x_N \geq 0$:

$$\begin{aligned} &U_{x_N} - G(x_N, \lambda, \xi'')U = F \\ &\Gamma U = 0 \text{ on } x_N = 0, \end{aligned} \tag{3.10}$$

where $\Gamma U = \mathcal{U}_+ - \mathcal{U}_-$, $F = \begin{pmatrix} \mathcal{F}_+ \\ -\mathcal{F}_- \end{pmatrix}$, $G(x_N, \lambda, \xi'') = \begin{pmatrix} \mathcal{G}_+ & 0 \\ 0 & -\mathcal{G}_- \end{pmatrix}$, and

$$\mathcal{G}_\pm(x_N, \lambda, \xi'') = \begin{pmatrix} A_N(\mathcal{U}^0(x_N)) & I \\ M(\pm x_N, \lambda, \xi'') & 0 \end{pmatrix}. \tag{3.11}$$

The boundary condition in (3.10) just expresses the continuity of $\mathcal{U}$ in (3.8) at $x_N = 0$.

Remark 3.1. The results described in this paper apply to more general, even nonlinear, viscosities, and are in the process of being extended to systems with degenerate viscosity like the compressible Navier-Stokes equations. We have chosen to work here with the artificial $\epsilon\triangle$ viscosity in order to simplify the exposition and keep the main ideas clear.

3.2. **The small viscosity error problem.** The reduction to a boundary problem on a half space is more subtle in the case of (SV) and begins with the introduction of an initially unknown surface $x_N = \Psi^\epsilon(x')$ close to $\mathcal{S}$ that we refer to as the "viscous front". Solutions to the parabolic problem (2.3) will be smooth in $x_0 > 0$, so the viscous front is not a surface of discontinuity. Instead, we can think of it roughly as the center of a viscous boundary layer with the property that $\mathcal{S}^\epsilon \to \mathcal{S}$ as $\epsilon \to 0$.

As in [GW] we make the change of coordinates

$$\tilde{x}' = x', \ \tilde{x}_N = x_N - \Psi^\epsilon(x'), \tag{3.12}$$

where the smooth function Ψ^ϵ remains to be determined. Set $\tilde{u}^\epsilon(\tilde{x}) = u^\epsilon(x)$, and drop the tildes to rewrite (2.3) (suppressing some epsilons) as

$$\sum_{j=0}^{N-1} A_j(u)\partial_{x_j} u + A_N(u, d\Psi)\partial_{x_N} u - \epsilon \sum_1^N (\partial_{x_j} - \partial_{x_j}\Psi\partial_{x_N})^2 u = 0, \text{ where} \tag{3.13}$$

$$A_N(u, d\Psi) = A_N(u) - \sum_0^{N-1} A_j(u)\partial_{x_j}\Psi. \tag{3.14}$$

On $\overline{\mathbb{R}}_+^{N+1} = \{x_N \geq 0\}$ define

$$u_\pm^\epsilon(x) = u^\epsilon(x', \pm x_N),$$

and note that u^ϵ satisfies the problem on $\mathbb{R}^{N+1}$ (2.3) if and only if $u_\pm^\epsilon$ satisfies the doubled parabolic boundary problem on $\overline{\mathbb{R}}_+^{N+1}$:

(3.15)

$$\begin{aligned}
&(a) \sum_0^{N-1} A_j(u_\pm)\partial_{x_j} u_\pm \pm A_N(u_\pm, d\Psi)\partial_{x_N} u_\pm \mp \epsilon \left(\sum_1^{N-1} \partial_{x_j}^2 \Psi\right) \partial_{x_N} u_\pm \\
&\qquad - \epsilon \left(\sum_1^{N-1} \partial_{x_j}^2 + C^\epsilon(x')\partial_{x_N}^2 \mp 2\sum_1^{N-1} \partial_{x_j}\Psi\partial_{x_j}\partial_{x_N}\right) u_\pm = 0 \\
&(b) u_+ - u_- = 0 \text{ on } x_N = 0 \\
&(c) \partial_{x_N} u_+ + \partial_{x_N} u_- = 0 \text{ on } x_N = 0,
\end{aligned}$$

where

$$C^\epsilon(x') = 1 + |\nabla_{x''}\Psi^\epsilon|^2\ . \tag{3.16}$$

At this stage, we decide to look for a function Ψ^ϵ which is polynomial with respect to ϵ, that is:

$$\Psi^\epsilon(x') = \psi_0(x') + \epsilon\psi_1(x') + \cdots + \epsilon^M\psi_M(x'), \tag{3.17}$$

where $\psi_1, \ldots, \psi_M$ remain to be determined.

In the next section we describe how to construct an approximate solution to (3.15) of the form $(\tilde{u}_\pm^\epsilon, \Psi^\epsilon)$ where Ψ^ϵ is given by (3.17), and $\tilde{u}_\pm^\epsilon(x) =$

$$\left(\mathcal{U}_\pm^0(x,z) + \epsilon\mathcal{U}_\pm^1(x,z) + \cdots + \epsilon^M\mathcal{U}_\pm^M(x,z)\right)|_{z=\frac{x_N}{\epsilon}}. \tag{3.18}$$

Here

$$\mathcal{U}_\pm^j(x,z) = U_\pm^j(x) + V_\pm^j(x', z),$$

$U_\pm^0(x)$ is the original shock and $V_\pm^j(x', z)$ are boundary layer profiles exponentially decreasing in z. The $\epsilon-$dependence is suppressed in the notation.

The approximate solution $(\tilde{u}^\epsilon_\pm, \Psi^\epsilon)$ is chosen to satisfy (3.15) to high order in the following sense. The boundary conditions are satisfied exactly and the interior equations hold with zero on the right side of (3.15) replaced by $\epsilon^M R^{\epsilon,M}_\pm(x)$ for smooth functions $R^{\epsilon,M}_\pm$.

We now fix once and for all such a high order approximate solution $(\tilde{u}^\epsilon_\pm, \Psi^\epsilon)$ and seek an exact solution to (3.15) of the form

$$u^\epsilon_\pm = \tilde{u}^\epsilon_\pm + w^\epsilon_\pm, \tag{3.19}$$

where $w^\epsilon_\pm$ satisfies a second order error problem with the same boundary conditions and a forcing term $\epsilon^M R^{\epsilon,M}_\pm(x)$. To determine unique $w^\epsilon_\pm$ we need to set initial conditions, and in order to obtain $w^\epsilon_\pm$ with high regularity we choose initial data of the form

$$w^\epsilon_\pm = \epsilon^{M'} \omega^\epsilon_{0,\pm}(x'', x_N) \text{ on } x_0 = 0, \tag{3.20}$$

which is corner-compatible (at the corner $x_0 = 0$, $x_N = 0$) to sufficiently high order with the interior forcing and boundary conditions.

