Nonclassical multidimensional viscous and inviscid shocks

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Abstract

Extending our earlier work on Lax-type shocks of systems of conservation laws, we establish existence and stability of curved multidimensional shock fronts in the vanishing viscosity limit for general Lax- or undercompressive-type shock waves of nonconservative hyperbolic systems with parabolic regularization. The hyperbolic equations may be of variable multiplicity and the parabolic regularization may be of "real", or partially parabolic, type. We prove an existence result for inviscid nonconservative shocks that extends to multidimensional shocks a one-dimensional result of X. Lin proved by quite different methods. In addition, we construct families of smooth viscous shocks converging to a given inviscid shock as viscosity goes to zero, thereby justifying the small viscosity limit for multidimensional nonconservative shocks.

In our previous work on shocks we made use of conservative form especially in parts of the low frequency analysis. Thus, most of the new analysis of this paper is concentrated in this area. By adopting the more general nonconservative viewpoint, we are able to shed new light on both the viscous and inviscid theory. For example, we can now provide a clearer geometric motivation for the low frequency analysis in the viscous case. Also, we show that one may, in the treatment of inviscid stability of nonclassical and/or nonconservative shocks, remove an apparently restrictive technical assumption made by Mokrane and Coulombel in their work on, respectively, shock-type nonconservative boundary problems and conservative undercompressive shocks. Another advantage of the nonconservative perspective is that Lax and undercompressive shocks can be treated by exactly the same analysis.

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1 Introduction

In this paper we develop a theory of multidimensional inviscid and viscous shocks without assuming conservative form for the underlying inviscid system. One motivation, of course, is to provide a theory of multidimensional nonconservative shocks. Another motivation is less obvious: by dropping the assumption of conservative form it turns out that we are able to treat both Lax shocks and undercompressive shocks by *exactly the same analysis.* Under the appropriate spectral stability conditions, the construction of curved inviscid shocks, and the proof that such shocks can be obtained as vanishing viscosity limits of viscous shocks, can be accomplished by an analysis that does not distinguish between Lax and undercompressive shocks.[1] We regard classical (i.e., conservative Lax) shocks as special

[1]We do not treat overcompressive shocks, which possess neither a well-defined inviscid limit nor theory. See [F1, F2, FL, ZS, Z1] for further discussion.

cases of nonclassical shocks and conservative shocks as special cases of nonconservative ones. Recall that systems in nonconservative form arise in models of deformation of elastic-plastic solids [TC], two-phase flow [SW, SGR, Se], spray dynamics [RS], and other applications.

In our previous work on shocks [GMWZ1]-[GMWZ4] we made use of conservative form especially in parts of the low frequency analysis. Thus, most of the new analysis of this paper is concentrated in this area. The nonconservative viewpoint provides a clearer geometric motivation for the low frequency analysis and leads, for example, to a simpler construction of high order approximate solutions. In passing, we simplify also the nonclassical inviscid theory, removing a technical assumption of [Mo, Cou]; see Remark 1.19. As we did for viscous boundary problems involving a given fixed boundary in [GMWZ5], we allow the underlying inviscid system to have characteristics of variable multiplicity; thus, our analysis is applicable, for example, to (Lax or undercompressive) MHD shocks.

When $d = 1$, let us recall the result of [BB] which applies to the vanishing viscosity approach of 1D quasilinear strictly hyperbolic systems (including non conservative ones), in a context which extends the usual one of "entropy weak solutions" of the conservative case. This result is stronger than the ones derived here in that it is for "unprepared data" (i.e., it states that viscous solutions for *any* nearby data converge to the inviscid solution) whereas ours are for "prepared data" (viscous solutions for *some* nearby data converge to the inviscid solution), is *global* in time (hence accommodates interaction of shock fronts), and is framed in a weaker norm (B.V. vs. H^s). On the other hand, the "Glimm-type" analysis of [BB] based on approximate decoupling of scalar modes uses strongly the assumptions of small variation of the background solution (hence small shock amplitude), Laplacian viscosity, strict hyperbolicity, and a single space dimension, whereas one of the main motivations for our approach is to remove such restrictions.

The approach we have used in [GMWZ2, GMWZ3, GMWZ4] to construct families of smooth, exact, viscous shocks converging to a given curved inviscid shock in the vanishing viscosity limit has four main steps.

The first step is to construct viscous profiles for planar shocks; this amounts to solving an ODE with prescribed endstates at $\pm\infty$. These exact solutions to the viscous problem describe the fast, shock layer, transition between two constant states.

The second step is to linearize the full, parabolic (or partially parabolic) problem about a profile solution and define appropriate spectral stability determinants or Evans functions for this linearized problem. Suitable nonvanishing conditions for these determinants give necessary (and sometimes sufficient) conditions for linear and nonlinear stability. We define two sorts of Evans functions, standard and modified; the first exhibits the usual translational degeneracy at zero frequency, while the second is typically nonvanishing at zero frequency. Part of this step is to clarify the connection between the two Evans functions (see Theorem 5.15, for example).

The third step is to compute the standard Evans function for a given profile and check whether the nonvanishing conditions are actually satisfied. This can be done numerically ([HZ], e.g.) and sometimes even analytically ([PZ, FS]).

The final step is to construct approximate, curved viscous shock solutions and then prove the linear and nonlinear stability of those solutions, assuming the Evans condition is satisfied. This step involves understanding the link between the viscous Evans function, the Lopatinski determinant that governs stability of the inviscid hyperbolic problem, and the transversality properties of profiles (see Theorem 5.2, for example). It also involves the construction of Kreiss-type symmetrizers and the proof of "maximal" estimates for the linearized parabolic problem.

Most of the work of this paper is concerned with the second and fourth steps. For some of the problems we consider, step one has not been done yet. For example, we are not aware of any viscous profile constructions for nonconservative, undercompressive shocks (except in some trivial cases). For such shocks the present paper reduces the full nonlinear stability problem to the construction of profiles and the verification of the Evans hypothesis.

In other problems step one has been done, but not step three. For example, viscous profiles have been constructed for nonconservative, Lax shocks ([S, Sa], for example), and for some conservative, undercompressive shocks ([AMPZ, AMPZ2, IMP, Sh, SSh1, SSh2]), but the Evans hypothesis has not yet been carefully verified. In these cases our paper reduces the full nonlinear stability problem to the verification of the Evans hypothesis. In some problems, such as the Navier-Stokes regularization of Euler shocks studied in [GMWZ4], all four steps are now complete.

Finally, let us clarify the relation of section 2 of this paper to the first step. If one **starts with** a single transversal profile for a given planar shock, the generalized Rankine-Hugoniot condition identifies "nearby" shocks for which profiles must also exist.

1.1 Inviscid $\mathcal{C}$-shocks

We begin by defining a notion of inviscid shock that is more general than the standard concept of nonconservative shock (used, e.g., in [Lin] and recalled below). Consider an $N \times N$ system on $\mathbb{R}^{d+1}$

$$\sum_{j=0}^{d} A_j(u)\partial_j u = 0. \tag{1.1}$$

The system is said to be *conservative* when $A_j(u) = f_j'(u)$ for $\mathbb{R}^N$-valued functions f_j. This requirement is dropped in the following assumptions.

The Assumptions 1.1 and 1.8 stated below are in force throughout the paper. The other Assumptions are made in a given Theorem, Proposition, etc., only if stated explicitly there.

Assumption 1.1. *(H0) The $A_j(u)$ are C^∞ functions from a connected open set $\mathcal{U}^* \subset \mathbb{R}^N$ to $\mathbb{R}^{N\times N}$. For all $u \in \mathcal{U}^*$ the matrix $A_0(u)$ is invertible.*

(H1)(Hyperbolicity near endstates) Let

$$\overline{A}_j := A_0^{-1}A_j \text{ and } \overline{A}(u,\xi) := \sum_{j=1}^{d} \xi_j \overline{A}_j(u). \tag{1.2}$$

For $u_\pm$ in connected open sets $\mathcal{U}_\pm \subset \mathcal{U}^*$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, the eigenvalues of $\overline{A}(u_\pm,\xi)$ are real.

For $(s,h) \in \mathbb{R}^d$ let

$$\mathcal{A}_d(u,s,h) := A_d(u) - sA_0(u) - \sum_{j=1}^{d-1} h_i A_i(u) \text{ and } \overline{\mathcal{A}}_d := A_0^{-1}\mathcal{A}_d. \tag{1.3}$$

Given $q = (u_+, u_-, s, h) \in \mathcal{U}_+ \times \mathcal{U}_- \times \mathbb{R}^d$, let $R_-(q)$ (resp. $L_+(q)$) denote the number of negative (resp. positive) eigenvalues of $\overline{\mathcal{A}}_d(u_+, s, h)$ (resp. $\overline{\mathcal{A}}_d(u_-, s, h)$).

Assumption 1.2. *Let k be an integer such that $0 \le k \le N+1$. We are given a connected $N+d-k$ dimensional C^∞ submanifold*

$$\mathcal{C} \subset \mathcal{U}_+ \times \mathcal{U}_- \times \mathbb{R}^d$$

such that for all $q = (u_+, u_-, s, h) \in \mathcal{C}$, $\overline{\mathcal{A}}_d(u_\pm, s, h)$ is invertible. R_- and L_+ are now independent of $q \in \mathcal{C}$ and we suppose $R_- + L_+ = N + 1 - k$. We'll refer to $\mathcal{C}$ as a shock manifold *and k as the* undercompressive index*.*

Definition 1.3. *Let $t = x_0$ and $y = (x_1, \dots, x_{d-1})$. For $T > 0$ an* inviscid $\mathcal{C}$-shock *on $[0,T] \times \mathbb{R}^d$ is a triple of functions*

$$(u_+(t, y, x_d), u_-(t, y, x_d), \psi(t, y)) \tag{1.4}$$

taking values in $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$, with $\psi \in C^1([0,T] \times \mathbb{R}^{d-1})$ and $u_\pm$ of class C^1 in $\pm(x - \psi(t,y)) \ge 0$ respectively, and satisfying

$$\begin{aligned} &\sum_{j=0}^{d} A_j(u_\pm)\partial_j u_\pm = 0 \ \textit{in} \ \pm(x_d - \psi(t,y)) \ge 0 \\ &(u_+(t,y,\psi(t,y)), u_-(t,y,\psi(t,y)), d\psi(t,y)) \in \mathcal{C} \ \textit{for} \ (t,y) \in [0,T] \times \mathbb{R}^{d-1}. \end{aligned} \tag{1.5}$$

When $k = 0$ the $\mathcal{C}$-shock is said to be of Lax type*; otherwise it is called* undercompressive*.*

The problem (1.5) can be regarded as a transmission problem with a free interface for the unknowns (1.4). To each point $(u_+, u_-, s, h) \in \mathcal{C}$ we can associate the *planar $\mathcal{C}$-shock*

$$(u_+, u_-, st + hy). \tag{1.6}$$

With slight abuse we'll sometimes refer to the point (u_+, u_-, s, h) itself as a planar $\mathcal{C}$-shock. Later we will often use the notation $(p, s, h) := (p_+, p_-, s, h)$ for planar $\mathcal{C}$-shocks.

Making the change of variable $\tilde{x} := x - \psi(t,y)$ we see that the problem (1.5) is equivalent to the transmission problem with flat interface

$$\begin{aligned} &(a) \ \sum_{j=0}^{d-1} A_j(u_\pm)\partial_j u_\pm + \mathcal{A}_d(u_\pm, d\psi)\partial_d u_\pm = 0 \text{ in } \pm x \ge 0 \\ &(b) \ (u_+(t,y,0), u_-(t,y,0), d\psi(t,y)) \in \mathcal{C} \text{ for } (t,y) \in [0,T] \times \mathbb{R}^{d-1}. \end{aligned} \tag{1.7}$$

We shall also refer to solutions of (u_+, u_-, ψ) of (1.7) as inviscid $\mathcal{C}$-shocks.

