Uniform stability estimates for constant-coefficient symmetric hyperbolic boundary value problems

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Abstract

Answering a question left open in [MZ2], we show for general symmetric hyperbolic boundary problems with constant coefficients, including in particular systems with characteristics of variable multiplicity, that the uniform Lopatinski condition implies strong L^2 well-posedness, with no further structural assumptions. The result applies, more generally, to any system that is strongly L^2 well-posed for at least one boundary condition. The proof is completely elementary, avoiding reference to Kreiss symmetrizers or other specific techniques. On the other hand, it is specific to the constant-coefficient case; at least, it does not translate in an obvious way to the variable-coefficient case. The result in the hyperbolic case is derived from a more general principle that can be applied, for example, to parabolic or partially parabolic problems like the Navier-Stokes or viscous MHD equations linearized about a constant state or even a viscous shock.

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1 Introduction

Consider a noncharacteristic, hyperbolic boundary value problem with constant coefficients on the half-space $\mathbb{R}^{d+1}_+ = \{(t,x) : x_d \geq 0\}$:

$$(1.1)\qquad \begin{aligned} &(a)\ Lu := u_t + \sum_{j=1}^{d} A^j u_{x_j} = f \\ &(b)\ \Gamma u(t,\tilde{x},0) = g, \end{aligned}$$

where $u \in \mathbb{R}^n$, $\det A_d \neq 0$, $\tilde{x} := (x_1, \ldots, x_{d-1})$, Γ is a constant $k \times n$ matrix, and the symbol $\sum_j^d A^j i\xi_j$ satisfies the hyperbolicity condition

(1.2)

$$\sum_j^d A^j i\xi_j \text{ has only pure imaginary, semisimple eigenvalues for all } \xi \in \mathbb{R}^d.$$

There are two distinct, but partially overlapping classes of systems for which the existence/stability theory is well developed, namely the Friedrichs symmetrizable hyperbolic systems with maximally dissipative boundary conditions and the Kreiss–Métivier class of strictly hyperbolic or constant-multiplicity systems with Γ satisfying a sharp spectral condition called the uniform Lopatinski condition.

Definition 1.1. *1. The operator L (1.1)(a) is called* Friedrichs symmetrizable *when there exists a positive symmetric matrix S such that SA^j is symmetric for $j = 1, \ldots, d$.*

2. Suppose L is Friedrichs symmetrizable with symmetrizer S. The boundary condition is maximally dissipative when $\operatorname{rank}\Gamma = k$*, SA_d is negative definite on* $\ker\Gamma$*, and $k =$dimension of the unstable subspace of A_d.*

Remark 1.2. *Let us recall a few well-known properties of the systems just defined (see, e.g., [Met4], Chapter 2).*

1. SA_d is negative definite on $\ker\Gamma$ *if and only if there are positive constants c and C such that for all $h \in \mathbb{C}^n$*

$$(1.3)\qquad -(SA_d h, h) \geq c|h|^2 - C|\Gamma h|^2.$$

2. One can define an adjoint problem (L^*, Γ^*) where

$$(1.4)\qquad \begin{aligned} &L^* = -\partial_t - \sum_1^d A_j^* \partial_{x_j}, \\ &\Gamma^* \textit{ is an } (n-k) \times n \textit{ matrix with } \ker\Gamma^* = (A_d \ker\Gamma)^{\perp}. \end{aligned}$$

The problem (L^*, Γ^*) *is symmetrizable and maximally dissipative in the backward sense; that is,* S^{-1} *is a symmetrizer for* $-L^*$ *and* $S^{-1}A_d^*$ *is positive definite on* $\ker \Gamma^*$*.*