The next step is to write the error problem satisfied by $w^\epsilon_\pm$ as a $4m \times 4m$ first order system. For $\gamma > 1$ and $L < M$ set

$$\begin{aligned} &\tilde{w}^\epsilon_\pm = \epsilon^{-L} w^\epsilon_\pm \\ &U_\pm = e^{-\gamma x_0}(\tilde{w}_\pm, \epsilon\partial_N \tilde{w}_\pm) \text{ and } U = (U_+, U_-). \end{aligned} \tag{3.21}$$

A straightforward computation based on examining the problems satisfied by $u^\epsilon_\pm$ and $\tilde{u}^\epsilon_\pm$ shows that the nonlinear error problem for $w^\epsilon_\pm$ can be rewritten as

$$\begin{aligned} &\partial_N U - \frac{1}{\epsilon} G U = e^{-\gamma x_0}\epsilon^K(\mathcal{F}_\epsilon(U, \partial'' U) + \mathbb{F}(x)), \\ &\Gamma U = 0 \text{ on } x_N = 0, \\ &U = 0 \text{ in } x_0 < 0, \end{aligned} \tag{3.22}$$

where the $2m \times 4m$ matrix Γ and the $4m \times 4m$ matrix G are defined by

$$\begin{aligned} \Gamma &= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \\ G &= \begin{pmatrix} \mathcal{G}_+ & 0 \\ 0 & \mathcal{G}_- \end{pmatrix}, \end{aligned} \tag{3.23}$$

with

$$\mathcal{G}_\pm = \begin{pmatrix} 0 & I \\ M_\pm & A_\pm \end{pmatrix}, \tag{3.24}$$

where (suppressing some epsilons)

$$M_\pm = C^\epsilon(x')^{-1}\left[\epsilon(\partial_0+\gamma)+\sum_1^{N-1}A_j(\tilde{u}_\pm)\epsilon\partial_j-\sum_1^{N-1}\epsilon^2\partial_j^2+E_\pm\right]$$

$$E_\pm w_\pm = \pm(\partial_u\mathcal{A}_N(\tilde{u}_\pm,d\Psi)w_\pm)\partial_z\tilde{u}_\pm+\sum_1^{N-1}(\partial_u A_j(\tilde{u}_\pm)w_\pm)\epsilon\partial_j\tilde{u}_\pm \tag{3.25}$$

$$A_\pm = C^\epsilon(x')^{-1}\left[\pm\mathcal{A}_N(\tilde{u}_\pm,d\Psi)\pm 2\sum_1^{N-1}\partial_j\Psi\epsilon\partial_j\right].$$

On the right side of (3.22) the power K can taken as large as desired provided the approximate solution was computed to high enough order, $\mathbb{F}\in H^J$ for J large, supp $\mathbb{F}\subset\{x_0\geq 0\}$, and

$$|\mathcal{F}_\epsilon|\leq C(|U|^2+|\partial_{x''}U||U|). \tag{3.26}$$

In reducing to the forward error problem (3.22) with regular forcing $\mathbb{F}$ we have used the compatibility conditions satisfied by (3.20) to transfer initial data to forcing by a maneuver much like that in the case of (LT).

Thus we are led to prove L^2 estimates for the linearized problem corresponding to (3.22):

$$\begin{aligned}&\partial_N U-\frac{1}{\epsilon}GU=F \text{ on } x_N\geq 0\\ &\Gamma U=0 \text{ on } x_N=0\\ &U=0 \text{ in } x_0<0.\end{aligned} \tag{3.27}$$

Remark 3.2. At this point we encourage the reader to take note of some of the similarities (and differences) of the $4m\times 4m$ matrices G appearing in (3.27) (SV) and (3.10) (LT). In particular note that if one "undoubles" (3.27) (to get a $2m\times 2m$ problem on $\mathbb{R}^{N+1}$) and then further rewrites the problem as an equivalent second order $m\times m$ system, the latter system is a linearization of (3.13) about $\tilde{u}$.

4. HIGH ORDER APPROXIMATE SOLUTIONS FOR (SV)

We obtain equations for the $\psi_j(x')$, $U^j_\pm$, $V^j_\pm(x',z)$ appearing in (3.17), (3.18) by plugging those expansions into the doubled boundary problem (3.15), collecting coefficients of equal powers of epsilon, and setting those coefficients equal to zero. For each power of ϵ this leads to a "slow" or outer layer problem for one of the $(U^j_\pm,\psi_j)$ and a "fast" or inner layer problem for $V^j_\pm$. For example, the interior problem for $V^0_\pm$

obtained by setting the fast coefficient of ϵ^{-1} equal to zero is

$$C_0(x')\partial_z^2 V_\pm^0 = \pm A_N(\mathcal{U}_\pm^0, d\psi_0)\partial_z V_\pm^0 \text{ on } z \geq 0, \tag{4.1}$$

where $C_0(x')$ appears when we write C^ϵ in (3.16) as

$$C^\epsilon(x') = C_0(x') + \epsilon C_1(x') + \cdots + \epsilon^{2M} C_{2M}(x'). \tag{4.2}$$

The corresponding boundary conditions on $x_N = 0$, $z = 0$ are

$$\begin{aligned} &(a)\ U_+^0 + V_+^0 = U_-^0 + V_-^0 \\ &(b)\ \partial_z V_+^0 = -\partial_z V_-^0. \end{aligned} \tag{4.3}$$

Observe that when we integrate (4.1) from $+\infty$ to z we obtain a doubled version of the travelling wave equation (2.4):

$$C_0(x')\partial_z V_\pm^0 = \pm\mathbb{G}(U_\pm^0 + V_\pm^0, d\psi_0) \mp \mathbb{G}(U_\pm^0, d\psi_0) \text{ on } z \geq 0. \tag{4.4}$$

We seek to solve this subject to the boundary conditions (4.3) and the requirement that $V_\pm^0(x', z)$ decrease exponentially to 0 as $z \to +\infty$. The boundary conditions clearly overdetermine the problem, but note that the Rankine-Hugoniot condition on $(U_\pm^0, d\psi_0)$ (which is assumed to hold) is the necessary **compatibility condition**. More precisely, assume that $V_\pm^0$ satisfies (4.4) and (4.3)(a). Then

$$\text{(4.3)(b) holds } \Leftrightarrow [\mathbb{G}(U_\pm^0, d\psi_0)] = 0 \text{ on } x_N = 0. \tag{4.5}$$

A solution to (4.4) satisfying (4.3)(a) (and thus (4.3)(b)) is easily found using the profile $\mathcal{U}^0(x', z)$ given in (2.4). The profile $\mathcal{U}^0(x', z)$ is related to $\mathcal{U}_\pm^0(x', x_N, z)$ by

$$\mathcal{U}^0(x', z) = \begin{cases} \mathcal{U}_+^0(x', 0, z), & z \geq 0 \\ \mathcal{U}_-^0(x', 0, -z), & z \leq 0 \end{cases}. \tag{4.6}$$

Remark 4.1. It is not hard to prove the existence of profiles $\mathcal{U}^0(x', z)$ for sufficiently weak Lax shocks. To handle the case of strong shocks, we have to assume the existence of profiles.

The interior slow problem corresponding to the order ϵ^0 turns out to be just the nonlinear hyperbolic problem (in doubled form) satisfied by the given inviscid shock $U_\pm^0$.

To see how the construction of high order profiles works, it will be enough just to consider the case of $V_\pm^1(x', z)$. The interior problems satisfied by $V_\pm^1$ are the fast problems at the order ϵ^0. As in the case of $V_\pm^0$, each problem is a second order ODE that can be integrated using the conservative structure to give a first order ODE. The latter equation is a linearization of (4.4) with forcing depending on previously determined functions. Again, there are two boundary conditions:

$$
\begin{aligned}
&(a)\ U^1_+ + V^1_+ = U^1_- + V^1_-\\
&(b)\ \partial_{x_N} U^0_+ + \partial_z V^1_+ = -(\partial_{x_N} U^0_- + \partial_z V^1_-),
\end{aligned}
\tag{4.7}
$$

so the first order problem for $V^1_\pm$ is overdetermined.