Definition 1.4. *Given a shock manifold $\mathcal{C}$ as in Assumption 1.2 and $q \in \mathcal{C}$, a* local defining function *for $\mathcal{C}$ near q is a C^∞ function $\chi : \mathcal{O} \to \mathbb{R}^{N+k}$, where $\mathcal{O} \subset \mathbb{R}^{2N+d}$ is an open neighborhood of q, such that $\nabla_q \chi(q)$ has full rank $N + k$ and*

$$\mathcal{C} \cap \mathcal{O} = \{(p_+, p_-, s, h) : \chi = 0\}. \tag{1.8}$$

Remark 1.5. 1. The point of introducing inviscid $\mathcal{C}$-shocks is to separate out the part of the construction of nonconservative inviscid shocks that can be done without reference to viscosity. We are mainly interested in the special cases where $\mathcal{C}$ represents the set of endstates $u_\pm$, speeds s, and directions h for which there exists an associated viscous connection, and where the defining function χ is derived from an associated Melnikov separation function (see Section 1.2).

2. In section 4.1 we define a spectral stability condition for the problem (1.7), the *uniform Lopatinski condition*, which generalizes the classical uniform stability condition of Majda [Ma1]. We show that the validity of the condition at a point $q \in \mathcal{C}$ depends only on the inviscid operator in (1.1) and the manifold $\mathcal{C}$; in particular, it is independent of the choice of local defining function for $\mathcal{C}$.

3. Observe that it follows immediately from Assumption 1.2 and Definition 1.3 that $\mathcal{C}$-shocks are always *noncharacteristic*.

In section 7 we prove the following existence theorem for inviscid $\mathcal{C}$-shocks using results of [MZ2, Mo, Cou, Ma2] and this paper. The theorem assumes the existence of a *K-family of smooth inviscid symmetrizers* for the linearization of the interior problem (1.7)(a) (see Remark 1.7.) In the following statement we set $\overline{\mathbb{R}}^d_\pm = \{(y, x_d) : \pm x_d \geq 0\}$ and let

$$H^s(\overline{\mathbb{R}}^d_\pm) = \{u \in H^s_{loc}(\overline{\mathbb{R}}^d_\pm) : u \text{ is constant outside some compact subset of } \overline{\mathbb{R}}^d_\pm\}. \tag{1.9}$$

We define H^s spaces on other unbounded domains similarly.

Theorem 1.6. *Suppose the inviscid operator satisfies Assumption 1.1 and, for a given integer k with $0 \leq k \leq N-1$, let $\mathcal{C}$ be an $N+d-k$ dimensional $\mathcal{C}$-shock manifold as in Assumption 1.2.*

1. Consider a planar shock $q = (p_+, p_-, s, h) \in \mathcal{C}$. Suppose that the uniform Lopatinski condition is satisfied at q and that a K-family of smooth inviscid symmetrizers exists on a neighborhood of q in $\mathcal{U}_+ \times \mathcal{U}_- \times \mathbb{R}^d$. Then for any finite $T_0 > 0$ there exist nonplanar $\mathcal{C}$-shocks (1.4) *on $[0, T_0] \times \mathbb{R}^d$ that are near $(p_+, p_-, st + hy)$ in C^1 norm.*

2. Let $s > \frac{d}{2} + 1$ and suppose that for $u^0_\pm(y, x_d) \in H^{s+1}(\overline{\mathbb{R}}^d_\pm)$, $\psi^0(y) \in H^{s+2}(\mathbb{R}^{d-1})$, there exists an $\mathbb{R}$-valued function $\sigma(y) \in H^{s+1}(\mathbb{R}^d)$ such that

$$(u^0_+(y, 0), u^0_-(y, 0), \sigma(y), \nabla_y \psi^0(y)) \in \mathcal{C} \text{ for } y \in \mathbb{R}^{d-1}, \tag{1.10}$$

and the uniform Lopatinski condition holds at all points (1.10). *Suppose also that $u^0_\pm(y, x_d)$, $\sigma(y)$, and $\psi^0(y)$ determine shock front initial data compatible to order $s-1$ (Definition 7.1). Then there exists a time $T_0 > 0$ and functions*

$$u_\pm(t, y, x_d) \in H^s([0, T_0] \times \overline{\mathbb{R}}^d_\pm), \quad \psi(t, y) \in H^{s+1}([0, T_0] \times \mathbb{R}^{d-1})$$

satisfying (1.7) *on $[0, T_0] \times \overline{\mathbb{R}}^d_\pm$ with initial data*

$$u_\pm|_{t=0} = u^0_\pm, \ \psi|_{t=0} = \psi^0. \tag{1.11}$$

Remark 1.7. 1. K-families of smooth inviscid symmetrizers are defined in [MZ2], Defn. 4.9, and sufficient conditions on the symbol of the linearized system are given there ([MZ2], Theorems 3.4 and 5.6) for the existence of such families. These conditions are satisfied, for example whenever the eigenvalues of $\overline{A}(u_\pm,\xi)$ (as in (1.2)) are real and semisimple with constant multiplicity for $u_\pm \in \mathcal{U}_\pm \subset \mathcal{U}^*$, $\xi \in \mathbb{R}^d \setminus \{0\}$. This is the case with Euler shocks [Ma1]. In addition, inviscid K-families exist in some cases where characteristics of variable multiplicity are allowed; for example, in Friedrichs symmetrizable hyperbolic systems where all real characteristic roots are either *geometrically regular* or *totally nonglancing* ([MZ2], Theorem 5.6). Examples of this sort are provided by both fast and slow inviscid MHD shocks under generically satisfied conditions on the size of the magnetic field ([MZ2], Appendix A.)

The requirement that all real characteristic roots are geometrically regular is shown in [MZ2], Theorem 3.4 to be equivalent to Majda's block structure condition [Ma1]. Note also that the sufficient conditions described above for the existence of inviscid K-families apply equally well to nonconservative problems.

The construction of a K-family of inviscid symmetrizers Σ_K is an intermediate step in the construction of a Kreiss symmetrizer that is independent of the boundary (or transmission) condition. When the uniform Lopatinski condition is satisfied, a Kreiss symmetrizer is obtained from a K-family by taking K sufficiently large ([MZ2], Prop. 4.10).

1.2 Inviscid $\mathcal{C}_\mathcal{B}$-shocks

We will be especially interested in the case where the shock manifold $\mathcal{C}$ is derived from a viscosity. Consider the $N \times N$ viscous system on $\mathbb{R}^{d+1}$

$$\sum_{j=0}^{d} A_j(u^\varepsilon)\partial_j(u^\varepsilon) - \varepsilon \sum_{j,k=1}^{d} \partial_j\big(B_{j,k}(u^\varepsilon)\partial_k u^\varepsilon\big) = 0, \tag{1.12}$$

where $\epsilon > 0$ and the A_j satisfy Assumption 1.1. Set

$$\overline{B}_{j,k} := A_0^{-1} B_{j,k} \text{ and } \overline{B}(u,\xi) := \sum_{j,k=1}^{d} \overline{B}_{j,k}(u)\xi_j\xi_k. \tag{1.13}$$

Assumption 1.8. *(H2) (Artificial Viscosity) The $B_{j,k}$ are C^∞ functions on $\mathcal{U}^*$ valued in $\mathbb{R}^{N\times N}$. There is a positive constant c such that for all $u \in \mathcal{U}^*$ and $\xi \in \mathbb{R}^d$ the eigenvalues μ of $\overline{B}(u,\xi)$ satisfy $\Re\mu \geq c|\xi|^2$.*

(H3) (Strict dissipativity near endstates) There is a positive constant c such that for all $u \in \mathcal{U}_\pm$ and $\xi \in \mathbb{R}^d$ the eigenvalues μ of $i\overline{A}(u,\xi) + \overline{B}(u,\xi)$ satisfy $\Re\mu \geq c|\xi|^2$.

Remark 1.9. In order to treat Navier-Stokes or viscous MHD shocks, for example, we must of course allow partially parabolic (or *real*) viscosities. The case of real viscosities involves substantial additional difficulties which have already been addressed in [GMWZ4] for conservative Lax shocks. These difficulties are confined to the high frequency analysis and are unaffected by the distinctions conservative/nonconservative or Lax/undercompressive. The exposition is lighter for artificial viscosity; we discuss the changes needed to handle real viscosities in an appendix.

We can look for exact travelling wave solutions to (1.12) of the form

$$u^\varepsilon(t,y,x_d) = w\left(\frac{x_d - st - hy}{\varepsilon}\right). \tag{1.14}$$

Setting $\nu = \nu(s,h) := (-s, -h_1, \dots, -h_{d-1}, 1)$ we note that

$$\mathcal{A}_d(u,s,h) = \sum_{j=0}^{d} A_j(u)\nu_j \text{ and define } \mathcal{B}_{d,d}(u,s,h) := \sum_{j,k=1}^{d} B_{j,k}(u)\nu_j\nu_k. \tag{1.15}$$

Then u^ε is a solution of the parabolic problem (1.12) if and only if $w(z)$ is a solution of the profile equation

$$\mathcal{A}_d(w,s,h)\partial_z w - \partial_z(\mathcal{B}_{d,d}(w,s,h)\partial_z w) = 0 \text{ on } -\infty < z < \infty. \tag{1.16}$$

Given $q = (u_+, u_-, s, h) \in \mathbb{R}^{2N+d}$, the solution $w(z)$ is said to be a *viscous profile associated to* q when in addition to (1.16) we have

$$\lim_{z \to \pm\infty} w(z) = u_\pm, \tag{1.17}$$

and we write $w(z) = W(z,q)$.

We are mainly interested in problems where Assumption 1.2 is replaced by the following stronger assumption.

Assumption 1.10. *For a fixed k we are given a shock manifold $\mathcal{C}$ as in Assumption 1.2 together with a C^∞ function $W(z,q)$ from $\mathbb{R}\times\mathcal{C}$ to $\mathcal{U}^*$ such that for all $q = (u_+, u_-, s, h) \in \mathcal{C}$, $W(z,q)$ satisfies* (1.16) *and* (1.17)*; that is, for all $q \in \mathcal{C}$, $W(z,q)$ is a viscous profile associated to q.*

Definition 1.11. *When the points of a shock manifold $\mathcal{C}$ are associated to viscous profiles $W(z,q)$ corresponding to a particular viscosity*

$$\mathcal{B}(u) := \sum_{j,k=1}^{d} \partial_j(B_{j,k}(u)\partial_k u) \tag{1.18}$$

as in Assumption 1.10, we'll write $\mathcal{C} = \mathcal{C}_\mathcal{B}$ and refer to the associated shocks (Definition 1.3) as inviscid $\mathcal{C}_\mathcal{B}$-shocks. *We call the condition*

$$(u_+(t,y,0), u_-(t,y,0), d\psi(t,y)) \in \mathcal{C}_\mathcal{B} \text{ for } (t,y) \in [0,T] \times \mathbb{R}^{d-1} \tag{1.19}$$

the generalized Rankine-Hugoniot (GRH) condition *determined by $\mathcal{B}$.*

Remark 1.12. *1.* The definition of inviscid $\mathcal{C}_\mathcal{B}$-shock given above is the same as the notion of *nonconservative shock* associated to a viscosity $\mathcal{B}$ used, for example, in [Lin]. In that paper Lin proves the existence of (what we call) one-dimensional inviscid $\mathcal{C}_\mathcal{B}$-shocks by methods that are quite different from those we use to prove Theorem 1.6.

2. It follows from (H2), (H3) that for $q = (u_+, u_-, s, h) \in \mathcal{C}$, the matrix

$$\mathcal{B}_{d,d}^{-1}(u_+, s, h)\mathcal{A}_d(u_+, s, h)$$

has no eigenvalues on the imaginary axis. The number of eigenvalues with negative real part is the same as for $\overline{\mathcal{A}}_d(u_+, s, h)$, namely R_-. The analogous statement holds for u_- with L_+

replacing R_- ([Me1], Lemma 5.1.3). Consequently, for q in a compact subset of $\mathcal{C}$, standard ODE theory (e.g., [Me1], Lemma 5.3.3) implies that the associated profiles $W(z,q)$ satisfy

$$|\partial_z W(z,q)| \le Ce^{-\delta|z|} \text{ for some } \delta > 0. \tag{1.20}$$

3. Planar Lax-type shock profiles may be constructed by center manifold reduction using the method of Schechter [S]. Both artificial and real viscosities are treated there. Small-amplitude undercompressive profiles might be constructible by a combination of the methods of Schecter [S] and the bifurcation analysis used in [AMPZ] to construct such profiles in the conservative case.[2] Constructions using fixed-point arguments are given in Sainsaulieu [Sa] for the Lax case. Starting with a single transversal profile associated to a planar shock $\underline{q} = (\underline{p}_+, \underline{p}_-, \underline{s}, \underline{h}) \in \mathbb{R}^{2N+d}$, we show in Proposition 2.8 how to construct a $\mathcal{C}_\mathcal{B}$-manifold near $\underline{q}$.