3. Given a Friedrichs symmetrizable operator L (1.1)*(a) one can always define a maximally dissipative boundary condition for it using the projector* π_+ *of* $\mathbb{C}^n$ *onto the unstable subspace* $\mathcal{U}$ *of* SA_d*. More precisely, if* $\dim \mathcal{U} = n_+$*, one can take* $\Gamma = T\pi_+$*, where* T *is linear isomorphism*

$$T : \mathcal{U} \to \mathbb{C}^{n_+}. \tag{1.5}$$

We invite the reader to check (1.3) *in that case.*

We will also consider more general boundary conditions of the form

$$\Gamma_\gamma u := e^{\gamma t}\Gamma(D_t, D_{\tilde{x}}, \gamma)e^{-\gamma t}u = g \tag{1.6}$$

where $\Gamma(D_t, D_{\tilde{x}}, \gamma)$ is a Fourier multiplier:

$$\widehat{\Gamma v}(\tau, \eta) := \Gamma(\tau, \eta, \gamma)\hat{v}(\tau, \eta) \tag{1.7}$$

defined by a continuous bounded $k \times n$ symbol $\Gamma(\tau, \eta, \gamma)$.

As described in [BT, MZ2], physical applications such as shock stability in magnetohydrodynamics (MHD) motivate the study of a third class consisting of symmetric hyperbolic problems with uniform Lopatinski boundary conditions but possibly variable-multiplicity characteristics. This class was treated in depth in [MZ2] under some additional structural assumptions on the system, satisfied in particular for MHD, at both the linearized (constant- and variable-coefficient) and nonlinear level, using a generalization of the symmetrizer techniques introduced by Kreiss [K] in the strictly hyperbolic setting. However, it was noted that these structural assumptions could be significantly relaxed in the constant-coefficient case for which symmetrizers need not be smooth. Indeed, the construction in this case hints of further generality, suggesting that for Friedrichs symmetrizable systems, the uniform Lopatinski condition alone is perhaps all that is needed for L^2 well-posedness (Definition 1.3).

The purpose of this note is to verify by a very simple argument, bypassing completely the symmetrizer constructions of [K, Met3, MZ2] that this conjecture is indeed correct. However, the argument does not, at least in an obvious fashion, carry through to the variable-coefficient or nonlinear case, for which the Kreiss symmetrizer approach remains up to now the only choice.

Definition 1.3. *We say that the problem* (L, Γ_γ) (1.1)*,* (1.6) *is* strongly L^2 well-posed *if there exists a* $C > 0$ *such that for* $\gamma > 0$*,* $f \in e^{\gamma t} L^2(\mathbb{R}^{d+1}_+)$*,* $g \in e^{\gamma t} L^2(\mathbb{R}^d)$ *there exists a unique solution* $u \in e^{\gamma t} L^2(\mathbb{R}^{d+1}_+)$*, and* u *satisfies the energy estimate*

$$(1.8)\qquad \begin{aligned} \gamma \int_{-\infty}^{\infty} e^{-2\gamma t} \|u(\cdot, t)\|_{L^2}^2 \, dt + \int_{-\infty}^{\infty} e^{-2\gamma t} |u(0,t)|^2 \, dt \le \\ C \left(\gamma^{-1} \int_{-\infty}^{\infty} e^{-2\gamma t} \|f(\cdot, t)\|_{L^2}^2 \, dt + \int_{-\infty}^{\infty} e^{-2\gamma t} |g(0,t)|^2 \, dt \right). \end{aligned}$$

The word strongly *is used to highlight the trace estimate of* u*.*

For Friedrichs symmetric systems with maximally dissipative boundary conditions, strong L^2 well-posedness follows by standard arguments (see, e.g., [Met4], Chapter 2) from an a priori estimate of the form (1.8) for the original problem (L, Γ) and an analogous estimate for the adjoint problem (L^*, Γ^*). The forward estimate, for example, is obtained using integration by parts, taking the L^2 inner product of Su against equation (1.1); maximal dissipativity of Γ (1.3) allows the resulting boundary term to be estimated in a straightforward way, yielding (1.8). The adjoint estimate is similar.