The necessary compability condition is arranged by solving the following linearized shock problem for $(U^1_\pm, \psi_1)$:

$$
\begin{aligned}
&(a) H^\pm(U^0_\pm)\partial U^1_\pm = P^0_\pm(x) \text{ on } x_N \geq 0\\
&(b)[\mathbb{B}(U^0_\pm)d\psi_1] + [A_N(U^0_\pm, d\psi_0)U^1_\pm] = Q^0(x') \text{ on } x_N = 0,
\end{aligned}
\tag{4.8}
$$

where the forcing and boundary data depend on previously determined functions. The interior problem in (4.8) is the slow problem at the order ϵ^1, the boundary operator is a linearization of the Rankine-Hugoniot conditions, and the boundary data is chosen precisely so that if $V^1_\pm$ satisfies the interior ODE described above together with (4.7)(a), then (4.7)(b) holds if and only if (4.8)(b) holds.

It is a consequence of the Evans assumption (H3) that the linearized shock problem (4.8) is *uniformly stable* in the sense of Majda [M2, M3], and thus solvable. We now have the functions $(U^0_\pm, \psi_0)$, $V^0_\pm$, $(U^1_\pm, \psi_1)$, and the next step is to solve for $V^1_\pm$. So we must choose initial data for $V^1_\pm$ at $z = 0$ so that both (4.7)(a) holds and the solution $V^1_\pm$ decays exponentially to 0 as $z \to +\infty$. We explain next how the Evans assumption (H3) allows us to do this.

Consider again the travelling wave equation (2.4) on $\mathbb{R}_z$, and note that $U^0_\pm$ are both rest points (use the RH conditions). Clearly, $\mathcal{U}^0(x', z)|_{z=0}$ belongs to both the stable manifold of $U^0_+(x', 0)$ and the unstable manifold of $U^0_-(x', 0)$. The Evans assumption implies these manifolds intersect transversally at $\mathcal{U}^0(x', z)|_{z=0}$. This implies in turn that the stable spaces for the linearized travelling wave equations satisfied by $V^1_\pm$ on $[0, +\infty)$ intersect transversally, and thus one can choose initial data for $V^1_\pm$ as described above.

One continues according to this pattern to solve for $(U^2_\pm, \psi_2)$, then $V^2_\pm$, then $(U^3_\pm, \psi_3)$, etc., always obtaining linearized Majda well-posed shock problems for $(U^j_\pm, \psi_j)$ whose boundary conditions are chosen as the compatibility conditions for the $V^j_\pm$.

5. The Evans assumption

It is easiest to define the Evans function first for the $2m \times 2m$ system (3.9) on $\mathbb{R}$ with $\mathcal{F} = 0$:

$$\mathcal{U}_{x_N} = \mathcal{G}\mathcal{U}. \tag{5.1}$$

Lemma 5.1 ([ZS])**.** *For $\xi'' \in \mathbb{R}^{N-1}$, $\Re\lambda > 0$, there exist bases of solutions*

$$\{\mathcal{U}_1^R, \cdots, \mathcal{U}_m^R\},\ \{\mathcal{U}_1^L, \cdots, \mathcal{U}_m^L\} \tag{5.2}$$

of (5.1) *with $\mathcal{F} = 0$, spanning the stable/unstable manifolds of the critical point $\mathcal{U} = 0$ such that*

$$\mathcal{D}(\lambda, \xi'') \equiv \det(\mathcal{U}_1^R, \cdots, \mathcal{U}_m^R, \mathcal{U}_1^L, \cdots, \mathcal{U}_m^L)|_{x_N=0} \tag{5.3}$$

is analytic in (λ, ξ'') for $(\lambda, \xi'') \in \mathbb{R}^{N-1} \times \{\Re\lambda \geq 0\} \setminus (0, 0)$.

Proof. A classical contraction mapping argument [Co] allows us to reduce to consideration of the limiting equations at $\pm\infty$. Letting

$$\mathcal{G}(\pm\infty, \lambda, \xi'') = \lim_{x_N \to \pm\infty} \mathcal{G}(x_N, \lambda, \xi''),$$

we look for solutions to

$$\mathcal{U}_{x_N} = \mathcal{G}(+\infty, \lambda, \xi'')\mathcal{U} \tag{5.4}$$

□

decaying at $+\infty$ of the form $\mathcal{U} = e^{\mu x_N}\mathcal{W}$, and do similarly for the problem at $-\infty$.

Note that μ is an eigenvalue of $\mathcal{G}(+\infty, \lambda, \xi'')$ if and only if

$$[\mu^2 - |\xi''|^2 - i\sum_1^{N-1} A_j(U_+^0)\xi_j - \mu A_N(U_+^0) - \lambda]v = 0 \tag{5.5}$$

for some nonzero v. Setting $\mu = i\xi_N$, $\xi_N \in \mathbb{R}$ yields

$$\det[-|\xi'', \xi_N|^2 - i\sum_1^{N} A_j(U_+^0)\xi_j - \lambda] = 0, \tag{5.6}$$

which has no solution with $\Re\lambda \geq 0$, except for $(\xi'', \xi_N) = 0$, $\lambda = 0$ (Remark 2.1). Thus, there are no eigenvalues with $\Re\mu = 0$ when $|\xi'', \lambda| > 0$, $\gamma \geq 0$, so the number of eigenvalues in each of $\Re\mu > 0$ and $\Re\mu < 0$ is constant then. We may choose $\xi_0 = 0$, $\xi'' = 0$ and γ large to obtain an obvious count of m eigenvalues in each of these regions. In particular, we obtain m stable ($\Re\mu < 0$) eigenvalues. Similarly, we obtain m unstable eigenvalues ($\Re\mu > 0$) for the problem at $-\infty$.

Definition 5.1. $\mathcal{D}$ is called the Evans-Lopatinski determinant (or *Evans function* for short) for the problem (5.1). Here and henceforth we always normalize the columns appearing in (5.3) so that they of are of size ~ 1 for $|\xi'', \lambda|$ near 0.

Remark 5.1. 1. Observe that nonvanishing of $\mathcal{D}$ in $\Re\lambda > 0$ is necessary even for linearized stability. Linear dependence of the columns in (5.3) implies existence of a solution $\mathcal{U}$ to (5.1) with $\mathcal{F} = 0$ decaying at both $\pm\infty$, and thus of an exponentially unstable solution $w = O(e^{\lambda t})$ to (3.5) with $f = 0$.

2. Set $\zeta = (\lambda, \xi'') = (\xi', \gamma)$ (with slight abuse), and introduce polar coordinates

$$\zeta = \rho\hat{\zeta}, \text{ where } \hat{\zeta} = (\hat{\xi}', \hat{\gamma}) \text{ and } \hat{\zeta} \in S^N. \tag{5.7}$$

We'll always take $\hat{\gamma} \geq 0$, so define $S^N_+ = S^N \cap \{\hat{\gamma} \geq 0\}$.

Observe that smooth functions $f(\zeta)$ can be rewritten as smooth functions $f(\hat{\zeta}, \rho)$. But if $g(\hat{\zeta}, \rho)$ is smooth with $g(\hat{\zeta}, 0)$ nonconstant, the function $g(\zeta)$ corresponding to $g(\hat{\zeta}, \rho)$ is not continuous at $\zeta = 0$.