4. In [Mo] Mokrane studies strictly hyperbolic nonconservative boundary problems involving an additional front-like unknown ψ, together with boundary conditions that are uniformly stable in the sense of Majda. This class of problems includes the case of conservative Lax shocks. Although Mokrane makes no reference to undercompressive shocks and although his class does not strictly include what we call inviscid $\mathcal{C}$-shocks (since his boundary conditions are not fully nonlinear in $\nabla\psi$), the estimates and methods contained in his proofs allow for the treatment of undercompressives in cases where block structure in the sense of Kreiss/Majda can be attained. In particular his discussion of the adjoint problem is fully adequate to yield existence of inviscid undercompressive shocks.

5. An interesting example of (conservative) undercompressive shocks arises in the study of isothermal phase transitions in a van der Waals fluid. Here the underlying hyperbolic problem is the system of isothermal Euler equations with the van der Waals pressure law

$$p(\rho) = P(v) = \frac{RT}{v-b} - \frac{a}{v^2}; \qquad v = \frac{1}{\rho},\ a > 0, b > 0. \tag{1.21}$$

In [B] Benzoni-Gavage proves the existence of planar phase transitions admissible in the sense of Slemrod's viscosity-capillarity criterion [S]. These phase transitions are undercompressive shock solutions to the Euler equations ($d = 2$ or 3) with undercompressive index $k = 1$, and they satisfy the uniform Lopatinski condition. They provide an example of $\mathcal{C}$-shocks where the shock manifold $\mathcal{C}$ comes from a viscosity-capillarity regularization term rather than a viscosity regularization term as in the case of $\mathcal{C}_\mathcal{B}$ shocks.

1.3 Viscous $\mathcal{C}_\mathcal{B}$-shocks

If we introduce an unknown front $x_d = \psi^\varepsilon(t,y)$ and change variables

$$\tilde{x}_d = x_d - \psi^\varepsilon(t,y), \tag{1.22}$$

[2] Schecter's analysis requires both strict hyperbolicity and genuine nonlinearity of the associated hyperbolic system. As pointed out in [AMPZ], at least one of these must fail in the undercompressive case by examination of characteristic speeds, leading to a more complicated, codimension-three bifurcation as compared to the codimension-one bifurcation of the Lax case.

the problem (1.12) can be rewritten

$$\sum_{j=0}^{d-1} A_j(u^\varepsilon)\partial_j u^\varepsilon + \mathcal{A}_d(u^\varepsilon, d\psi^\varepsilon)\partial_{\tilde{x}_d} u^\varepsilon - \varepsilon \sum_{j,k=1}^{d} D_j(B_{j,k}(u^\varepsilon)D_k u^\varepsilon) = 0 \text{ on } \mathbb{R}^{d+1}, \tag{1.23}$$

where $D_j = \partial_j - (\partial_j \psi^\varepsilon)\partial_{\tilde{x}_d}$, for $j = 1, \ldots, d-1$ and $D_d = \partial_{\tilde{x}_d}$. The introduction of the *viscous front* ψ^ε allows us as in [GW, GMWZ3] to reformulate (1.22) as a parabolic transmission problem:

$$\begin{aligned} &\sum_{j=0}^{d-1} A_j(u^\varepsilon)\partial_j u^\varepsilon + \mathcal{A}_d(u^\varepsilon, d\psi^\varepsilon)\partial_{\tilde{x}_d} u^\varepsilon - \varepsilon \sum_{j,k=1}^{d} D_j(B_{j,k}(u^\varepsilon)D_k u^\varepsilon) = 0 \text{ on } \overline{\mathbb{R}}^{d+1}_{\pm}, \\ &[u^\varepsilon] = 0, [\partial_{\tilde{x}_d} u^\varepsilon] = 0 \text{ on } \tilde{x}_d = 0. \end{aligned} \tag{1.24}$$

We can regard ψ^ε as an additional unknown, and then we are forced to add an extra transmission condition as in (6.1) to obtain a well-posed problem [GMWZ3].

Henceforth we drop the tildes appearing in (1.24).

Definition 1.13. *Let $(u_+(t, y, x_d), u_-(t, y, x_d), \psi(t, y))$ be an inviscid $\mathcal{C}_{\mathcal{B}}$-shock satisfying the hyperbolic transmission problem* (1.7)*. A* viscous $\mathcal{C}_{\mathcal{B}}$-shock *is a family of exact solutions $(u^\varepsilon_\pm, \psi^\varepsilon)$ of* (1.24) *such that*

$$\begin{aligned} &u^\varepsilon_\pm \to u_\pm \ \textit{in} \ L^2_{loc}([0,T] \times \overline{\mathbb{R}}^d_\pm) \\ &u^\varepsilon_\pm \to u_\pm \ \textit{in} \ L^\infty_{loc}([0,T] \times \mathbb{R}^d_\pm) \\ &\psi^\varepsilon \to \psi \ \textit{in} \ L^\infty_{loc}([0,T] \times \mathbb{R}^{d-1}) \end{aligned} \tag{1.25}$$

as $\varepsilon \to 0$. When the inviscid shock is planar, the associated viscous shock is called a planar viscous $\mathcal{C}_{\mathcal{B}}$-shock.

Example 1.14. *Let $q = (p_+, p_-, s, h) \in \mathcal{C}_{\mathcal{B}}$ be a planar inviscid $\mathcal{C}_{\mathcal{B}}$-shock. Then the family*

$$\begin{aligned} &u^\varepsilon_\pm(t, y, x_d) := W\left(\frac{x_d}{\varepsilon}, q\right)|_{\pm x_d \geq 0} \\ &\psi^\varepsilon(t, y) = st + hy \end{aligned} \tag{1.26}$$

is a planar viscous $\mathcal{C}_{\mathcal{B}}$-shock satisfying (1.25)*.*

We are interested in the existence of nonplanar viscous $\mathcal{C}_{\mathcal{B}}$ shocks. In section 4.2 we recall the definition of the *standard Evans function* $D_s(q, \hat{\zeta}, \rho)$. For $q \in \mathcal{C}_{\mathcal{B}}$ this is a spectral stability function for the transmission problem obtained by linearizing (1.24) with respect to u^ε about $W(\frac{x_d}{\varepsilon}, q)$.

Definition 1.15. *We say that the* standard uniform Evans condition *is satisfied at q when there exist positive constants c, ρ_0 such that*

$$\begin{aligned} &(a)\ |D_s(q, \hat{\zeta}, \rho)| \geq c\rho \ \textit{for} \ \hat{\zeta} \in \overline{S}^d_+,\ 0 < \rho \leq \rho_0 \\ &(b)\ D_s(q, \hat{\zeta}, \rho) \neq 0 \ \textit{for} \ \rho > 0. \end{aligned} \tag{1.27}$$

Here $\overline{S}^d_+ = \{\zeta = (\tau, \gamma, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-1} : |\zeta| = 1, \gamma \geq 0\}$.

The next Theorem extends the main result of [GMWZ3], which applied to conservative Lax shocks in cases where the hyperbolic characteristics were of constant multiplicity and the viscosity was fully parabolic, to nonclassical (nonconservative, undercompressive) shocks. The result works in certain cases where characteristics have variable multiplicities. The extension to real viscosity is discussed in an appendix.

Theorem 1.16. *Consider the viscous transmission problem* (1.24) *under the structural Assumptions 1.1 and 1.8, and let the shock manifold $\mathcal{C} = \mathcal{C}_{\mathcal{B}}$ be as in Assumption 1.10. Let $(u_+(t, y, x_d), u_-(t, y, x_d), \psi(t, y))$ be an inviscid $\mathcal{C}_{\mathcal{B}}$-shock satisfying the hyperbolic transmission problem* (1.7) *on $[-T_0, T_0] \times \overline{\mathbb{R}}^d_\pm$. Let*

$$q(t, y) := (u_+(t, y, 0), u_-(t, y, 0), d\psi(t, y)) \in \mathcal{C}_{\mathcal{B}} \text{ for all } (t, y) \in [-T_0, T_0] \times \mathbb{R}^{d-1} \tag{1.28}$$

be constant outside a compact set. Suppose for all (t, y) that the standard uniform Evans condition holds at $q(t, y)$ and that a K-family of smooth viscous symmetrizers exists on a neighborhood of $q(t, y)$ in $\mathcal{U}_+ \times \mathcal{U}_- \times \mathbb{R}^d$. Then provided (u_+, u_-, ψ) is sufficiently smooth, there exists a family of viscous $\mathcal{C}_{\mathcal{B}}$-shocks $(u^\varepsilon_+, u^\varepsilon_-, \psi^\varepsilon)$ satisfying (1.25).

Remark 1.17. K-families of smooth viscous symmetrizers are defined in [GMWZ5], Definition 3.5. Under the structural assumptions of sections 1.1 and 1.2 (or section 1.1 and Appendix A), such families always exist when the hyperbolic characteristics are real and semisimple with constant multiplicity. This is the case for Euler shocks with artificial viscosity or Navier-Stokes regularization [GMWZ3, GMWZ4].

More generally, the main result of [GMWZ5], Theorem 3.7, shows that viscous K-families exist in either one of the following two situations, both of which allow variable multiplicities and nonconservative form:

(a) all real characteristic roots satisfy a *block structure condition* ([GMWZ5], Defn. 4.9, condition BS*),*

(b) the system is symmetric dissipative in the sense of Kawashima ([GMWZ5], Defn. 2.5) and real characteristic roots are either totally nonglancing [GMWZ5], Defn. 4.3) or satisfy the above block structure condition.

The block structure condition just referred to is now more complicated than the one in the inviscid case. In the inviscid case the condition is the same as Majda's condition [Ma1] and is shown in [MZ2] to be equivalent to geometric regularity of the characteristic root ([MZ2], Theorem 3.4). In the viscous case, in addition to geometric regularity one must require that the viscous perturbation "respect" the decoupling between incoming and outgoing modes ([GMWZ5], Defn. 4.9).

An example of situation (b) is that of fast Lax shocks for the equations of viscous MHD ([GMWZ5], section 8). It is also shown in [GMWZ5] that smooth K-families do *not* exist for slow shocks for the viscous MHD equations. In fact, *viscous continuity* (see Defn. 4.22 below) is a necessary condition for the existence of smooth K-families of viscous symmetrizers, and this condition is shown to fail for slow MHD shocks.

Fast Lax shocks with a small magnetic field are perturbations of acoustic, gas dynamical shocks, so there is good reason to expect the standard Evans condition to be satisfied in this case, at least for ideal gas laws.

1.4 Overview of the main results

In most of the paper starting with section 2 we restrict our exposition to the case of parabolic systems of the form

$$\sum_{j=0}^{d} A_j(u)\partial_j u - \varepsilon \triangle u = 0 \text{ with } A_0(u) = I. \tag{1.29}$$

The general fully parabolic case where the $B_{j,k}$ are just required to satisfy Assumption 1.8 does, in fact, involve substantial additional difficulties in comparison with (1.29), but these have already been dealt with in [MZ3, GMWZ5]. In particular, the reduction to generalized block structure ([GMWZ5], Defn. 4.22) is much harder in the general case. Since these additional difficulties are unrelated to the distinctions Lax/undercompressive or conservative/nonconservative, we prefer to lighten the exposition and focus on the case of Laplacian viscosity. For the part of the theory we are presenting here, only a straightforward notational adjustment is needed for passage to the general fully parabolic case. In Appendix A we discuss the further changes needed to treat real (partially parabolic) viscosities.