Maximally dissipative boundary conditions are clearly quite special. In order to define boundary conditions satisfying the more general uniform Lopatinski condition, we first apply to (1.1),(1.6) the Laplace transform in the temporal variable t and the Fourier transform in tangential spatial variables $\tilde{x} := (x_1, \ldots, x_{d-1})$ to obtain the resolvent equation:

$$(1.9)\qquad \begin{aligned} \hat{u}' - G(\Lambda)\hat{u} &= \hat{f}, \\ \Gamma(\Lambda)\hat{u}(0) &= \hat{g}. \end{aligned}$$

Here $\hat{u}$, $\hat{f}$, and $\hat{g}$ denote the Laplace–Fourier transforms of u, f, and g,

$$(1.10)\qquad \Lambda = (\tau, \eta, \gamma) \in \mathcal{P} := \{(\tau, \eta, \gamma) : (\tau, \eta) \in \mathbb{R}^d, \gamma > 0\},$$

and

$$(1.11)\qquad G(\Lambda) := -A_d^{-1} \left(\gamma + i\tau + i \sum_{j=1}^{d-1} \eta_j A^j \right).$$

Recall that the Laplace transform of a function $f(t) \in e^{\gamma t} L^2(t)$ is the Fourier transform of $e^{-\gamma t} f$.

From hyperbolicity, (1.2), we find easily the result of Hersch [H] that, for $\gamma > 0$, $G(\Lambda)$ has no pure imaginary eigenvalues. For, existence of an eigenvalue $i\kappa$, $\kappa \in \mathbb{R}$ of G would imply existence of an eigenvalue $\gamma + i\tau$ with nonzero real part γ of the matrix symbol $\sum_{j=1}^{d} A^j i\xi_j$, where $\xi = (\tilde{\xi}, \xi_d) := (\eta, -\kappa) \in \mathbb{R}^d$. Thus, defining $\mathbb{E}_-(\Lambda)$ to be the stable subspace of $G(\Lambda)$, we have that $\dim \mathbb{E}_-(\Lambda)$ is constant for all $\gamma > 0$, and (taking η, $\tau = 0$, $\gamma = 1$)

$$\dim \mathbb{E}_-(\Lambda) \equiv n_+ \quad \text{for } \gamma > 0, \tag{1.12}$$

where n_+ denotes the dimension of the unstable eigenspace of A^d.

Definition 1.4. *A system* (L, Γ_γ) (1.1), (1.2), (1.6) *is said to satisfy the* uniform Lopatinski condition *when*

$$\begin{aligned} &(i) k = \operatorname{rank} \Gamma(\Lambda) = \dim E_-(\Lambda) \textit{ for all } \Lambda \in \mathcal{P} \\ &(ii) |v| \le C|\Gamma(\Lambda)v| \text{ for } v \in \mathbb{E}_-(\Lambda), \textit{ for } C > 0 \textit{ independent of } \Lambda \in \mathcal{P}. \end{aligned} \tag{1.13}$$

Kreiss [K] showed that the uniform Lopatinski condition can be derived as a necessary condition for strong L^2 well-posedness. The existence part of Definition 1.3 applied to the transformed problem (1.9) implies

$$\dim E_-(\Lambda) \ge k, \tag{1.14}$$

since, when $\hat{f} = 0$, solutions of (1.9) in $L^2(x_d)$ must have boundary data in $E_-(\Lambda)$. Plancherel's theorem yields an estimate similar to (1.8) for the transformed problem (with, e.g., $L^2(t, \tilde{x})$ norms replaced by $L^2(\tau, \eta)$ norms). In fact, by studying special solutions of (1.1), (1.6) built from plane waves, this can be pushed further to obtain estimates for (1.9) uniform with respect to Λ:

$$\gamma \|\hat{u}\|^2_{L^2(x_d)} + |\hat{u}(0)|^2 \le C(\|\hat{f}\|^2_{L^2(x_d)}/\gamma + |\hat{g}|^2) \tag{1.15}$$

for $\gamma > 0$ and C independent of $\Lambda \in \mathcal{P}$, where $|\cdot|$ is the standard complex modulus (e.g., see [Met4], Prop. 6.2.2). Taking $\hat{f} = 0$ in (1.15) we deduce

$$|v| \le C|\Gamma(\Lambda)| \text{ for } v \in E_-(\Lambda),$$

which implies

$$\dim E_-(\Lambda) \le \operatorname{rank} \Gamma(\Lambda) \le k \text{ for all } \Lambda \in \mathcal{P}. \tag{1.16}$$

With (1.14) this shows the uniform Lopatinski condition is necessary for strong L^2 well-posedness.