3. In $\rho > 0$ we may write $\mathcal{D}(\zeta) = \mathcal{D}(\hat{\zeta}, \rho)$. Lemma 5.1 implies $\mathcal{D}(\hat{\zeta}, \rho)$ is analytic in $\{\hat{\gamma} > 0, \rho > 0\}$. It is shown in [ZS] that $\mathcal{D}$ and $\mathcal{D}_\rho$ are continuously extendible to $\{\hat{\gamma} \geq 0, \rho \geq 0\}$ and that

$$\mathcal{D}(\hat{\zeta}, \rho) = C\kappa\Delta(\hat{\zeta})\rho + o(\rho) \tag{5.8}$$

as $\rho \to 0$, for some $C \neq 0$. Here κ is nonvanishing if and only if the stable/unstable manifolds for $U^0_\pm$ of the travelling wave ODE (2.4) are transverse at the connection $\mathcal{U}^0(x', 0)$. $\Delta(\hat{\zeta})$ is the Lopatinski-Kreiss-Majda determinant for the ideal shock problem linearized at $(U^0_\pm, d\psi_0)$. The shock is *uniformly stable* in the sense of Majda [M2, M3] precisely when $\Delta(\hat{\zeta}) \neq 0$ for $\hat{\zeta} \in S^N_+$.

4. The vanishing of $\mathcal{D}(\hat{\zeta}, 0)$ reflects the fact that at $\rho = 0$, the equation (5.1) with $\mathcal{F} = 0$ has the solution $(\phi(x_N), 0)$, where $\phi(x_N) = \partial_{x_N}\mathcal{U}^0$ (differentiate (2.4) twice). This solution is fast-decaying at both $\pm\infty$. It will be convenient later to normalize

$$\mathcal{U}^R_1(x_N, \hat{\zeta}, 0) = \mathcal{U}^L_m(x_N, \hat{\zeta}, 0) = (\phi(x_N), 0). \tag{5.9}$$

5.1. Assumption on the viscous profile.

(H3) $\mathcal{D}(\hat{\zeta}, \rho)$ vanishes to precisely first order at $\rho = 0$ (where it must vanish) for all $\hat{\zeta} \in S^d_+$, and has no other zeros in $S^d_+ \times \overline{\mathbb{R}}_+$.

In view of the above remarks $\mathcal{D}(\hat{\zeta}, \rho)$ vanishes to precisely first order at $\rho = 0$ if and only if both $\beta \neq 0$ and $\Delta(\hat{\zeta}) \neq 0$ on S^d_+.

Remark 5.2. It has been shown recently in independent work by [FS] and [PZ] that profiles for weak Lax shocks do satify (H3), under the structural assumptions of symmetrizability plus strict convexity of the characteristic associated with the shock. These assumptions are satisfied by several of the physically important systems.

5.2. **Evans function for (SV).** Parallel to (5.1) consider the following $2m \times 2m$ system on $\mathbb{R}_z$ in which x' is viewed as a smoothly varying parameter and

$$\beta = (\beta', \gamma') \equiv \epsilon\zeta = (\epsilon\xi', \epsilon\gamma) : \tag{5.10}$$

$$\partial_z U - \mathcal{G}U = 0, \text{ where} \tag{5.11}$$

(5.12)

$$\mathcal{G}(x', z, \beta) = \begin{pmatrix} 0 & I \\ M^0 & A^0 \end{pmatrix}, \text{ with}$$

$$M^0(x', z, \beta) =$$

$$C^0(x')^{-1}\left[(i\beta_0 + \gamma') + \sum_1^{N-1} A_j(\mathcal{U}^0(x', z))i\beta_j + \sum_1^{N-1} \beta_j^2 + E^0(x', z)\right],$$

$$E^0(x', z)w = (\partial_u A_N(\mathcal{U}^0(x', z), d\psi_0(x'))w)\, \partial_z\mathcal{U}^0(x', z),$$

$$A^0(x', z, \beta) = C^0(x')^{-1}\left[A_N(\mathcal{U}^0(x', z), d\psi_0(x')) + 2\sum_1^{N-1} \partial_j\psi_0(x')i\beta_j\right].$$

We observe that (5.11) is the problem obtained if one "undoubles" (3.27), freezes the tangential variable x', Fourier-Laplace transforms the tangential ($\partial_{x'}$) derivatives, and rescales by setting $z = \frac{x_N}{\epsilon}$, $\beta = \epsilon\zeta$. In addition we have evaluated the coefficients at the leading part of $(\tilde{u}, d\Psi)$. For $\beta' \in \mathbb{R}^N$, $\gamma' > 0$ we obtain the exact analogue of Lemma (5.1) by the same proof as before, where now the Evans function $\mathcal{D} = \mathcal{D}(x', \beta)$. Remark 5.1 continues to apply, but in place of (5.8) we now have (writing $\beta = \rho\hat{\beta}$):

$$\mathcal{D}(x', \hat{\beta}, \rho) = C\kappa(x')\Delta(x', \hat{\beta})\rho + o(\rho) \tag{5.13}$$

as $\rho \to 0$. Hypothesis (H3) takes the same form as before for (SV), and Remark 5.2 still applies.

In particular, note that the transversality and uniform stability properties needed in the construction of the high order approximate solution are implied by (H3).

6. CONJUGATION TO A LIMITING PROBLEM

In the next few sections we focus mainly on conjugation and symmetrizer construction for (LT), since in that case we can work entirely with Fourier multipliers and there is no need for a pseudodifferential calculus. The constructions that we describe below for (LT) are in fact essentially identical to the constructions at the *principal symbol level* that are needed for (SV), where the variable x' in the coefficients is simply frozen and carried along as a parameter. The symbols thereby constructed are then quantized using a semiclassical or mixed (classical-semiclassical) pseudodifferential calculus, and these operators serve as the conjugators and symmetrizers. The quantization procedure is described in section ().

Return now to the $4m \times 4m$ doubled boundary problem (3.10) and consider the limiting matrix

(6.1)

$$G(\infty,\lambda,\xi'') = \lim_{x_N\to+\infty} G(x_N,\lambda,\xi'') = \begin{pmatrix} \mathcal{G}_+(\infty,\lambda,\xi'') & 0 \\ 0 & -\mathcal{G}_-(\infty,\lambda,\xi'') \end{pmatrix},$$

where

$$\mathcal{G}_\pm(\infty,\lambda,\eta) = \begin{pmatrix} A_N(U^0_\pm) & I \\ M(\pm\infty,\lambda,\xi'') & 0 \end{pmatrix}.$$

It is already clear from Lemma 5.1 that the spectral properties of $G(\infty,\lambda,\xi'')$ play an important role in the analysis. We summarize them here:

Proposition 6.1 (Spectral properties of $G(\infty,\lambda,\xi'')$, [Z],[ZS])**.**
1. When $\rho > 0$ and $\gamma \geq 0$, $G(\infty,\lambda,\xi'')$ has $2m$ eigenvalues counted with multiplicities in $\Re\mu > 0$ and $2m$ eigenvalues in $\Re\mu < 0$.

*2. $G(\infty,0,0)$ has 0 as a semisimple eigenvalue of multiplicity $2m$. The nonvanishing eigenvalues (*fast modes*) are those of $A_N(U^0_+)$ (k positive, $m-k$ negative) and $-A_N(U^0_-)$ (l positive, $m-l$ negative).*

3. Consider the multiple zero eigenvalue of $G(\infty,\hat{\zeta},0)$ (polar coordinates). For $\hat{\gamma} > \delta > 0$, this eigenvalue splits for $\rho > 0$ small into $k+l = m-1$ slow decaying modes

$$\mu = c_\delta \rho + O(\rho^2) \text{ where } \Re c_\delta < 0 \tag{6.2}$$

and $(m-k)+(m-l) = m+1$ slow growing modes ($\Re c_\delta > 0$).

Here "decaying" and "growing" refer to the corresponding exponential solutions $e^{\mu x_N} v$.

Proof. We focus on $\mathcal{G}_+(\infty,\lambda,\xi'')$; a parallel argument handles $-\mathcal{G}_-(\infty,\lambda,\xi'')$.