1.4.1 Constructing $\mathcal{C}_\mathcal{B}$ and characterizing $T_q\mathcal{C}_\mathcal{B}$

With $\mathcal{B}(u) = \triangle u$ now and starting with a single given transversal viscous profile $\underline{w}(z)$ (Definition 2.14) associated to a planar shock $\underline{q} = (\underline{p}_+, \underline{p}_-, 0, 0)$, we show in Proposition 2.8 how to construct a shock manifold $\mathcal{C}_\mathcal{B}$ near $\underline{q}$, thereby providing a local verification of Assumption 1.10. Regarding the profile equation equivalently as a transmission problem

$$\begin{aligned} &(a)\ (1+|h|^2)w'' = \mathcal{A}_d(w,s,h)w' \text{ on } \pm z \geq 0 \\ &(b)\ [w] = 0,\ [w_z] = 0 \text{ on } z = 0 \end{aligned} \tag{1.30}$$

with unknowns $(w_\pm(z), s, h)$, we first obtain in Proposition 2.2 a convenient parametrization of all possible solutions of (1.30)(a) with endstates $p_\pm$ for $q = (p_+, p_-, s, h)$ near $\underline{q}$. The parametrization is given in terms of the functions $\phi_\pm(z, p_\pm, s, h, a_\pm)$ defined in (2.26). After adding a third transmission condition to (1.30),

$$s + w_+ \cdot \underline{w}_z - \underline{w}_z \cdot \underline{w} = 0 \text{ on } z = 0, \tag{1.31}$$

chosen so that certain rank conditions (2.37) are satisfied, we construct $\mathcal{C}_\mathcal{B}$ by a direct application of the implicit function theorem. This yields local defining functions $\chi(q)$ for $\mathcal{C}_\mathcal{B}$. In an appendix we prove that the manifold $\mathcal{C}_\mathcal{B}$ obtained by this process is independent of the choice of the third transmission condition.

Just as membership in $\mathcal{C}_\mathcal{B}$ defines the GRH condition, membership in $T_q\mathcal{C}_\mathcal{B}$ defines the linearized GRH condition at q. One has, of course, the obvious characterization of $T_q\mathcal{C}_\mathcal{B}$ in terms of the implicitly defined function χ as

$$T_q\mathcal{C}_\mathcal{B} = \{(\dot p_+, \dot p_-, \dot s, \dot h) \in \mathbb{R}^{2N+d} : \chi'(q)(\dot p_+, \dot p_-, \dot s, \dot h) = 0\}. \tag{1.32}$$

For the later analysis of the relationship between the standard and modified Evans functions in the low frequency regime, it is important to have a more explicit description of $T_q\mathcal{C}_\mathcal{B}$.

Such a description can be obtained by considering the *full* linearization with respect to (w, s, h) of (1.30), (1.31) at $(W(z,q), s, h)$:

$$(1.33)\quad \begin{aligned}&(a)\mathcal{L}_0(z,q,\partial_z)\dot{w} := -(1+|h|^2)\dot{w}'' + \mathcal{A}_d(W,s,h)\dot{w}' + \partial_w \mathcal{A}_d(W,s,h)\dot{w}W' = \\ &\qquad \dot{s}A_0(W)W' + \sum_{j=1}^{d-1} \dot{h}_j A_j(W)W' + 2\sum_{j=1}^{d-1} h_j \dot{h}_j W'' := \mathcal{L}_{0,1}(z,q)(\dot{s},\dot{h}) \\ &(b)\Gamma_a(\dot{w},\dot{s},\dot{h}) := \begin{pmatrix} [\dot{w}] \\ [\dot{w}_z] \\ \dot{s} + \dot{w}_+ \cdot \underline{w}_z \end{pmatrix} = 0 \text{ on } z = 0\end{aligned}$$

Using the fact that solutions of (1.33)(a) can be constructed from the derivatives $\nabla_{p_\pm,s,h,a_\pm}\phi_\pm$, in Proposition 3.12 we show

(1.34)

$$T_q\mathcal{C}_\mathcal{B} = \{(\dot{p}_+, \dot{p}_-, \dot{s}, \dot{h}) : \text{ there exists a solution } (\dot{w}_+, \dot{w}_-, \dot{s}, \dot{h}) \text{ of (1.33) with } \lim_{z\to\pm\infty} \dot{w}_\pm = \dot{p}_\pm\}.$$

The proof of (1.34) also involves the construction (Prop. 3.2) of smooth functions $\mathcal{R}_\pm(z,q,\dot{s},\dot{h})$, exponentially decaying to zero as $z \to \pm\infty$ and linear in $(\dot{s},\dot{h})$, such that if we define

$$(1.35)\qquad \dot{v} := \dot{w} - \mathcal{R}(z,q,\dot{s},\dot{h}),$$

then $(\dot{w},\dot{s},\dot{h})$ satisfies (1.33) if and only if $(\dot{v},\dot{s},\dot{h})$ satisfies

$$(1.36)\quad \begin{aligned}&(a)\ \mathcal{L}_0(z,q,\partial_z)\dot{v} = 0 \text{ on } \pm z \geq 0 \\ &(b)\Gamma_b(q)(\dot{v},\dot{s},\dot{h}) := \begin{pmatrix} [\dot{v}] + [\mathcal{R}(z,q,\dot{s},\dot{h}] \\ [\dot{v}_z] + [\mathcal{R}_z(z,q,\dot{s},\dot{h})] \\ \dot{v}_+ \cdot \underline{w}_z \end{pmatrix} = 0 \text{ on } z = 0.\end{aligned}$$

This reduces the study of the fully linearized profile problem to the study of the partially linearized problem, but with a more complicated boundary condition. All solutions to (1.36)(a) may be constructed by rewriting the problem as a $2N \times 2N$ first order system as in (3.14), and then conjugating to block form as in (3.27).

Next we describe a *reduced* transmission operator, $\Gamma_{0,red}(q)$, constructed in a simple way from the explicit operator $\Gamma_b(q)$, with the property

$$(1.37)\qquad \ker \Gamma_{0,red}(q) = T_q\mathcal{C}_\mathcal{B}.$$

Let us denote by $\mathcal{S}_0(q)$ the space of solutions of (1.36)(a) that decay to 0 as $z \to \pm\infty$ and by $\mathcal{S}(q)$ the space of bounded solutions of (1.36)(a). By Remark 3.5 we have

$$(1.38)\qquad \dim \mathcal{S}_0(q) = N + 1 - k, \quad \dim \mathcal{S}(q) = 2N + (N + 1 - k).$$

Consider for a moment the partial linearization with respect to w of (1.30) at $W(z,q)$:

$$(1.39)\quad \begin{aligned}&(a)\ \mathcal{L}_0(z,q,\partial_z)\dot{v} = 0 \text{ on } \pm z \geq 0 \\ &(b)\ \Gamma_c(\dot{v}) := \begin{pmatrix} [\dot{v}] \\ [\dot{v}_z] \end{pmatrix} = 0 \text{ on } z = 0.\end{aligned}$$

Transversality of $W(z,q)$ and translation invariance imply that the restriction $\Gamma_c|_{\mathcal{S}_0(q)}$ has a nontrivial kernel spanned by $W_z(z,q)$ (Prop. 2.12). Since $W_z(z,q)$ is near $\underline{w}_z$ for q near $\underline{q}$, we deduce immediately that

$$\mathbb{F}_P(q) := \{\Gamma_b(q)(\dot{v},0,0) : \dot{v} \in \mathcal{S}_0(q)\} \subset \mathbb{R}^{2N+1} \tag{1.40}$$

has dimension $N+1-k$, since the third transmission condition "removes the kernel" of $\Gamma_c|_{\mathcal{S}_0(q)}$. Choose an arbitrary $N+k$ dimensional complementary subspace $\mathbb{F}_{H,\mathcal{R}}(q)$ such that

$$\mathbb{R}^{2N+1} = \mathbb{F}_{H,\mathcal{R}}(q) \oplus \mathbb{F}_P(q), \tag{1.41}$$

and let $\pi_{H,\mathcal{R}}(q)$, $\pi_P(q)$ be the associated projections. Letting $\dot{v}(z,\dot{p}_\pm)$ denote *any* solution of (1.36)(a) satisfying $\lim_{z\to\pm\infty}\dot{v} = \dot{p}_\pm$, we have a well-defined map

$$\Gamma_{0,red}(q) : \mathbb{R}^{2N+d} \to \mathbb{F}_{H,\mathcal{R}}(q) \subset \mathbb{R}^{2N+1} \tag{1.42}$$

given by

$$\Gamma_{0,red}(q)(\dot{p}_+,\dot{p}_-,\dot{s},\dot{h}) := \pi_{H,\mathcal{R}}(q)\left(\Gamma_b(q)(\dot{v}(z,\dot{p}_\pm),\dot{s},\dot{h})\right). \tag{1.43}$$

Transversality implies that $\Gamma_{0,red}(q)$ has full rank $N+k$ even when restricted to the subspace $\dot{h}=0$ (Cor. 3.11). Finally, it is readily shown that $\ker\Gamma_{0,red}(q)$ is equal to the right side of (1.34), and this gives (1.37) (Prop. 3.12). We remark that the solutions $\dot{v}(z,\dot{p}_\pm)$ can be expressed explicitly in terms of block form coordinates (u_H,u_P) (3.34).

1.4.2 Stability determinants

At the beginning of section 4 we return to the general context of $\mathcal{C}$ shocks and define the Lopatinski and modified Lopatinski determinants, $D_{Lop}(q,\hat{\zeta})$ and $D_{Lop,m}(q,\hat{\zeta})$. The corresponding uniform (resp., modified uniform) Lopatinski condition at $q\in\mathcal{C}$ is the condition that

$$|D_{Lop}(q,\hat{\zeta})| \geq c \text{ (resp. } |D_{Lop,m}(q,\hat{\zeta})| \geq c) \tag{1.44}$$

for some $c>0$ independent of $\hat{\zeta} \in S^d_+ = \overline{S}^d_+ \cap \{\hat{\gamma}>0\}$. The determinant D_{Lop} is the one that naturally governs the stability of inviscid $\mathcal{C}$ shocks. The need to define $D_{Lop,m}$ becomes apparent only later in the low frequency analysis of the modified Evans function. Although the definition of D_{Lop} itself is not independent of the choice of defining function χ for $\mathcal{C}$, we show in Proposition 4.5 that the validity of the uniform Lopatinski condition is independent of the choice of χ, and thus depends just on the inviscid operator and $\mathcal{C}$. Moreover, we show (Prop. 4.8):

Proposition 1.18. *Suppose the uniform Lopatinski condition holds at $q\in\mathcal{C}$ and χ is any defining function for $\mathcal{C}$ near q. Then*
(a) $\chi'_{p,s}(q)$ has full rank $N+k$;
(b) if $d\geq 2$, $\chi'_p(q)$ has full rank $N+k$.

Remark 1.19. This Proposition shows that it is not necessary, as in the treatments [Mo, Cou], to introduce the full rank condition in part (b) as an extra hypothesis when $d \geq 2$. For $d = 1$, the full rank condition is not necessary for other reasons, since the analysis may be carried out by alternative methods that do not require it.[3] At the expense of further effort, one may dispense with the rank condition altogether by defining a pseudodifferential adjoint problem as in [GMWZ6]. (The rank condition is used only to cleverly define an adjoint equation that is differential.)

We show that the uniform Lopatinski condition at q always implies the modified Lopatinski condition at q, and in Proposition 4.7 we give geometric conditions under which the converse holds. As a corollary of these results for $\mathcal{C}$-shocks, in the case when $\mathcal{C} = \mathcal{C}_{\mathcal{B}}$ and $W(z, q)$ is the viscous profile associated to $q \in \mathcal{C}_{\mathcal{B}}$, we show that certain transversality assumptions on $W(z, q)$ imply the equivalence of the two Lopatinski conditions (Cor. 4.9).