A major contribution of Kreiss [K] was to show, in the strictly hyperbolic case, by an ingenious construction of frequency-dependent symmetrizers, that the uniform Lopatinski condition is in fact equivalent to strong L^2 well-posedness, a result later generalized to constant multiplicity hyperbolic systems through the work of Majda–Osher [MO] and Metivier [Met2], and to certain variable-multiplicity hyperbolic systems in [MZ2].

Our main result is the following extension to general Friedrichs symmetrizable systems in the constant coefficient case:

Theorem 1.5. *Consider a constant coefficient Friedrichs symmetrizable system* L (1.1)*(a) with boundary condition* Γ_γ (1.6)*. The system* (L, Γ_γ) *is strongly* L^2 *well-posed if and only if it satisfies the uniform Lopatinski condition.*

Remark 1.6. *1. For constant coefficient systems one might try to obtain the Kreiss estimate* (1.8) *by direct estimation of solutions constructed by Fourier-Laplace transform. As far as we know this has been done successfully only under more restrictive hypotheses than the ones we make here (restrictions on multiplicities, order of glancing points, etc.). Weaker bounds (Hadamard well-posedness: in effect, estimates exhibiting a loss of several derivatives) have been established by this approach in great generality [H].*

2. For the constant coefficient systems we consider here (Friedrichs symmetrizable with uniform Lopatinski boundary conditions), the Kreiss estimate (1.8) *has been obtained by a simple integration by parts argument when* $f = 0$ *([S], p. 199). However, that argument does not appear to extend to the case* $f \neq 0$*. On the other hand, an estimate losing one-half derivative may easily be obtained by subtracting out the solution* w *of the Cauchy problem extended to the whole space and solving the residual problem with zero interior data and boundary data* $g - \Gamma w(0)$*, controlling* $|w(0)|$ *by the standard trace estimate* $|w(0)| \leq C_1 |w|_{H^{1/2}} \leq C_2 |f|_{H^{1/2}}$*.*

3. Theorem 2.6 was established using symmetrizers in [MZ2] under the additional structural assumption that, at frequencies ξ_0 *in the vicinity of which the eigenvalues* $a_j(\xi)$ *of the symbol* $\sum_{j=1}^d A^j i\xi_j$ *are of variable multiplicity, crossing eigenvalues are either geometrically regular in the sense that eigenvalues and eigenprojections are both analytic, totally nonglancing in the sense that* $\partial a_j / \partial \xi_d$ *have a common, nonzero sign for all* a_j *involved, or linearly separating in the sense that crossing eigenvalues* $a_j(\xi)$ *separate to linear order in the distance of* ξ *from a smooth manifold where they agree. The new content of Theorem 1.5 is that these additional assumptions may be dropped.*

As sketched briefly in Section 4, the same argument yields an analogous result for the linearized equations about a planar viscous shock or boundary layer with "real", or physical, viscosity. Thus, the general principle contained in Proposition 2.6 can be also be applied to parabolic or partially parabolic problems. However, as discussed in Section 3, our results do not apply to the nonlinear or variable-coefficient case, either in the hyperbolic or viscous–hyperbolic context.