Statement (1) follows from the proof of Lemma 5.1 and (2) is clear since $\mathcal{G}_+(\infty,0,0)=\begin{pmatrix}A_N(U_+^0) & I\\ 0 & 0\end{pmatrix}$.

(3) Consider the characteristic equation in polar coordinates (drop the hats)

$$[\mu^2-\rho^2|\xi''|^2-\mu A_N(U_+^0)-i\rho\sum_1^{N-1}A_j(U_+^0)\xi_j-\rho\lambda]v=0, \tag{6.3}$$

and posit the expansions

$$\mu=c\rho+O(\rho^2),\quad v=r+O(\rho). \tag{6.4}$$

Compare terms of order ρ to obtain

$$\begin{gathered}(cA_N(U_+^0)+i\sum_1^{N-1}A_j(U_+^0)\xi_j+\lambda)r=0,\ \text{ or}\\ \left[c+\left(i\sum_1^{N-1}A_j(U_+^0)\xi_j+\lambda\right)A_N(U_+^0)^{-1}\right]A_N(U_+^0)r=0.\end{gathered} \tag{6.5}$$

Thus, c is an eigenvalue of $-(i\sum_1^{N-1}A_j(U_+^0)\xi_j+\lambda)A_N(U_+^0)^{-1}$, which by hyperbolicity has no center subspace for $\gamma>0$. So the stable/unstable roots $\Re c<0/\Re c>0$ separate to first order in ρ. They may be counted by setting $\xi''=0$, and using the fact that $A_N(U_+^0)$ has k positive eigenvalues. $\square$

The conjugation argument is based on the following lemma [MZ]:

Lemma 6.1. *Let $\Omega=\{(\lambda,\xi''):|\lambda,\xi''|\le C,\gamma\ge 0\}$. There is a matrix $W(x_N,\lambda,\xi'')$ defined and smooth on $[0,\infty)\times\Omega$ such that*

(a) W^{-1} is uniformly bounded and there is a $\theta>0$ such that

$$W(x_N,\lambda,\xi'')=I+O(e^{-\theta x_N}). \tag{6.6}$$

(b) W satisfies (suppressing some parameters)

$$\partial_{x_N}W=G(x_N)W(x_N)-W(x_N)G(\infty). \tag{6.7}$$

Sketch of proof. The right side of (6.7) can be written

$$\mathcal{L}W+\triangle GW$$

where $\mathcal{L}$ is the constant coefficient operator $\text{ad}G(\infty)=[G(\infty),\cdot]$ and $\triangle G$ is left multiplication by $G-G(\infty)=O(e^{-\delta x_N})$. Eigenvalues of $\mathcal{L}$ are differences of eigenvalues of $G(\infty,\lambda,\xi'')$.

Clearly, the identity matrix is an eigenvector of $\mathcal{L}$ associated to the eigenvalue 0 and a solution of the limiting problem corresponding to (6.7). Since 0 is an eigenvalue of high multiplicity for $\rho=0$ which splits for $\rho > 0$ in a manner consistent with Proposition 6.1, real parts of eigenvalues cross as parameters are varied. Thus, the classical contraction mapping argument used to find solutions of variable coefficient ODEs asymptotically near solutions of limiting constant coefficient ODEs (e.g., [Co]) does not apply. Nevertheless, the Gap Lemma [GZ] allows one to take advantage of the exponential convergence of G to $G(\infty)$ to adapt the classical argument to prove the existence of solutions depending smoothly on parameters and satisfying (6.6), as long as the eigenvalues of $\mathcal{L}$ remain separated by a line $\Re\mu = -\kappa$ for some $\kappa \in (0, \delta]$. This is true locally.

□

The substitution $U = WV$ transforms the equation (3.10) into the constant coefficient problem

$$\begin{aligned} &V_{x_N} - G(\infty, \lambda, \xi'')V = W^{-1}F \\ &\tilde{\Gamma}(0, \lambda, \xi'')V = 0 \text{ on } x_N = 0, \end{aligned} \tag{6.8}$$

where $\tilde{\Gamma}(x_N, \lambda, \xi'')V = \Gamma W(x_N, \lambda, \xi'')V$. Thus, estimates for (6.8) imply estimates for (3.10).

We'll refer to W as *the MZ conjugator* [MZ].

7. Block structure

Recall from Remark 5.1 the notation $\zeta = (\xi', \gamma) = \rho\hat{\zeta}$. The first step in the construction of symmetrizers is to conjugate $G(\infty, \zeta)$ to a block form that clearly separates the fast blocks ($P_\pm(\zeta)$ below) from the slow block $H_B(\hat{\zeta}, \rho)$, and further decomposes $\frac{1}{\rho}H_B(\hat{\zeta}, \rho)$ into subblocks corresponding to eigenvalues with real parts bounded away from or near 0.

Proposition 7.1 (Block structure)**.** *For all $\underline{\hat{\zeta}}$ with $\underline{\hat{\gamma}} \geq 0$ there is a neighborhood ω of $(\underline{\hat{\zeta}}, 0)$ in $S^N \times \overline{\mathbb{R}}_+$ and there are C^∞ matrices $T(\hat{\zeta}, \rho)$ on ω such that $T^{-1}G(\infty)T$ has the following block diagonal structure*

$$T^{-1}G(\infty)T = \begin{bmatrix} P_+(\zeta) & 0 & 0 \\ 0 & P_-(\zeta) & 0 \\ 0 & 0 & H_B(\hat{\zeta}, \rho) \end{bmatrix} \equiv G_B(\infty). \tag{7.1}$$

Here the eigenvalues of P_+ (resp. P_-) belong to a compact set in $\Re\mu > 0$ (resp. $\Re\mu < 0$) and in addition

$$\Re P_+ = \frac{1}{2}(P_+ + P_+^*) \geq cI \text{ and } -\Re P_- \geq cI \text{ on } \omega \tag{7.2}$$

for some $c > 0$.

We have $H_B(\hat\zeta,\rho)=\rho\hat H_B(\hat\zeta,\rho)$ *with*

$$\hat H_B(\hat\zeta,\rho)=\begin{bmatrix}Q_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & Q_p\end{bmatrix}(\hat\zeta,\rho). \tag{7.3}$$

The blocks Q_k *are* $\nu_k\times\nu_k$ *matrices which satisfy one of the following conditions:*

i) $\Re Q_k(\hat\zeta,\rho)$ *is positive definite.*

ii) $\Re Q_k(\hat\zeta,\rho)$ *is negative definite.*

iii) $\nu_k=1$, $\Re Q_k(\hat\zeta,\rho)=0$ *when* $\hat\gamma=\rho=0$, *and* $\partial_{\hat\gamma}(\Re Q_k)\partial_\rho(\Re Q_k)>0$.

iv) $\nu_k>1$, $Q_k(\hat\zeta,\rho)$ *has purely imaginary coefficients when* $\hat\gamma=\rho=0$, *there is* $\mu_k\in\mathbb{R}$ *such that*

$$Q_k(\underline{\hat\zeta},0)=i\begin{bmatrix}\mu_k & 1 & 0 & \\ 0 & \mu_k & \ddots & 0\\ & \ddots & \ddots & 1\\ & & \cdots & \mu_k\end{bmatrix}, \tag{7.4}$$

and the lower left corner a *of* Q_k *satisfies* $\partial_{\hat\gamma}(\Re a)\partial_\rho(\Re a)>0$.