Section 4 continues with the definition of the standard and modified Evans functions, $D_s(q, \hat{\zeta}, \rho)$ and $D_m(q, \hat{\zeta}, \rho)$, for viscous $\mathcal{C}_{\mathcal{B}}$ shocks. We start with the rescaled transmission problem

$$
\begin{aligned}
&\sum_{j=0}^{d-1} A_j(u)\partial_j u + \mathcal{A}_d(u, d\psi)\partial_z u - \sum_{j=1}^{d}(\partial_j - \partial_j\psi\partial_z)^2 u = 0 \text{ on } \pm z \geq 0 \\
&[u] = 0,\ [u_z] = 0 \text{ on } z = 0
\end{aligned} \tag{1.45}
$$

for which we have an exact solution given by a profile $W(z, q)$ and front $\psi = st + hy$. The determinant D_s is defined by considering the (Fourier-Laplace transform of the) partial linearization of (1.45) with respect to u about $W(z, q)$,

$$
\begin{aligned}
&\mathcal{L}(z, q, \zeta, \partial_z)u = f \text{ on } \pm z \geq 0 \\
&[u] = 0, [u_z] = 0 \text{ on } z = 0,
\end{aligned} \tag{1.46}
$$

while D_m is defined by considering the full linearization of (1.45) with respect to (u, ψ) about $(W(z, q), st + hy)$ and adding a third transmission condition:

$$
\begin{aligned}
&\mathcal{L}(z, q, \zeta, \partial_z)u - \psi\mathcal{L}_1(z, q, \zeta) = f \\
&[u] = 0,\ [u_z] = 0,\ c_0(\zeta)\psi + \underline{w}_z(0) \cdot u^+ = 0.
\end{aligned} \tag{1.47}
$$

Here $\mathcal{L}$ and $\mathcal{L}_1$ are given explicitly in (3.9) and $c_0(\zeta) = i\tau + \gamma + |\eta|^2$. Moreover, we suppose q is near a basepoint $\underline{q}$ and $\underline{w}(z) := W(z, \underline{q})$. It is important to note the correspondence between the operators appearing in (1.47) and those ($\mathcal{L}_0$ and $\mathcal{L}_{0,1}$) appearing in the fully linearized profile transmission problem (1.33). Writing $\mathcal{L}_1(z, q, \zeta) = \rho\check{\mathcal{L}}_1(z, q, \zeta, \rho)$ and $c_0(\zeta) = \rho\check{c}_0(\hat{\zeta}, \rho)$, we have

$$
\begin{aligned}
&\mathcal{L}_0(z, q, \partial_z) = \mathcal{L}(z, q, 0, \partial_z),\ \mathcal{L}_{0,1}(z, q)(i\hat{\tau} + \hat{\gamma}, i\hat{\eta}) = \check{\mathcal{L}}_1(z, q, \hat{\zeta}, 0), \\
&\text{and } i\hat{\tau} + \hat{\gamma} = \check{c}_0(\hat{\zeta}, 0).
\end{aligned} \tag{1.48}
$$

[3] For example, one may (after doubling to obtain a problem on a half-space) replace the nonstandard front variable ψ with a standard interior variable $v := \Psi_t$, where $\Psi := \psi e^{-z}$ is an extension to $z > 0$ of the front variable ψ (defined only at $z = 0$), and add the corresponding artificial interior equation $v_z + v = 0$, to obtain a standard hyperbolic initial–boundary-value problem in (u, v) (no longer involving ψ), with linear boundary condition $b_1 u + b_2 v = b_0$ (no longer involving ψ_t), treatable by the techniques of [CP].

Corresponding to the two Evans functions we have the standard (Defn. 4.12) and modified (Defn. 4.20) uniform Evans conditions. The standard uniform Evans condition is the one that is easier to verify analytically [PZ, FS] or numerically [B, HZ]. The vanishing of $D_s(q, \hat{\zeta}, \rho)$ at $\rho = 0$ reflects the translational degeneracy of (1.39) pointed out earlier. On the other hand the modified uniform Evans condition, which was introduced in [GMWZ3], implies maximal L^2 estimates for the fully linearized problem (1.47) and is essential for our construction of viscous $\mathcal{C}_{\mathcal{B}}$ shocks. Thus, a key result of this paper, discussed below, is the following theorem:

Theorem 1.20. *Under Assumptions 1.1, 1.8, and 1.10, the standard uniform Evans condition at $q \in \mathcal{C}_{\mathcal{B}}$ implies the modified uniform Evans condition at q.*

This was proved for conservative problems with (hyperbolic) characteristics of constant multiplicity in [GMWZ3], but a new argument is needed for nonconservative, variable multiplicity systems.

The third transmission condition in (1.47) yields a well-posed problem and, roughly speaking, removes the translational degeneracy of (1.46) at $\rho = 0$. Parallel to the earlier passage from (1.33) to (1.36), the fully linearized problem (1.47) is most easily studied by reducing it to the partially linearized problem (1.46) with modified transmission conditions. In Proposition 4.18 we recall from [GMWZ3] the construction of functions $\check{\boldsymbol{R}}_\pm(z, q, \hat{\zeta}, \rho)$ with the property that (u, ψ) satisfies (1.47) if and only if (v, ϕ) defined by

$$\phi := \rho\psi, \qquad v_\pm = u_\pm - \phi \check{\boldsymbol{R}}_\pm \tag{1.49}$$

satisfy

$$\begin{aligned} &\mathcal{L}(z, q, \zeta, \partial_z) v_\pm = f_\pm \quad \text{on } \pm z \geq 0\,, \\ &[v(0)] + \phi[\check{\boldsymbol{R}}(0)] = 0 \quad [\partial_z v(0)] + \phi[\partial_z \check{\boldsymbol{R}}(0)] = 0,. \\ &\underline{w}_z(0) \cdot v^+(0) = 0\,. \end{aligned} \tag{1.50}$$

Moreover, the functions $\mathcal{R}_\pm(z, q, \dot{s}, \dot{h})$ appearing in (1.35) and the functions $\check{\boldsymbol{R}}_\pm(z, q, \hat{\zeta}, \rho)$ can be chosen to satisfy

$$\mathcal{R}_\pm(z, q, i\hat{\tau} + \hat{\gamma}, i\hat{\eta}) = \check{\boldsymbol{R}}(z, q, \hat{\zeta}, 0), \tag{1.51}$$

as might be anticipated from (1.48).

Remark 1.21. Considering the equality (1.51), we see that the introduction of the functions $\mathcal{R}_\pm(z, q, \dot{s}, \dot{h})$ and their use in characterizing $T_q\mathcal{C}_{\mathcal{B}}$ can be viewed as a "missing step" that helps to clarify and provide a geometric motivation for some of the low frequency analysis in [GMWZ3]. This step was not needed there, and hence overlooked, because of the assumption of conservative form.

At the end of section 4 we discuss conditions for the continuity of decaying eigenspaces of the operators that define the linearized inviscid and viscous problems. Inviscid continuity implies, for example, that the uniform Lopatinski condition must hold for $q \in \mathcal{C}$ near $\underline{q}$ when it holds at $\underline{q}$. Similarly viscous continuity implies that the standard uniform Evans condition holds for $q \in \mathcal{C}$ near $\underline{q}$ when it holds at $\underline{q}$. Moreover, when inviscid or viscous continuity holds, sometimes a converse can be proved (as in the nonconservative Zumbrun-Serre theorem, Theorem 5.2) or a proof can be substantially simplified (see Remark 5.16). Viscous continuity (and thus inviscid continuity) always holds when the linearized hyperbolic problem has characteristics of constant multiplicity [MZ1]. A sufficient condition for inviscid (resp., viscous) continuity at $\underline{q}$ is the existence of a smooth K-family of inviscid (resp., viscous) symmetrizers near $\underline{q}$. In any given proposition we do not assume inviscid or viscous continuity unless we explicitly say so.

The first main result of section 5 is the nonconservative Zumbrun-Serre theorem, Theorem 5.2:

Theorem 1.22. *Consider a shock profile $\underline{w}(z) = W(z, \underline{q})$, where $\underline{q} = (\underline{p}, 0, 0)$, and suppose the low frequency standard Evans condition (1.27)(a) holds at $\underline{q}$. Then $\underline{w}(z)$ is transversal (Defn. 2.14) and the uniform Lopatinski condition holds at $\underline{q}$. In fact, for $\hat{\gamma} > 0$*

$$D_s(\underline{q}, \hat{\zeta}, \rho) = \rho\alpha(\underline{q}) D_{Lop}(\underline{q}, \hat{\zeta}) + O_{\hat{\gamma}}(\rho^2), \tag{1.52}$$

where $O_{\hat{\gamma}}(\rho^2) \leq C_{\hat{\gamma}}\rho^2$ and $\alpha(\underline{q})$ is a constant whose nonvanishing is equivalent to a-transversality of $\underline{w}$ (Defn. 2.14).

This theorem does not assume the prior existence of a shock manifold $\mathcal{C}_{\mathcal{B}}$. However, the transversality conclusion allows us to construct a unique $\mathcal{C}_{\mathcal{B}}$ manifold near $\underline{q}$ as in Proposition 2.8. The proof of Theorem 1.22 is quite different from the argument in the conservative case [ZS] and relies heavily on the functions $\phi_{\pm}(z, p_{\pm}, s, h, a_{\pm})$ (2.26). One advantage of the new argument is that it generalizes almost verbatim to the case of real viscosities.

Corollary 1.23. *Even in situations, like that of slow MHD shocks, where viscous continuity fails, the uniform Lopatinski condition is a necessary condition for the standard uniform Evans condition to hold.*

The remainder of section 5 is mainly concerned with obtaining block decompositions of the modified and standard Evans functions for ρ small, Proposition 5.11 and Theorem 5.14. Roughly speaking, the blocks in the decomposition of the determinants correspond to the H and P blocks in the conjugated form (3.20) of the linearized parabolic problem. The block decompositions are useful for relating the standard and modified Evans to each other and to the Lopatinski and modified Lopatinski determinants. Theorem 5.14 implies, for example:

Theorem 1.24. *Assume the profile $W(z, q)$ is strongly transversal (Defn. 2.14). Then for some $\rho_0 > 0$*

$$D_s(q, \hat{\zeta}, \rho) = \rho\alpha(q, \zeta) D_m(q, \hat{\zeta}, \rho) + O(\rho^2), \tag{1.53}$$

where the error is uniform for $\hat{\zeta} \in \overline{S}^d_+$, $0 < \rho \leq \rho_0$ and $\alpha(q, \zeta)$ is C^∞ and bounded away from zero for ρ small.

In combination with Theorem 1.22 and Proposition 1.18 this implies Theorem 1.20. A result like Theorem 1.24 was proved in [GMWZ3] in the conservative, constant multiplicity case by giving a low frequency expansion of both D_s and D_m in terms of D_{Lop}. If viscous continuity fails, this much simpler type of argument does not work.

In Theorem 5.15 we summarize a number of the connections that hold between the different Evans and Lopatinski determinants.

Section 6 is devoted to the construction of high order approximate solutions (which converge as viscosity tends to zero to a given curved inviscid $\mathcal{C}_\mathcal{B}$ shock) to the nonlinear small viscosity transmission problem (6.1). The characterizations of transversality and of $T_q\mathcal{C}_\mathcal{B}$ in terms of properties of the fully linearized profile problem given in Propositions 3.6 and 3.12 lead to a simpler and shorter construction of higher order profiles than was given in [GW, GMWZ4]. In section 7 we complete the construction of curved inviscid $\mathcal{C}$-shocks (Theorem 1.6), and of families of curved viscous $\mathcal{C}_\mathcal{B}$-shocks converging to a given inviscid $\mathcal{C}_\mathcal{B}$-shock as viscosity approaches zero (Theorem 1.16). In particular, Theorem 1.16 shows that the approximate solutions constructed in section 6 are close to true exact solutions of the parabolic transmission problem (6.1).

Finally, in Appendix A we explain the changes needed to handle the partially parabolic case of real viscosities, and in Appendix B we prove the local uniqueness of $\mathcal{C}_\mathcal{B}$ manifolds.

2 Generalized Rankine-Hugoniot condition

We let $\mathcal{B}(u) = \triangle u$ now and set $\underline{q} = (\underline{p}_+, \underline{p}_-, 0, 0)$, where $\underline{p}_\pm \in \mathcal{U}_\pm$ with $A_d(\underline{p}_\pm)$ invertible. Suppose $\underline{w}(z)$ is a viscous profile associated to $\underline{q}$. Our main task in this section is to determine when and how we can construct a shock manifold $\mathcal{C}_\mathcal{B}$ near $\underline{q}$.