2 Generalized resolvent-type equations

It remains to prove the sufficiency of the uniform Lopatinski condition in Theorem 1.5. We'll deduce this from the theory of maximally dissipative problems together with a new result, Proposition 2.6, for constant-coefficient "generalized resolvent-type" equations

$$\begin{aligned} L(\Lambda)u &:= u' - G(\Lambda)u = f, \\ \Gamma(\Lambda)u(0) &= g. \end{aligned} \tag{2.1}$$

on the half-line $x \in \mathbb{R}^+$. Here Λ is a parameter confined to a connected open set $\mathcal{P}$, and $\Gamma(\Lambda)$ is a $k \times n$ matrix. Initially, the *only* assumption we make about the $n \times n$ matrix $G(\Lambda)$ is that it has no pure imaginary eigenvalues for $\Lambda \in \mathcal{P}$. The parameter Λ might represent Laplace and/or Fourier frequencies, model variables, etc.. If we define $\mathbb{E}_-(\Lambda)$ to be the stable subspace of $G(\Lambda)$, these hypotheses imply that $\dim \mathbb{E}_-(\Lambda)$ is independent of $\Lambda \in \mathcal{P}$.

Definition 2.1. *Relative to some choice of $\alpha = \alpha(\Lambda) > 0$, the system* (2.1) *is* uniformly stable *if there exists $C > 0$ such that for any $u \in H^1(\mathbb{R}_+)$ and $\Lambda \in \mathcal{P}$ we have the a priori estimate*

$$\alpha\|u\|^2 + |u(0)|^2 \le C(\|L(\Lambda)u\|^2/\alpha + |\Gamma(\Lambda)u(0)|^2), \tag{2.2}$$

where $\|\cdot\|$ denotes the $L^2(x)$ norm and $|\cdot|$ the norm in $\mathbb{C}^k$.

Definition 2.2. *System* (2.1) *satisfies the uniform Lopatinski condition if*

$$\begin{aligned} &(i) k = \operatorname{rank} \Gamma(\Lambda) = \dim \mathbb{E}_-(\Lambda) \\ &(ii) |v| \le C|\Gamma(\Lambda)v| \text{ for } v \in \mathbb{E}_-(\Lambda), \end{aligned} \tag{2.3}$$

for some $C > 0$ independent of $\Lambda \in \mathcal{P}$.

Lemma 2.3. *Condition (ii) of the uniform Lopatinski condition* (2.3) *is a necessary and sufficient condition for $L^2(x)$ solutions of $L(\Lambda)u = 0$ to satisfy the trace estimate*

$$|u(0)|^2 \le C|\Gamma(\Lambda)u(0)|^2 \tag{2.4}$$

with C independent of $\Lambda \in \mathcal{P}$.

Proof. The L^2 solutions $u(x)$ of the constant coefficient problem $L(\Lambda)u = 0$ are precisely the functions

$$u(x) = e^{xG(\Lambda)}u_0, \tag{2.5}$$

where $u_0 \in E_-(\Lambda)$.

□

The key assumption on $G(\Lambda)$ is the following one:

Assumption 2.4. *For* some *constant $k \times n$ matrix $\tilde{\Gamma}$ the system*

$$\begin{aligned} u' - G(\Lambda)u &= f, \\ \tilde{\Gamma}u(0) &= g. \end{aligned} \tag{2.6}$$

has a unique solution $u \in L^2(x)$ for any given $f \in L^2(x)$, $g \in \mathbb{C}^k$, and u satisfies

$$\alpha\|u\|^2 + |u(0)|^2 \le C(\|f\|^2/\alpha + |g|^2) \tag{2.7}$$

with C independent of f, g, and $\Lambda \in \mathcal{P}$.

Example 2.5. *It follows from the discussion in the introduction that this assumption is satisfied by any $G(\Lambda)$ obtained as in* (1.9) *by Laplace-Fourier transform of a Friedrichs symmetrizable system* (1.1)*(a). In this case Λ and $\mathcal{P}$ are defined as in* (1.10)*, $\alpha(\Lambda) = \gamma$, and we take $\tilde{\Gamma}$ to be a maximally dissipative boundary condition as described in Remark 1.2, part 3.*

We will prove Theorem 1.5 using the following general principle together with Example 2.5. The idea is that existence of a boundary condition for which good estimates hold already encodes structural properties relevant to the stability analysis.