Sketch of proof. There is a C^∞ invertible matrix $T_1(\zeta)$ defined on a neighborhood of $\zeta=0$ such that $T_1^{-1}G(\infty)T_1$ has the block diagonal form

$$T_1^{-1}G(\infty)T_1=\begin{pmatrix}P_R & 0 & 0 & 0\\ 0 & H_R & 0 & 0\\ 0 & 0 & P_L & 0\\ 0 & 0 & 0 & H_L\end{pmatrix}, \tag{7.5}$$

where H_R, H_L, P_R, and P_L are C^∞ functions of ζ satisfying

$$\begin{aligned}&H_R(0)=0,\ H_L(0)=0,\ P_R(0)=A_N(U_+^0),\ P_L(0)=-A_N(U_-^0)\\ &H_R(\zeta)=-M(+\infty,\zeta)A_N(U_+^0)^{-1}+O(|\zeta|^2)\\ &H_L(\zeta)=M(-\infty,\zeta)A_N(U_-^0)^{-1}+O(|\zeta|^2),\end{aligned} \tag{7.6}$$

and

$$T(0)=\begin{pmatrix}I & -A_N(U_+^0)^{-1} & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & A_N(U_-^0)^{-1}\\ 0 & 0 & 0 & I\end{pmatrix}. \tag{7.7}$$

The eigenvalues of $P_R(\zeta)$ and $P_L(\zeta)$ satisfy $|\Re\mu|>C>0$ on some neighborhood of $\zeta=0$.

Another conjugation by a constant coefficient matrix yields a matrix with the same blocks in the new order (P_R, P_L, H_R, H_L). Next, using projectors

$$\mathcal{P}(\zeta) = \frac{1}{2\pi i}\int_{\mathcal{C}} (\xi_N - \begin{pmatrix} P_R & 0 \\ 0 & P_L \end{pmatrix})^{-1} d\xi_N \tag{7.8}$$

for appropriate contours $\mathcal{C} \subset \mathbb{C}$, one reduces (P_R, P_L) to (P_+, P_-).

The blocks H_R and H_L are conjugated separately to block structure after localization to ω. Thus, there is a k_0 such that the blocks $Q_1, \dots, Q_{k_0}$ in $\hat{H}_B$ correspond to H_R, while blocks $Q_{k_0+1}, \dots, Q_p$ correspond to H_L. One first uses projectors as above to separate out blocks corresponding to eigenvalues with (respectively) positive, negative, or zero real parts. A further change of basis in cases (iii),(iv) puts $Q_k(\underline{\hat{\zeta}}, 0)$ in Jordan form. Changing basis again using Ralston's Lemma [Ra] makes Q_k pure imaginary in cases (iii),(iv) when $\hat{\gamma} = \rho = 0$. Observe that (by hyperbolicity) blocks satisfying conditions (iii) or (iv) only arise when $\underline{\hat{\gamma}} = 0$.

The crucial sign condition in (iii) and (iv) allows one to construct symmetrizers by a modification of the ansatz used in [K]: an extra term is added to the kth block of the symmetrizer corresponding to the extra ρ parameter. We'll discuss the sign condition later in the case where blocks of type (iv) are at most of size 2×2.

$\square$

Definition 7.1. Blocks satisfying condition (iv) in the above theorem will be referred to as *glancing blocks.* These correspond to coalescing eigenvalues.

8. Nondegenerate symmetrizers

Let $\mathcal{T} = WT$, where W is the MZ conjugator and T is the conjugator of $G(\infty)$ to block structure. Conjugation by $\mathcal{T}$ allows us to prove estimates for (3.10) by proving estimates for

$$\begin{aligned} &U_x - G_B(\infty, \hat{\zeta}, \rho)U = F \\ &\Gamma_1(\hat{\zeta}, \rho)U = 0 \text{ on } x_N = 0, \end{aligned} \tag{8.1}$$

where $\Gamma_1 = \Gamma W \mathcal{T}$.

Here we wish to illustrate the use of Kreiss symmetrizers to prove estimates in a simpler situation where Γ_1 is replaced by an artificial boundary condition Γ_a that satisfies the uniform Lopatinski condition near a basepoint $\underline{X} = (\underline{\hat{\zeta}}, \underline{\rho})$.

Definition 8.1. For $\hat{\gamma} > 0$, $\rho > 0$ let $\mathcal{E}_\pm(\hat{\zeta}, \rho)$ denote the space of boundary values at $x_N = 0$ of decaying solutions to the homogeneous $4m \times 4m$ problem

$$U_{x_N} - G(x_N, \hat{\zeta}, \rho)U = 0 \text{ on } x_N \geq 0, \tag{8.2}$$

and define $E_\pm(\hat{\zeta}, \rho)$ by $\mathcal{E}_\pm(\hat{\zeta}, \rho) = W(0, \hat{\zeta}, \rho)T(\hat{\zeta}, \rho)E_\pm(\hat{\zeta}, \rho)$.

By Proposition 6.1 the spaces $E_\pm(\hat{\zeta}, \rho)$, which are simply the growing/decaying generalized eigenspaces of $G_B(\infty, \hat{\zeta}, \rho)$ have dimension $2m$ and satisfy

$$\mathcal{C}^{4m} = E_+(\hat{\zeta}, \rho) \oplus E_-(\hat{\zeta}, \rho). \tag{8.3}$$

The projections in (8.3) are not uniformly bounded near points $X_0 = (\underline{\hat{\zeta}}, 0)$ such that $\hat{H}_B(X_0)$ has one or more glancing blocks. The spaces $E_\pm$ vary smoothly in $\{\rho > 0, \hat{\gamma} > 0\}$ and extend continuously to $\{\rho \geq 0, \hat{\gamma} \geq 0$ (see () for a special case and [Met1] for the general case).

Definition 8.2. A boundary operator $\Gamma_a(\hat{\zeta}, \rho)$ depending continuously on $(\hat{\zeta}, \rho)$ is said to satisfy the uniform Lopatinski condition at $\underline{X} = (\underline{\hat{\zeta}}, \underline{\hat{\rho}}) \in S^d_+ \times \overline{\mathbb{R}}_+$ if there exists $C > 0$ such that

$$|\Gamma_a(\underline{X})u| \geq C|u| \tag{8.4}$$

for $u \in E_-(\underline{X})$.

Now let $X_0 = (\underline{\hat{\zeta}}, 0)$ and assume (8.4) at $\underline{X} = X_0$. The symmetrizer for the problem

$$\begin{aligned} &U_x - G_B(\infty)U = F \\ &\Gamma_a U = g \text{ on } x_N = 0 \end{aligned} \tag{8.5}$$

is a $4m \times 4m$ matrix constructed by blocks in a neighborhood of X_0

$$S(\hat{\zeta}, \rho) = \begin{pmatrix} S_+(\zeta) & & & & \\ & S_-(\zeta) & & & \\ & & S_1(\hat{\zeta}, \rho) & & \\ & & & \ddots & \\ & & & & S_p(\hat{\zeta}, \rho) \end{pmatrix}, \tag{8.6}$$

where the $S_\pm$, S_j are C^∞ functions of their arguments. We'll sometimes write

$$S = \begin{pmatrix} S_P & \\ & S_H \end{pmatrix}, \tag{8.7}$$

where each block is of size $2m$.

8.1. **Decomposition of $\mathbb{C}^{4m}$.** The block form (7.1) of $G_B(\infty, \hat{\zeta}, \rho)$ determines a partition of $U \in \mathbb{C}^{4m}$ as $U = (u_+, u_-, u_1, \dots, u_p)$. Denote by α_j the number of eigenvalues of Q_j with $\Re\mu < 0$ for $\hat{\gamma} > 0$ (or $\rho > 0$), and write

$$u_j = (u_{j-}, u_{j+}) \tag{8.8}$$

where u_{j-} consists of the first α_j components of u_j.