2.1 The connection problem as a transmission problem

Consider again the profile transmission problem (1.30):

$$\begin{aligned}&(a)\ w'' = (1+|h|^2)^{-1}\mathcal{A}_d(w,s,h)w' \text{ on } \pm z \geq 0\\&(b)\ [w]=0,\ [w_z]=0 \text{ on } z=0.\end{aligned}\tag{2.1}$$

As in (2.1) we'll often suppress $\pm$ subscripts on unknowns. Set

$$\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}w\\w'\end{pmatrix};\quad b(h) := (1+|h|^2)^{-1}\tag{2.2}$$

and rewrite (2.1) as a $2N \times 2N$ first-order system:

(2.3)

$$(a)\ \begin{pmatrix}u\\v\end{pmatrix}' = \begin{pmatrix}0 & 1\\0 & G_d(u,s,h)\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}v\\G_d(u,s,h)v\end{pmatrix},\ \text{where } G_d(u,s,h) := b(h)\mathcal{A}_d(u,s,h)$$

$$(b)\ \Gamma\begin{pmatrix}u_+\\u_-\\v_+\\v_-\end{pmatrix} = \begin{pmatrix}[u]\\[v]\end{pmatrix} = 0 \text{ on } z=0,\ \Gamma : \mathbb{R}^{4N} \to \mathbb{R}^{2N}.$$

We seek conditions on $q = (p_+, p_-, s, h)$ that will allow us to find solutions to (2.3) such that

$$\lim_{z\to\pm\infty} u(z) = p_\pm. \tag{2.4}$$

Notation 2.1. *Given a matrix $G : \mathbb{C}^p \to \mathbb{C}^p$ with no eigenvalues on the imaginary axis, let $\mathbb{E}_\pm(G)$ denote the invariant subspace of $\mathbb{C}^p$ generated by the generalized eigenvectors of G associated to eigenvalues μ such that $\pm\Re\mu > 0$. We denote by $\Pi_\pm$ the corresponding spectral projectors associated to the decomposition*

$$\mathbb{C}^p = \mathbb{E}_+(G) \oplus \mathbb{E}_-(G). \tag{2.5}$$

When G is real (that is, when each entry of G is real), the $\Pi_\pm$ are real and so we can also regard the $\mathbb{E}_\pm(G)$ as real vector spaces. This gives a decomposition

$$\mathbb{R}^p = \mathbb{E}_+(G) \oplus \mathbb{E}_-(G). \tag{2.6}$$

Note that the spaces appearing in (2.5) *are just the complexifications of the corresponding spaces in* (2.6).

In the first step we ignore the transmission conditions and construct solutions $u_\pm$ to the interior problems in $\pm z \geq 0$ such that (2.4) holds. We do this by regarding (2.3)(a) in $\pm z \geq 0$ as a perturbation, quadratic in $(u_\pm - p_\pm, v_\pm)$, of

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} v \\ G_d(p_\pm, s, h)v \end{pmatrix}. \tag{2.7}$$

Let $\Pi_-(p_+, s, h)$ be the projection on $\mathbb{E}_-(G_d(p_+, s, h))$ with respect to the decomposition

$$\mathbb{R}^N = \mathbb{E}_-(G_d(p_+, s, h)) \oplus \mathbb{E}_+(G_d(p_+, s, h)), \tag{2.8}$$

and define $\Pi_+(p_-, s, h)$ similarly. For $p_\pm$ in relatively compact open neighborhoods $\omega_\pm \subset \mathcal{U}_\pm$ of $\underline{p}_\pm$ and for $|s, h| \leq \epsilon_\omega$, where ϵ_ω is a small enough positive constant, we fix isomorphisms linear in $a_\pm \in \mathbb{E}_\mp(G_d(\underline{p}_\pm, 0, 0))$ and C^∞ in $(p_\pm, s, h)$:

$$\alpha_\pm(p_\pm, s, h; a_\pm) : \mathbb{E}_\mp(G_d(\underline{p}_\pm, 0, 0)) \to \mathbb{E}_\mp(G_d(p_\pm, s, h)). \tag{2.9}$$

Proposition 2.2. *(a) Let $\omega_\pm$, ϵ_ω, and $\alpha_\pm$ be as just defined. There are positive constants R and r such that for all $p_\pm \in \omega_\pm$, $|s, h| \leq \epsilon_\omega$, and $a_\pm \in \mathbb{E}_\pm(G_d(\underline{p}_\pm, 0, 0))$ with $|a_\pm| \leq r$, the equation* (2.1)*(a) has a unique solution $u_\pm = \Phi_\pm(z, p_\pm, s, h, a_\pm)$ satisfying*

$$\begin{aligned} &(i)\ \lim_{z\to\pm\infty} u_\pm = p_\pm \\ &(ii)\ \Pi_\mp(p_\pm, s, h)\partial_z u_\pm(0) = \alpha_\pm(p_\pm, s, h; a_\pm) \\ &(iii)\ \|\partial_z u_\pm\|_{L^1(\pm z\geq 0)} \leq R \text{ and } \|\partial_z u_\pm\|_{L^\infty(\pm z\geq 0)} \leq R. \end{aligned} \tag{2.10}$$

The function $\Phi_\pm$ is C^∞ on $\{\pm z \geq 0\} \times \Omega_\pm$, where

$$\Omega_\pm = \{(p_\pm, s, h, a_\pm) : p_\pm \in \omega_\pm, |s, h| \leq \epsilon_\omega, \text{and } a_\pm \in \mathbb{E}_\pm(G_d(\underline{p}_\pm, 0, 0)) \text{ with } |a_\pm| \leq r\}. \tag{2.11}$$

It satisfies

$$\Phi_\pm(z,p_\pm,s,h,a_\pm)=p_\pm+e^{zG_d(p_\pm,s,h)}G_d^{-1}(p_\pm,s,h)\alpha(p_\pm,s,h;a_\pm)+O(|a_\pm|^2) \tag{2.12}$$

uniformly with respect to $(z,p_\pm,s,h)$*. Moreover, there exist positive constants* δ *and* C_β *such that for all* $\pm z\geq 0$ *and* $(p_\pm,s,h,a_\pm)\in\Omega_\pm$*:*

$$|\partial^\beta_{z,p_\pm,s,h,a_\pm}(\Phi_\pm(z,p_\pm,s,h,a_\pm)-p_\pm)|\leq C_\beta e^{-\delta|z|},\ \ |\beta|\leq 2. \tag{2.13}$$

We will also denote by $\Phi_\pm(z,p_\pm,s,h,a_\pm)$ *the maximal extension of* $\Phi_\pm$ *to* $\pm z\leq 0$ *as a solution of* (2.1)*(a).*

(b) Suppose u_+ *is a solution of* (2.1)*(a)*$_+$, (2.10)*(i)*$_+$ *on* $[z_1,+\infty)$ *for some* $p_+\in\omega_+$ *and* $|s,h|\leq\epsilon_\omega$*. Then for all* $z_0\geq z_1$ *large enough, we have*

$$u_+(z)=\Phi_+(z-z_0,p_+,s,h,a_+)\ \text{with}\ \alpha(p_+,s,h;a_+)=\Pi_-(p,s,h)\partial_z u_+(z_0). \tag{2.14}$$

We may (and shall) always take $z_0\geq 0$*. The analogous statement holds for solutions* u_- *on* $(-\infty,z_2]$*.*

Proof. **1.** The proof of part (a) (resp., (b)) is essentially the same as that of Prop. 5.3.5 (resp., Proposition 5.3.6) in [Me1], to which we refer for more detail. We provide a sketch for the + case that describes the slight differences.

2. Let $\Pi^+(p_+,s,h)$ denote the projection on the second summand in (2.8). Note there is a $\theta>0$ such that

$$\begin{aligned}&|e^{(z-y)G_d(p_+,s,h)}\Pi_-(p_+,s,h)|\leq Ce^{-\theta(z-y)},\ \text{for}\ z\geq y\\&|e^{(z-y)G_d(p_+,s,h)}\Pi^+(p_+,s,h)|\leq Ce^{-\theta|z-y|},\ \text{for}\ z\leq y.\end{aligned} \tag{2.15}$$

Define integral operators

$$\begin{aligned}&I(v)(z)=-\int_z^{+\infty}v(y)dy\\&\mathcal{I}_0(F)(z)=\\&\quad\int_0^z e^{(z-y)G_d(p_+,s,h)}\Pi_-(p_+,s,h)F(y)dy-\int_z^{+\infty}e^{(z-y)G_d(p_+,s,h)}\Pi^+(p_+,s,h)F(y)dy.\end{aligned} \tag{2.16}$$

With $F(u,v):=(G_d(u,s,h)-G_d(p_+,s,h))\,v$ we contruct $(u_+,\partial_z u_+)$ as a fixed point of the map

$$\mathcal{T}_+(u_+,v_+):=\Big(p_++I(v_+),e^{zG_d(p_+,s,h)}\alpha_+(p_+,s,h;a_+)+\mathcal{I}_0(F(u,v))\Big) \tag{2.17}$$

on

$$B_R=\{(u_+,v_+):\|u_+-p_+\|_{L^\infty(z\geq 0)}+\|v_+\|_{L^1\cap L^\infty}\leq R\}. \tag{2.18}$$

For R and r small enough $\mathcal{T}_+$ is a contraction on B_R.

Uniqueness follows from the fixed point theorem. By construction the solution satisfies $\|v_+\|_{L^1} = O(|a_+|)$; thus, $\|u_+ - p_+\|_{L^\infty} = O(|a_+|)$. Since $F(u_+, v_+) = O(|u_+ - p_+||v_+|)$ it follows that

$$\|v_+ - e^{zG_d(p_+,s,h)}\alpha_+(p_+, s, h; a_+)\|_{L^1\cap L^\infty} = O(|a_+|^2), \tag{2.19}$$

which in turn yields

$$\|u_+ - p_+ - e^{zG_d(p_+,s,h)}G_d^{-1}(p_+, s, h)\alpha_+(p_+, s, h; a_+)\|_{L^\infty} = O(|a_+|^2). \tag{2.20}$$

Exponential decay as in (2.13) follows by a similar application of the fixed point theorem in spaces $e^{-\delta z}L^\infty$.

3. To prove part (b) we note that if u_+ is a solution of (2.1)(a)$_+$, (2.10)(a)$_+$ on $[z_1, +\infty)$, then for all $z_0 \geq z_1$ large enough there holds

$$\begin{aligned} &\|\partial_z u_+\|_{L^1(z\geq z_0)} + \|\partial_z u_+\|_{L^\infty(z\geq z_0)} \leq R, \\ &\Pi_-(p_+, s, h)\partial_z u_+(z_0) = \alpha_+(p_+, s, h; a_+) \text{ for some } |a_+| \leq r, \end{aligned} \tag{2.21}$$

where R and r are the constants determined in part (a). By translation invariance $u_+(z+z_0)$ satisfies (2.1)(a)$_+$, (2.10)(a)$_+$ on $z \geq 0$, and using part (a) we obtain

$$u_+(z + z_0) = \Phi_+(z, p_+, s, h, a_+) \text{ for } z \geq 0. \tag{2.22}$$

□

Definition 2.3. *Let $R_\pm$ be the number of eigenvalues μ of $\mathcal{A}_d(\underline{p}_+, 0, 0)$ with $\pm\mu > 0$ (R refers to the right endstate $\underline{p}_+$). Let $L_\pm$ be the number of eigenvalues μ of $\mathcal{A}_d(\underline{p}_-, 0, 0)$ with $\pm\mu > 0$.*

Definition 2.4. *We shall only consider shocks satisfying $R_- + L_+ = N + 1 - k$, where N is the dimension of the system* (1.29) *and $k \geq 0$ is defined to be the* undercompressive index. *Thus,*

$$\dim\left(\mathbb{E}_-(G_d(p_+, s, h)) \times \mathbb{E}_+(G_d(p_-, s, h)\right) := \mathcal{N}_- = R_- + L_+ = N + 1 - k. \tag{2.23}$$

When $k = 0$ the shock is called a Lax shock; when $k > 0$ it is called undercompressive.