Proposition 2.6. *Consider the resolvent-type problem $(L(\Lambda), \Gamma(\Lambda))$ as in* (2.1)*, and suppose that $G(\Lambda)$ satisfies Assumption 2.4 for some choice of $\alpha(\Lambda)$. If the system $(L(\Lambda), \Gamma(\Lambda))$ satisfies the uniform Lopatinski condition, then it is uniformly stable relative to $\alpha(\Lambda)$.*

Proposition 2.6 and its corollary Theorem 1.5, proved below, extend and greatly simplify the results of [MZ2] for constant-coefficient symmetrizable systems.

Proof of Proposition 2.6. Let $u \in H^1(\mathbb{R}_+)$, set

$$L(\Lambda)u := f, \quad \Gamma(\Lambda)u(0) := g \tag{2.8}$$

and for $\tilde\Gamma$ as in Assumption 2.4 introduce the auxiliary problem

$$\begin{aligned} w' - G(\Lambda)w &= f, \\ \tilde\Gamma w(0) &= 0. \end{aligned} \tag{2.9}$$

By Assumption (2.4), there exists a unique L^2 solution w satisfying

$$\alpha\|w\|^2 + |w(0)|^2 \le \tilde C\|f\|^2/\alpha. \tag{2.10}$$

Now consider the residual $e := u - w \in L^2$, satisfying

$$\begin{aligned} e' - G(\Lambda)e &= 0, \\ \Gamma(\Lambda)e(0) &= \Gamma(\Lambda)(u(0) - w(0)) = g - \Gamma(\Lambda)(w(0)). \end{aligned} \tag{2.11}$$

By the uniform Lopatinski assumption and Lemma 2.3,

$$\begin{aligned} |e(0)|^2 &\le C|\Gamma(\Lambda)e(0)|^2 \\ &\le C(|g| + |\Gamma(\Lambda)w(0)|)^2 \\ &\le 2C(|g|^2 + C_1\tilde C\|f\|^2/\alpha). \end{aligned} \tag{2.12}$$

On the other hand, we may equally well consider (2.11) as

$$\begin{aligned} e' - G(\Lambda)e &= 0, \\ \tilde\Gamma e(0) &=: \tilde g. \end{aligned} \tag{2.13}$$

Applying Assumption 2.4 again, we thus have

$$\begin{aligned} \alpha\|e\|^2 + |e(0)|^2 &\le \tilde C|\tilde g|^2 \\ &= \tilde C|\tilde\Gamma e(0)|^2, \end{aligned} \tag{2.14}$$

which, by (2.12), gives

$$\alpha\|e\|^2 + |e(0)|^2 \le 2\tilde C C_2 C(|g|^2 + C_1\tilde C\|f\|^2/\alpha), \tag{2.15}$$

where C_1 is the matrix norm of Γ and C_2 of $\tilde\Gamma$. Adding (2.10) and (2.15), we obtain the result $\square$

Proof of Theorem 1.5. It remains to prove the sufficiency of the uniform Lopatinski condition.

Let $u(x,t) \in e^{\gamma t} H^1(\mathbb{R}^{d+1}_+)$ and set

$$Lu := f \quad \Gamma_\gamma u(0) := g. \tag{2.16}$$

The strong L^2 well-posedness of the system (L, Γ_γ) follows by standard arguments (e.g., [CP], Chapter 7) from an a priori estimate of the form (1.8) for (L, Γ_γ) and an analogous estimate for the adjoint problem (L^*, Γ^*_γ). For the definition of the adjoint boundary condition and the verification that the adjoint problem necessarily satisfies the (backward) uniform Lopatinski condition provided the forward problem satisfies the (forward) Lopatinski condition (Definition 1.4), we refer to [CP], Chapter 7.