Next set

$$\begin{aligned} U_{P+} &= (u_+, 0, 0, \dots, 0) \\ U_{P-} &= (0, u_-, 0, \dots, 0) \\ U_{H+} &= (0, 0, (0, u_{1+}), \dots, (0, u_{p+})) \\ U_{H-} &= (0, 0, (u_{1-}, 0), \dots, (u_{p-}, 0)), \end{aligned} \tag{8.9}$$

and write

$$\begin{aligned} U &= U_{P_+} + U_{P_-} + U_{H_+} + U_{H_-} \\ U_\pm &= U_{P_\pm} + U_{H_\pm} \\ U_P &= U_{P_+} + U_{P_-} \\ U_H &= U_{H_+} + U_{H_-}. \end{aligned} \tag{8.10}$$

Corresponding to (8.10) we have the decompositions

$$\begin{aligned} &\mathbb{C}^{4m} = F_{P_+} \oplus F_{P_-} \oplus F_{H_+} \oplus F_{H_-}, \text{ where} \\ &F_{P_+} = \{(u_+, 0, \dots, 0) : u_+ \in \mathbb{C}^{m-1}\} \text{ etc., } \quad \text{and} \\ &F_{H_\pm} = \oplus_{j=1}^{p} F_{H_{j\pm}}, \text{ where} \\ &F_{H_{j-}} = \{(0, 0, 0, \dots, (u_{j-}, 0), 0, \dots) : u_{j-} \in \mathbb{C}^{\alpha_j}\}, \text{ etc..} \end{aligned} \tag{8.11}$$

Proposition 6.1 shows these subspaces have dimensions

$$\begin{aligned} \dim E_{P_+} &= k + l = m - 1 \\ \dim E_{P_-} &= (m - k) + (n - l) = m + 1 \\ \dim E_{H_+} &= (m - k) + (m - l) = m + 1 \\ \dim E_{H_-} &= k + l = m - 1. \end{aligned} \tag{8.12}$$

8.2. **Estimates.** In the following discussion $U = U(x_N, \zeta)$, $\langle\ ,\ \rangle$ denotes the inner product in $\mathbb{C}^{4m}$,

$$(U(x_N, \zeta), V(x_N, \zeta)) \equiv \int_0^\infty \langle U(x_N, \zeta), V(x_N, \zeta) \rangle dx_N, \tag{8.13}$$

and

$$\begin{aligned}&|U|_2 = |U(x_N,\zeta)|_{L^2(x_N)}\\ &|U| = |U(0,\zeta)|.\end{aligned}\tag{8.14}$$

The S_j are constructed so that $S = S^*$, with interior estimates

$$\begin{aligned}&(\text{Re } SG_B(\infty)U_P, U_P) \geq C|U_P|_2^2\\ &(\text{Re } SG_B(\infty)U_{H_j}, U_{H_j}) \geq (\gamma+\rho^2)|U_{H_j}|_2^2,\end{aligned}\tag{8.15}$$

as well as boundary estimates

$$\begin{aligned}&(a)\ (SU_P, U_P) \geq C|U_{P_+}|^2 - |U_{P_-}|^2\\ &(b)\ (SU_{H_j}, U_{H_j}) \geq C|U_{H_{j+}}|^2 - |U_{H_{j-}}|^2\end{aligned}\tag{8.16}$$

both holding uniformly near the basepoint X_0.

Note that S_P can be taken to be simply

$$S_P = \begin{pmatrix} CI & \\ & -I\end{pmatrix}\tag{8.17}$$

for some large $C > 0$. The construction of S_H is discussed in section ().

Assuming Γ_a satisfies the uniform Lopatinski condition at X_0 we have

$$|U_-|^2 \leq C|\Gamma_a U_-|^2 \leq C(|\Gamma_a U|^2 + |U_+|^2)\tag{8.18}$$

at X_0 and in fact uniformly near X_0 by continuity.

Using the previous two estimates we obtain

$$\begin{aligned}(SU,U) &\geq C|U_+|^2 - |U_-|^2 = C|U_+|^2 + |U_-|^2 - 2|U_-|^2\\ &\geq C|U_+|^2 + |U_-|^2 - C_1(|\Gamma_a U|^2 + |U_+|^2)\\ &\geq C_2|U_+|^2 + |U_-|^2 - C_1|\Gamma_a U|^2,\end{aligned}\tag{8.19}$$

provided C was big enough.

From (8.15), (8.19), and the identity

(8.20)

$$-\langle SU(0), U(0)\rangle = \int_0^\infty \partial_{x_N}\langle SU, U\rangle dx_N = (2\Re SG_B U, U) + 2\Re(SF, U),$$

we obtain

$$\begin{aligned}&(|U_P|_2^2 + (\gamma+\rho^2)|U_H|_2^2) + |U|^2 \leq\\ &\qquad C\left(|F_P|_2^2 + \frac{1}{(\gamma+\rho^2)}|F_H|_2^2\right) + C|\Gamma_a U|^2,\end{aligned}\tag{8.21}$$

uniformly near X_0. Here we've used

$$|(SF,U)| \leq (C_\delta |F_P|_2^2 + \delta |U_P|_2^2) + \left(\frac{C_\delta}{(\gamma+\rho^2)} |F_H|_2^2 + \delta(\gamma+\rho^2)|U_H|_2^2 \right). \tag{8.22}$$

9. Degenerate symmetrizers

Return to the block structure problem $(G_B(\hat{\zeta},\rho),\Gamma_1(\hat{\zeta},\rho))$ (8.1), where $\Gamma_1 = \Gamma W \mathcal{T}$. It follows from (H3) that near any point $\underline{X}$ with $\underline{\rho} > 0$, Γ_1 satisfies the uniform Lopatinski condition (8.4) (Prop. 7.1, [GMWZ1]). Near $\rho = 0$ the condition fails "to first order":

Proposition 9.1. *For any $R > 0$ there is a constant C_R such that for $0 \leq \rho \leq R$,*

$$|\Gamma_1 U| \geq C_R \rho |U| \text{ for } U \in E_-(\hat{\zeta},\rho). \tag{9.1}$$

Indeed, this follows by transport via $W\mathcal{T}$ of the corresponding statement for $(G(x_N,\hat{\zeta},\rho),\Gamma)$:

$$|\Gamma U| \geq C_R \rho |U| \text{ for } U \in \mathcal{E}_-(\hat{\zeta},\rho). \tag{9.2}$$

Sketch of proof. The reason for the degeneracy is clear if one recalls part (4) of Remark 5.1. At $\rho = 0$ the problem $(G(x_N,\hat{\zeta},\rho),\Gamma)$ on $x_N \geq 0$ is equivalent to (5.1). Setting $\mathcal{U}^*_\pm(x_N) = (\phi_\pm(x_N),0)$ and $U^*(x_N) = (\mathcal{U}^*_+,\mathcal{U}^*_-)$ on $x_N \geq 0$, we have

$$\begin{aligned} &U^*_{x_N} - G(x_N,\hat{\zeta},0)U^* = 0 \\ &\Gamma U^* = 0 \text{ on } x_N = 0. \end{aligned} \tag{9.3}$$