Observe that if we take the given profile $\underline{w}(z)$ and define

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \underline{w}(z) \\ \underline{w}_z(z) \end{pmatrix}, \tag{2.24}$$

then $\begin{pmatrix} u \\ v \end{pmatrix}$ satisfies (2.3)(a)and (2.4) with $(s, h) = (0, 0)$ and $p_\pm = \underline{p}_\pm$. Thus, by part (b) of Prop. 2.2 there exist $\underline{z} = -z_0 \leq 0$ and $\underline{a}_\pm$ such that

$$\underline{w}(z) = \begin{cases} \Phi_+(z + \underline{z}, \underline{p}_+, 0, 0, \underline{a}_+), & z \geq 0 \\ \Phi_-(z - \underline{z}, \underline{p}_-, 0, 0, \underline{a}_-), & z \leq 0 \end{cases}. \tag{2.25}$$

Notation 2.5. *1. For $\Phi_\pm$ as constructed in Prop. 2.2 it is convenient to define*

$$\begin{aligned} &\phi_\pm(z,p_\pm,s,h,a_\pm) := \Phi_\pm(z \pm \underline{z}, p_\pm, s, h, a_\pm), \\ &\phi(z,p,s,h,a) := \begin{cases} \phi_+(z,p_+,s,h,a_+), \ z \geq 0 \\ \phi_-(z,p_-,s,h,a_-), \ z \leq 0 \end{cases} . \end{aligned} \tag{2.26}$$

2. Let

$$\mathbb{E}_-(G_d(p,s,h)) := \mathbb{E}_-(G_d(p_+,s,h)) \times \mathbb{E}_+(G_d(p_-,s,h). \tag{2.27}$$

The following corollary of Proposition 2.2 will be used in the proofs of Propositions 2.12 and 3.10.

Corollary 2.6. *For $(\dot{p}_\pm, \dot{s}, \dot{h}, \dot{a}_\pm) \in \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{E}_\mp(G_d(\underline{p}_\pm, 0, 0))$ we have*

$$\lim_{z \to \pm\infty} \nabla_{p_\pm,s,h,a_\pm} \phi_\pm(z,p_\pm,s,h,a_\pm)(\dot{p}_\pm, \dot{s}, \dot{h}, \dot{a}_\pm) = \dot{p}_\pm. \tag{2.28}$$

Decreasing r in the definition of $\Omega_\pm$ (2.11) if necessary, we have for $(p_\pm,s,h,a_\pm) \in \Omega_\pm$ that the map

$$(\dot{p}_\pm, \dot{s}, \dot{h}, \dot{a}_\pm) \to \begin{pmatrix} \nabla_{p_\pm,s,h,a_\pm} \phi_\pm(z,p_\pm,s,h,a_\pm)(\dot{p}_\pm, \dot{s}, \dot{h}, \dot{a}_\pm) \\ \dot{s} \\ \dot{h} \end{pmatrix} \tag{2.29}$$

is injective.

Proof. Consider the + case. The limit (2.28) follows directly from (2.13). Thus, if the image of $(\dot{p}_+, \dot{s}, \dot{h}, \dot{a}_+)$ in (2.29) is 0 we find $(\dot{p}_+, \dot{s}, \dot{h}) = 0$. Since (2.12) implies

$$\nabla_{a_+} \phi_+(-\underline{z}, p_+, s, h, a_+)\dot{a}_+ = G_d^{-1}(p_+,s,h)\alpha(p_+,s,h;\dot{a}) + O(|a_+|)\dot{a}_+, \tag{2.30}$$

we conclude for r small enough that if the right side of (2.30) is 0, then $\dot{a}_+ = 0$. □

2.2 Extra transmission condition

For $p := (p_+, p_-)$ near $\underline{p} = (\underline{p}_+, \underline{p}_-)$, (s,h) near $(0,0)$, and $a := (a_+, a_-)$ near $\underline{a} = (\underline{a}_+, \underline{a}_-)$, define

$$\Psi(p,s,h,a) = \Gamma \begin{pmatrix} \phi_\pm(0,p_\pm,s,h,\pm a) \\ \phi_{\pm,z}(0,p_\pm,s,h,a_\pm) \end{pmatrix} = \begin{pmatrix} \phi_+(0,\cdot) - \phi_-(0,\cdot) \\ \phi_{+,z}(0,\cdot) - \phi_{-,z}(0,\cdot) \end{pmatrix} (p,s,h,a) \in \mathbb{R}^{2N}, \tag{2.31}$$

and note that since $\underline{w}(z)$ is a connection, we have

$$\Psi(\underline{p}, 0, 0, \underline{a}) = 0. \tag{2.32}$$

We shall add a nonhomogeneous boundary condition to the one in (2.3) in order to remove the translational indeterminacy present in the shock case. Anticipating the later low frequency analysis, we consider the augmented boundary condition

$$\tilde{\Gamma}\begin{pmatrix} u_\pm \\ v_\pm \\ s \\ h \end{pmatrix} = \begin{pmatrix} u_+ - u_- \\ v_+ - v_- \\ s + u_+ \cdot \underline{w}_z - \underline{w} \cdot \underline{w}_z \end{pmatrix} \text{ on } z = 0. \tag{2.33}$$

Parallel to (2.31) define

(2.34)

$$\tilde{\Psi}(p, s, h, a) =$$

$$\tilde{\Gamma}\begin{pmatrix} \phi_\pm(0, p_\pm, s, h, a_\pm) \\ \phi_{\pm,z}(0, p_\pm, s, h, a_\pm) \\ s \\ h \end{pmatrix} = \begin{pmatrix} \phi_+(0, \cdot) - \phi_-(0, \cdot) \\ \phi_{+,z}(0, \cdot) - \phi_{-,z}(0, \cdot) \\ s + \phi_+(0, \cdot) \cdot \underline{w}_z(0) - \underline{w}(0) \cdot \underline{w}_z(0) \end{pmatrix} (p, s, h, a) \in \mathbb{R}^{2N+1}.$$

Observe that

$$\tilde{\Psi}(\underline{p}, 0, 0, \underline{a}) = 0. \tag{2.35}$$

Remark 2.7. The function $\tilde{\Psi}$ plays a role here similar to that of the extended Melnikov separation function defined in [ZS] for the undercompressive conservative case. Namely, it provides a convenient full rank (implicit) representation of the defining function χ.

2.3 Rank conditions and the manifold $\mathcal{C}_\mathcal{B}$

With $q = (p_+, p_-, s, h)$ the next Proposition gives conditions under which there exist connections $W(z, q)$ near $\underline{w}(z) = W(z, \underline{q})$ satisfying (2.1) and

$$\lim_{z \to \pm\infty} W(z, q) = p_\pm. \tag{2.36}$$

Proposition 2.8 (Connections near a given one).
(1) Let $\underline{w}(z)$ be a connection corresponding to $(p, s, h) = (\underline{p}, 0, 0)$ and suppose that

$$\begin{aligned} &(a)\text{rank } \nabla_a \tilde{\Psi}(\underline{p}, 0, 0, \underline{a}) = \mathcal{N}_- = N + 1 - k, \\ &(b)\text{rank } \nabla_{a,p} \tilde{\Psi}(\underline{p}, 0, 0, \underline{a}) = 2N + 1. \end{aligned} \tag{2.37}$$

Then in a neighborhood $\mathcal{O} \subset \mathbb{R}^{2N} \times \mathbb{R}^d$ of $(\underline{p}, 0, 0)$ there is a smooth manifold $\mathcal{C}_\mathcal{B}$ of dimension $d + N - k$ and there are smooth mappings $W_\pm(z, p, s, h)$ on $\{\pm z \geq 0\} \times \mathcal{C}_\mathcal{B}$ such that $(W_\pm, W_{\pm,z})$ satisfies the profile equation (2.3), the endstate condition (2.4), and the boundary condition

$$\tilde{\Gamma}\begin{pmatrix} W_\pm \\ W_{\pm,z} \\ s \\ h \end{pmatrix} = 0 \textit{ on } z = 0. \tag{2.38}$$

With $q := (p, s, h) = (p_+, p_-, s, h) \in \mathcal{C}_{\mathcal{B}}$*, if we set*

$$W(z,q) = \begin{cases} W_+(z,q), \ z \geq 0 \\ W_-(z,q), \ z \leq 0 \end{cases} , \tag{2.39}$$

then $W(z,q)$ *satisfies* (2.1)*,* (2.36)*.*

The manifold $\mathcal{C}_{\mathcal{B}}$ *can be defined by a* generalized Rankine-Hugoniot condition

$$\chi(p,s,h) = 0 \tag{2.40}$$

for a smooth function $\chi : \mathcal{O} \to \mathbb{R}^{N+k}$ *such that* rank $\chi_p(\underline{p}, 0, 0) = N + k$*.*

(2) The same conclusions hold if the second rank condition (2.37)*(b) is replaced by*

$$\text{rank } \nabla_{a,p,s} \tilde{\Psi}(\underline{p}, 0, 0, \underline{a}) = 2N + 1. \tag{2.41}$$

The only difference is that now we have rank $\chi_{p,s}(\underline{p}, 0, 0) = N + k$*.*

Proof. There is a reordering $(p_\alpha, p_\beta) \in \mathbb{R}^{N+k} \times \mathbb{R}^{N-k}$ of the original (p_+, p_-) coordinates such that $\nabla_{a,p_\alpha} \tilde{\Psi}(\underline{p}, 0, 0, \underline{a})$ is an isomorphism. Applying the implicit function theorem to $\tilde{\Psi}(p, s, h, a) = 0$ yields functions $p_\alpha(p_\beta, s, h)$, $a_\pm(p_\beta, s, h)$ and a manifold $\mathcal{C}_{\mathcal{B}}$ parametrized in the new coordinates by $(p_\alpha(p_\beta, s, h), p_\beta, s, h)$. Reordering again yields smooth functions $p_\pm(p_\beta, s, h)$ such that $\mathcal{C}_{\mathcal{B}}$ is given in the original coordinates by

$$\mathcal{C}_{\mathcal{B}} = \{(p_+(p_\beta, s, h), p_-(p_\beta, s, h), s, h) : (p_\beta, s, h) \text{ near } (\underline{p}_\beta, 0, 0)\}. \tag{2.42}$$

We define

$$W_\pm(z, p_+(p_\beta, s, h), p_-(p_\beta, s, h), s, h) := \phi_\pm(z, p_\pm(p_\beta, s, h), s, h, a_\pm(p_\beta, s, h)). \tag{2.43}$$

A generalized RH condition is given by the defining equation for $\mathcal{C}_{\mathcal{B}}$ in (p, s, h) space:

$$\chi(p, s, h) := p_\alpha - p_\alpha(p_\beta, s, h) = 0. \tag{2.44}$$

The proof of part (2) is essentially the same.

□

Remark 2.9. ***1.*** Using the condition (2.44) we formulate later a nonconservative analogue of the curved inviscid shock problem and construct curved multidimensional nonconservative shocks.

2. When $k = 0$, if in place of (2.37)(b) we assume

$$\text{rank } \nabla_{a,p_+} \tilde{\Psi}(\underline{p}, 0, 0, \underline{a}) = 2N + 1, \tag{2.45}$$

then we can prescribe (p_-, s, h) and solve for p_+ as in the conservative case.

We proceed to restate the hypotheses (2.37) equivalently in terms of rank conditions on Ψ. We expect to be able to do this, since $\tilde{\Psi}$ is just designed to remove the translational indeterminacy left by Ψ. A first step is to relate such hypotheses to properties of solutions of the linearized problem.

Consider the linearization of (2.3)(a) with respect to (u, v) at $(\underline{w}, \underline{w}')$, $(s, h) = (0, 0)$. Write this homogeneous $2N \times 2N$ linearized system as

$$\mathcal{L}\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} := \begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix}' - \mathcal{G}(z)\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = 0 \text{ on } \pm z \geq 0, \tag{2.46}$$

where

$$\mathcal{G}(z) = \begin{pmatrix}0 & I\\ O(e^{-\delta|z|}) & A_d(\underline{w})\end{pmatrix}. \tag{2.47}$$

The following conjugation lemma shows that solutions $U := (\dot{u}, \dot{v})$ of (2.46) can be conjugated to solutions V of the limiting constant-coefficient systems

$$\partial_z V = \mathcal{G}_\ell(\underline{p}_\pm)V, \text{ where } \mathcal{G}_\ell(\underline{p}_\pm) := \lim_{z\to\pm\infty}\mathcal{G}(z) = \begin{pmatrix}0 & I\\ 0 & A_d(\underline{p}_\pm)\end{pmatrix}. \tag{2.48}$$

Lemma 2.10 ([MZ3], Lemma 2.6). *For $\delta > 0$ as in* (1.20), *there exist $2N \times 2N$ matrices $Y_\pm(z)$ on $\pm z \geq 0$ and positive constants C, $\delta' < \delta$ such that*

(i) $Y_\pm$ and $Y_\pm^{-1}$ are C^∞ and bounded with bounded derivatives,

(ii) $|Y_\pm(z) - I| + |\partial_z Y_\pm(z)| \leq Ce^{-\delta'|z|}$,

(iii) $Y_\pm$ satisfy

$$\partial_z Y_\pm = \mathcal{G}(z)Y_\pm - Y_\pm\mathcal{G}_\ell(\underline{p}_\pm) \text{ on } \pm z \geq 0. \tag{2.49}$$

Observe that U satisfies (2.46) on $\pm z \geq 0$ if and only if V defined by $U = YV$ satisfies (2.48) on $\pm z \geq 0$.