The forward estimate is an immediate consequence of Proposition 2.6, Example 2.5, and Plancherel's Theorem. The backward estimate follows by a parallel argument, since as we noted in Remark 1.2, if $(L, \tilde{\Gamma})$ is a symmetrizable, maximally dissipative problem, then $(L^*, \tilde{\Gamma}^*)$ is symmetrizable and maximally dissipative in the backward sense. Thus, the $G(\Lambda)$ matrix that appears in the Laplace-Fourier transformed adjoint problem satisfies Assumption 2.4, and Proposition 2.6 can be applied to that problem as well. □

3 The variable-coefficient case

For the study of nonlinear hyperbolic boundary-value problems, it is important to treat also the variable-coefficient analog of (1.1),

$$\begin{aligned} &L(t,x,\partial_t,\partial_x)u := u_t + \sum_{j=1}^d A^j(t,x) u_{x_j} = f, \\ &\Gamma(t,\tilde{x}) u(t,\tilde{x},0) = g, \end{aligned} \tag{3.1}$$

where L is Friedrichs symmetrizable and $\Gamma(t,\tilde{x})$ is a $k \times n$ matrix or, more generally, a pseudodifferential operator $\Gamma_\gamma(t,\tilde{x},D_t,D_{\tilde{x}})$ of degree zero.

Strong L^2 well-posedness is defined for (3.1) as in the constant coefficient case. Following Kreiss [K] we define the uniform Lopatinski condition for (3.1) as uniform Lopatinski for the family of frozen-coefficient problems

$$\begin{aligned} &u_t + \sum_{j=1}^d A^j(q,0) u_{x_j} = f \\ &\Gamma(q) u(t,\tilde{x},0) = g, \end{aligned} \tag{3.2}$$

with parameter $q = (t, \tilde{x})$ varying in $\mathbb{R}^d$, where the constant $C > 0$ is now required to be uniform in both $\gamma > 0$ and the parameter q. The variable-coefficient analogue of Theorem 1.5, extending the result proved in [K, CP, MZ2] for hyperbolic constant-multiplicity systems, would be as follows:

The system (L, Γ) (3.1) is strongly L^2 well-posed if and only if it satisfies the frozen uniform Lopatinski condition.

So far we have been unable to prove the sufficiency of the uniform Lopatinski condition. The main obstacle, curiously, is to obtain a variable-coefficient analogue of the elementary Lemma 2.3. More precisely, we would like to show that if (L, Γ) satisfies the frozen uniform Lopatinski condition, then solutions $u \in L^2(\mathbb{R}^{d+1}_+)$ of

$$L(t, x, \partial_t, \partial_x)u = 0 \tag{3.3}$$

satisfy uniform trace estimates

$$|u(t, \tilde{x}, 0)|^2_{L^2(\mathbb{R}^d)} \leq C|\Gamma u(t, \tilde{x}, 0)|^2_{L^2(\mathbb{R}^d)}. \tag{3.4}$$

For L as in (3.1) one can always find a boundary condition $\tilde{\Gamma}(t, x)$ for which $(L, \tilde{\Gamma})$ is strongly L^2 well-posed (as in Remark 1.2, part 3), so if we had (3.4) we could work in the original (t, x) variables and simply repeat the argument of Proposition 2.6, with $\|\cdot\|_{L^2(\mathbb{R}^{d+1}_+)}$ replacing $\|\cdot\|_{L^2(\mathbb{R}_+)}$ now, to derive the needed a priori estimates. In fact, in place of (3.4) it would be sufficient to establish

$$|u(t, \tilde{x}, 0)|^2_{L^2(\mathbb{R}^d)} \leq C\left(|\Gamma u(t, \tilde{x}, 0)|^2_{L^2(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^{d+1}_+)}\right).$$

However, to do this using the tools available appears to be as difficult as finding an actual Kreiss symmetrizer, yielding the full estimate for general f. That is, the exact computation of Lemma 2.3 does not seem to be robust under lower-order perturbations: there is no apparent advantage to small f over the general case.

4 Viscous shock and boundary layers

In this final section we sketch how the general principle of Proposition 2.6 can be applied in a parabolic (or partially parabolic) problem.