Moreover, $U^*(x_N) = (\mathcal{U}^R_{1+},\mathcal{U}^L_{m-})(x_N,\hat{\zeta},0)$ and

$$(W(0,\hat{\zeta},\rho)\mathcal{T}(\hat{\zeta},\rho))^{-1} \begin{pmatrix} \mathcal{U}^R_{1+} \\ \mathcal{U}^L_{m-} \end{pmatrix} |_{(0,\hat{\zeta},\rho)} \in F_{P_-}, \tag{9.4}$$

so we see that Γ_1 degenerates on the one dimensional subspace $F_{P_{1-}}(\hat{\zeta},\rho) \subset F_{P_-}$ spanned by the element in (9.4). In view of the first order vanishing of $\mathcal{D}(\hat{\zeta},\rho)$ at $\rho = 0$ (H3), this is the only way $\Gamma_1|_{F_{P_-}}$ can degenerate. □

When $\hat{H}_B(\hat{\zeta},\rho)$ in (7.3) has no glancing blocks then

$$F_- \equiv F_{P_-} \oplus F_{H_-} = E_-(\hat{\zeta},\rho) \tag{9.5}$$

near the basepoint X_0. If Q_j is a glancing block, the special form (7.4) of $Q_j(X_0)$ implies that $F_{H_{j-}} \subset E_-(X_0)$, but this is not necessarily true away from X_0. But one can choose a continuous matrix $T_j(\hat{\zeta},\rho)$ with $T_j(X_0)=I$ such that

$$F_{H_{j-,c}} \equiv T_j(\hat{\zeta},\rho)F_{H_{j-}} \subset E_-(\hat{\zeta},\rho) \text{ near } X_0. \tag{9.6}$$

Define $F_{H_{j+,c}}$ similarly, extend the definitions to all j by letting $T_j=I$ if Q_j is not glancing, and set

$$F_{H_{\pm,c}} = \bigoplus_{j=1}^{p} F_{H_{j\pm,c}}, \tag{9.7}$$

so that

$$F_{P_-} \oplus F_{H_{\pm,c}} = E_-(\hat{\zeta},\rho). \tag{9.8}$$

With this preparation we can now state a more precise version of Proposition 9.1, where we write $U = U_{P_+} + U_{P_-} + U_{H_{+,c}} + U_{H_{-,c}}$.

Proposition 9.2. *There exists a constant $\delta > 0$ such that for ρ sufficiently small we have*

$$|\Gamma_1 U_{-,c}| \geq \delta(|U_{H_{-,c}}| + \rho|U_{P_-}|). \tag{9.9}$$

uniformly near X_0.

We are now in a position to construct a degenerate symmetrizer for the problem

$$\begin{aligned} U_{x_N} - G_B(\infty,\zeta)U &= F \\ \Gamma_1 U &= g \text{ on } x=0, \end{aligned} \tag{9.10}$$

where $\Gamma_1 = \Gamma W \mathcal{T}$.

As before we construct a symmetrizer $S=S^*$ of the form (8.6),(8.7) for $G_B(\infty)$ working block by block. The main difference here is that we take the S_P block to be degenerate

$$S_P = \begin{pmatrix} CI & 0 \\ 0 & -\rho^2 \end{pmatrix}, \tag{9.11}$$

where the two subblocks have sizes $m-1$ and $m+1$ respectively.

The construction of the S_H block proceeds just as before, except that now in place of (8.16)(b) we need

$$(SU_{H_j}, U_{H_j}) \geq C|U_{H_{j+,c}}|^2 - |U_{H_{j-,c}}|^2 \tag{9.12}$$

uniformly near the basepoint X_0. Summing (9.12) gives

$$(SU_H, U_H) \geq C|U_{H_{+,c}}|^2 - |U_{H_{-,c}}|^2. \tag{9.13}$$

Thus, we obtain interior estimates

$$\begin{aligned}(\text{Re } SG_B(\infty)U_P, U_P) &\geq C|U_{P_+}|_2^2 + \rho^2|U_{P_-}|_2^2\\ (\text{Re } SG_B(\infty)U_H, U_H) &\geq (\gamma+\rho^2)|U_H|_2^2,\end{aligned} \tag{9.14}$$

as well as boundary estimates

$$\begin{aligned}(SU_P, U_P) &\geq C|U_{P_+}|^2 - \rho^2|U_{P_-}|^2\\ (SU_H, U_H) &\geq C|U_{H_{+,c}}|^2 - |U_{H_{-,c}}|^2,\end{aligned} \tag{9.15}$$

uniformly near X_0.

Now, Proposition 9.2 implies

$$|U_{H_{-,c}}|^2 + \rho^2|U_{P_-}|^2 \leq C|\Gamma_1 U_{-,c}|^2 \leq C(|\Gamma_1 U|^2 + |U_{+,c}|^2), \tag{9.16}$$

where $U_{+,c} = U_{P_+} + U_{H_{+,c}}$. Using (9.15) and (9.16) we obtain for ρ small

$$\begin{aligned}(SU,U) &\geq C|U_{+,c}|^2 - (|U_{H_{-,c}}|^2 + \rho^2|U_{P_-}|^2)\\ &= C|U_{+,c}|^2 + (|U_{H_{-,c}}|^2 + \rho^2|U_{P_-}|^2) - 2(|U_{H_{-,c}}|^2 + \rho^2|U_{P_-}|^2)\\ &\geq C|U_{+,c}|^2 + |U_{H_{-,c}}|^2 + \rho^2|U_{P_-}|^2 - C_1(|\Gamma_1 U|^2 + |U_{+,c}|^2)\\ &\geq C_2|U_{+,c}|^2 + |U_{H_{-,c}}|^2 + \rho^2|U_{P_-}|^2 - C_1|\Gamma_1 U|^2\\ &\geq C_3(|U_+|^2 + |U_{H_-}|^2) + \rho^2|U_{P_-}|^2 - C_1|\Gamma_1 U|^2\end{aligned} \tag{9.17}$$

provided C was big enough.

In addition we have

$$\begin{aligned}|(SF,U)| &\leq |(SF_{P_+}, U_{P_+})| + |(SF_{P_-}, U_{P_-})| + |(SF_H, U_H)|\\ &\leq (C_\delta|F_{P_+}|_2^2 + \delta|U_{P_+}|_2^2) + \rho^2(C_\delta|F_{P_-}|_2^2 + \delta|U_{P_-}|_2^2)\\ &\quad + \left(\frac{C_\delta}{(\gamma+\rho^2)}|F_H|_2^2 + \delta(\gamma+\rho^2)|U_H|_2^2\right).\end{aligned} \tag{9.18}$$

Plugging these estimates into the usual symmetrizer argument (recall (8.20)), we obtain after absorbing terms in the usual way the key *small frequency estimate*

$$\begin{aligned}&(|U_{P_+}|_2^2 + \rho^2|U_{P_-}|_2^2 + (\gamma+\rho^2)|U_H|_2^2)\\ &\qquad\qquad + (|U_+|^2 + |U_{H_-}|^2 + \rho^2|U_{P_-}|^2) \leq\\ &C\left(|F_{P_+}|_2^2 + \rho^2|F_{P_-}|_2^2 + \frac{1}{(\gamma+\rho^2)}|F_H|_2^2\right) + C|\Gamma_1 U|^2\end{aligned} \tag{9.19}$$

uniformly near X_0.

Assuming $\Gamma_1 U = 0$ as in (8.1) we deduce immediately from (9.19) our main estimate with F as forcing

$$|U|_2^2 \leq C \frac{|F|_2^2}{\rho^2(\gamma + \rho^2)}. \tag{9.20}$$

In particular with ρF forcing we obtain

$$|U|_2^2 \leq C \frac{|F|_2^2}{(\gamma + \rho^2)}. \tag{9.21}$$

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