In the study of noncharacteristic viscous shock or boundary layers, one linearizes the compressible Navier-Stokes equations about a function of one variable, say $w(x_d)$, which describes the shock or boundary layer. After symmetrizing and applying a conjugating transformation to remove dependence on the variable x_d in the coefficients (see, e.g., the introduction to [GMWZ1] or [GMWZ4]), we reduce to the study of a constant coefficient, second-order, boundary value problem on the half-space $\mathbb{R}^{d+1}_+$,

$$
\begin{aligned}
A^0 u_t + \sum_{j=1}^{d} A^j u_{x_j} - \sum_{j,k=1}^{d} B^{jk} u_{x_j,x_k} = f,\\
\Gamma u(t,\tilde{x},0) = g,
\end{aligned}
\tag{4.1}
$$

where

$$
u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad A^j = \begin{pmatrix} A^j_{11} & A^j_{12} \\ A^j_{21} & A^j_{22} \end{pmatrix}, \quad B^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & B^{jk}_{22} \end{pmatrix},
$$

with $\det A^d \neq 0$, A^0 positive definite, A^j symmetric, and

$$
\Re \sum_{jk} \xi_j \xi_k B^{jk}_{22} \geq \theta |\xi|^2.
$$

Applying as before the Laplace transform in the temporal variable t and the Fourier transform in $\tilde{x}$, we obtain the generalized resolvent equation (with hats dropped)

$$
\begin{aligned}
\lambda A^0 u + A^d u' + \sum_{j=1}^{d-1} i\eta_j A^j u \\
- B^{dd} u'' - \sum_{j=1}^{d-1} i\eta_j (B^{j1} + B^{1j}) u' + \sum_{j,k=1}^{d-1} \eta_j \eta_k B^{jk} u &= f,\\
\Gamma u(0) = g,
\end{aligned}
\tag{4.2}
$$

which may be written after some rearrangement as a first-order system with a redefined Γ

$$
\begin{aligned}
U' - \mathcal{G}(\Lambda) U = F,\\
\Gamma U(0) = G,
\end{aligned}
\tag{4.3}
$$

in the variable $U := (u, u_2')$.

Taking "dissipative" boundary conditions in the class identified by Rousset [R3], $\tilde{\Gamma} U = (\Gamma_1 u_1, u_2)$, with Γ_1 maximally dissipative for the hyperbolic problem $A^0_{11} v_t + \sum_j A^j_{11} v_{x_j} = 0$, we obtain by integration by parts (after forming the L^2 inner product of u with (4.2)) estimates that are nearly of the form (2.2). The difference is that several weights $\alpha_k(\gamma, \tau, \eta)$ appear and u and u'_2 coordinates are weighted differently. We use this estimate to define *uniform viscous stability*, the analogue of Definition 2.1. For this choice of weights and $\tilde{\Gamma}$, Assumption 2.4 is then satisfied for $\mathcal{G}$ as in (4.3). The explicit estimates/weights are given in [GMWZ4, GMWZ5].

A review of the proof of Proposition 2.6 reveals that the new weights do not affect the arguments there. Thus, we obtain the analogous result that uniform viscous stability follows from the uniform Lopatinski condition. The latter condition is called in the viscous context the *uniform Evans condition.* This extends results of [MZ2, GMWZ6] in the variable-multiplicity case, in particular for MHD.

Unfortunately, this result, though suggestive, does not yield nonlinear stability, either for small viscosity, which requires variable-coefficient estimates, or for large time, which requires $L^1 \to L^2$ estimates between norms [GMWZ1].

Remark 4.1. A finer point of the analysis is that the conjugating transformation yields uniform estimates only for a *compact* set of frequencies, so a different analysis must be used in the high-frequency regime, as discussed in [GMWZ4, GMWZ5, GMWZ6]. In particular, the Evans condition must be required to hold uniformly under an appropriate high-frequency rescaling. However, this high-frequency part of the analysis has already been carried out in [GMWZ4, GMWZ6] without any assumptions on multiplicity of hyperbolic characteristics. Thus, the bounded-frequency argument just presented is precisely what is needed to extend to the general, variable-multiplicity case.

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