Let $\mathcal{S}_\pm$ denote the space of bounded solutions of (2.46) in $\pm z \geq 0$. We'll refer to

$$\mathcal{S} = \mathcal{S}_+ \times \mathcal{S}_- \tag{2.50}$$

as the space of bounded solutions of (2.46). Similarly, let $\mathcal{S}_\pm^0$ denote the space of solutions of (2.46) that decay to 0 as $z \to \pm\infty$ and set $\mathcal{S}_0 = \mathcal{S}_+^0 \times \mathcal{S}_-^0$.

Proposition 2.11. *(a) Let R_-, L_+ be as in Definition 2.3. The dimensions of $\mathcal{S}_\pm$ are $N + R_-$ and $N + L_+$ respectively. Thus,* $\dim\mathcal{S} = 2N + (N + 1 - k)$.

(b) The dimensions of $\mathcal{S}_\pm^0$ are R_- and L_+ respectively. Thus, $\dim\mathcal{S}_0 = N + 1 - k$.

Proof. Using the conjugators $Y_\pm$ we can obtain the Proposition immediately by proving the analogous statements for solutions $V = (v_1, v_2)$ of the limiting problem (2.48). Solutions in $z \geq 0$ are given by

$$V_+ = \begin{pmatrix}v_{1+}\\ v_{2+}\end{pmatrix} = \begin{pmatrix}v_{1+}(0) + (e^{zA_d(\underline{p}_+)} - I)A_d^{-1}(\underline{p}_+)v_{2+}(0)\\ e^{zA_d(\underline{p}_+)}v_{2+}(0)\end{pmatrix}. \tag{2.51}$$

Clearly, V_+ is bounded in $z \geq 0$ if and only if $v_{2+}(0) \in \mathbb{E}_-(A_d(\underline{p}_+))$ (with $v_{1+}(0) \in \mathbb{R}^N$ arbitrary), and V_+ decays to 0 as $z \to +\infty$ if and only if

$$v_{2+}(0) \in \mathbb{E}_-(A_d(\underline{p}_+)) \text{ and } v_{1+}(0) - A_d^{-1}(\underline{p}_+)v_{2+}(0) = 0. \tag{2.52}$$

The "$-$" case is similar.

□

In the next Proposition we consider rank conditions on Ψ as in (2.31). With Γ as in (2.3) observe that

$$(\underline{w}', \underline{w}'') \tag{2.53}$$

satisfies the problem

$$\mathcal{L}\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = 0 \text{ on } \pm z \geq 0,\ \Gamma\begin{pmatrix}\dot u_\pm\\ \dot v_\pm\end{pmatrix} = 0 \text{ on } z=0,\ \lim_{z\to\pm\infty}\dot u = 0. \tag{2.54}$$

Proposition 2.12. *(a)The condition*

$$\mathrm{rank}\nabla_a\Psi(\underline{p},0,0,\underline{a}) = \mathcal{N}_- - 1 = N-k \tag{2.55}$$

holds if and only if the problem (2.54) *has a one dimensional kernel spanned by* (2.53).

(b) The condition

$$\mathrm{rank}\nabla_{a,p}\Psi(\underline{p},0,0,\underline{a}) = 2N \tag{2.56}$$

holds if and only if for all $g\in\mathbb{R}^{2N}$ the problem

$$\mathcal{L}\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = 0,\ \Gamma\begin{pmatrix}\dot u_\pm\\ \dot v_\pm\end{pmatrix} = g \tag{2.57}$$

has a bounded solution.

Proof. Proposition 2.11 shows that the space of $\mathcal{S}$ of bounded solutions of $\mathcal{L}\dot U = 0$ has dimension $2N+(N+1-k)$, while $\mathcal{S}_0$ has dimension $N+1-k$. Thus, Corollary 2.6 implies that the map

$$(\dot p,\dot a)\to\begin{pmatrix}\nabla_a\phi(z,\underline{p},0,0,\underline{a})\dot a + \nabla_p\phi(z,\underline{p},0,0,\underline{a})\dot p\\ \nabla_a\phi_z(z,\underline{p},0,0,\underline{a})\dot a + \nabla_p\phi_z(z,\underline{p},0,0,\underline{a})\dot p\end{pmatrix} \tag{2.58}$$

is an isomorphism of $\mathbb{R}^{2N}\times\mathbb{E}_-(G_d(\underline{p},0,0))$ onto $\mathcal{S}$, and that

$$\dot a\to\begin{pmatrix}\nabla_a\phi(z,\underline{p},0,0,\underline{a})\dot a\\ \nabla_a\phi_z(z,\underline{p},0,0,\underline{a})\dot a\end{pmatrix} \tag{2.59}$$

is an isomorphism onto $\mathcal{S}_0$. It follows that the dimension of the space of solutions of (2.54) is the dimension of

$$\{\dot a\in\mathbb{E}_-(G_d(\underline{p},0,0)) : \nabla_a\Psi(\underline{p},0,0,\underline{a})\dot a = 0\}. \tag{2.60}$$

This dimension is one if and only if $\mathrm{rank}\nabla_a\Psi(\underline{p},0,0,\underline{a}) = N-k$.

Similarly, the map $\dot U\to\Gamma\dot U_\pm(0)$ from $\mathcal{S}$ to $\mathbb{R}^{2N}$ is onto if and only if

$$\nabla_{a,p}\Psi(\underline{p},0,0,\underline{a}) \tag{2.61}$$

has rank $2N$.

□

Proposition 2.13.

$$(a)\text{rank } \nabla_a \tilde{\Psi}(\underline{p},0,0,\underline{a}) = N+1-k \Leftrightarrow \text{rank } \nabla_a \Psi(\underline{p},0,0,\underline{a}) = N-k$$
$$(b)\text{rank } \nabla_{a,p} \tilde{\Psi}(\underline{p},0,0,\underline{a}) = 2N+1 \Leftrightarrow \text{rank } \nabla_{a,p} \Psi(\underline{p},0,0,\underline{a}) = 2N$$
$$(c)\text{rank } \nabla_{a,p,s} \tilde{\Psi}(\underline{p},0,0,\underline{a}) = 2N+1 \Leftrightarrow \text{rank } \nabla_{a,p,s} \Psi(\underline{p},0,0,\underline{a}) = 2N.$$

Proof. **1**. Using the isomorphism (2.59) we find an element $\underline{\dot{a}} \in \mathbb{E}_-(G_d(\underline{p},0,0))$ such that

$$\nabla_a \phi(z,\underline{p},0,0,\underline{a})\underline{\dot{a}} = \underline{w}_z(z). \tag{2.62}$$

We always have

$$\underline{\dot{a}} \in \ker \nabla_a \Psi(\underline{p},0,0,\underline{a}). \tag{2.63}$$

2. Statement (a) is equivalent (suppressing evaluation at $(\underline{p},0,0,\underline{a})$) to

$$\dim \ker \nabla_a \Psi = 1 \Leftrightarrow \dim \ker \nabla_a \tilde{\Psi} = 0. \tag{2.64}$$

Also observe from (2.34) that

$$\dot{a} \in \ker \nabla_a \tilde{\Psi} \Leftrightarrow \dot{a} \in \ker \nabla_a \Psi \text{ and } (\nabla_a \phi_+ \, \dot{a}) \cdot \underline{w}_z(0) = 0. \tag{2.65}$$

Now, if $\dim \ker \nabla_a \Psi = 1$, then $\ker \nabla_a \Psi$ is spanned by $\underline{\dot{a}}$. But by (2.62)

$$\nabla_a \phi_+(0,\underline{p},0,0,\underline{a})\underline{\dot{a}} = \underline{w}_z(0), \tag{2.66}$$

so $\dim \ker \nabla_a \tilde{\Psi} = 0$.

Suppose $\dim \ker \nabla_a \tilde{\Psi} = 0$. Then by (2.65) the linear functional defined on $\ker \nabla_a \Psi$ by

$$T\dot{a} = (\nabla_a \phi_+ \, \dot{a}) \cdot \underline{w}_z(0) \tag{2.67}$$

satisfies $\dim \ker T = 0$. The rank of T is one, so $\dim \ker \nabla_a \Psi = 1$. This proves (a).

3. Statement (b) is equivalent to

$$\dim \ker \nabla_{a,p} \Psi = N+1-k \Leftrightarrow \dim \ker \nabla_{a,p} \tilde{\Psi} = N-k. \tag{2.68}$$

Observe that $\ker \nabla_{a,p} \tilde{\Psi} \subset \ker \nabla_{a,p} \Psi$ and

$$(\dot{a},\dot{p}) \in \ker \nabla_{a,p} \tilde{\Psi} \Leftrightarrow (\dot{a},\dot{p}) \in \ker \nabla_{a,p} \Psi \text{ and } (\nabla_a \phi_+ \, \dot{a} + \nabla_p \phi_+ \, \dot{p}) \cdot \underline{w}_z(0) = 0. \tag{2.69}$$

We have

$$(\underline{\dot{a}},0) \in \ker \nabla_{a,p} \Psi \setminus \ker \nabla_{a,p} \tilde{\Psi}. \tag{2.70}$$

Suppose $\dim \ker \nabla_{a,p} \Psi = N+1-k$ and let a basis for that kernel be $\{(\underline{\dot{a}},0), e_1, \dots, e_{N-k}\}$. Using (2.66) and (2.69) we obtain a basis for $\ker \nabla_{a,p} \tilde{\Psi}$ of the form $\{e_i + s_i(\underline{\dot{a}},0) : i = 1,\dots,N-k\}$ for appropriate $s_i \in \mathbb{R}$.

Suppose $\dim \ker \nabla_{a,p} \tilde{\Psi} = N-k$. By (2.69) the functional defined on $\ker \nabla_{a,p} \Psi$ by

$$S(\dot{a},\dot{p}) = (\nabla_a \phi_+ \, \dot{a} + \nabla_p \phi_+ \, \dot{p}) \cdot \underline{w}_z(0) \tag{2.71}$$

has rank one and kernel of dimension $N-k$. So $\ker \nabla_{a,p} \Psi$ has dimension $N+1-k$.

The proof of (c) is similar to that of (b). □

Definition 2.14. *1. When both of the rank conditions*

$$\begin{aligned}\text{rank}\nabla_a\tilde{\Psi}(\underline{p},0,0,\underline{a}) &= N+1-k \;\; (a-transversality)\\ \text{rank}\nabla_{a,p,s}\tilde{\Psi}(\underline{p},0,0,\underline{a}) &= 2N+1 \;\; ((a,p,s)-transversality)\end{aligned} \tag{2.72}$$

are satisfied, we say the profile $\underline{w}(z)$ *is* transversal.

2. The profile $\underline{w}$ *is said to be* strongly transversal *if both a-transversality and*

$$\text{rank}\nabla_{a,p}\tilde{\Psi}(\underline{p},0,0,\underline{a}) = 2N+1 \;\; ((a,p)-transversality) \tag{2.73}$$

are satisfied.

3. For $q=(p,s,h)\in\mathcal{C}$ *the profile* $W(z,q)$ *is transversal (resp., strongly transversal) if conditions* (2.72) *(resp., (a) and (b) of Prop. 2.13) hold with* $(\underline{p},0,0,\underline{a})$ *replaced by* (p,s,h,a)*, where* $a=a(p_\beta,s,h)$ *is as in* (2.43)*.*

Remark 2.15. ***1.*** Transversality (resp., strong transversality) of $\underline{w}$ implies transversality (resp., strong transversality) of $W(z,q)$ for q near $\underline{q}$.

2. Geometric transversality. Consider the first-order system (2.3) as a system on $\mathbb{R}$ for the moment. Let W_s, W_{cs} be the stable and center-stable manifolds of the rest point $(\underline{p}_+,0)$, and let W_u, W_{cu} be the unstable and center-unstable manifolds of the rest point $(\underline{p}_-,0)$. Then a-transversality corresponds to the statement $W_s \pitchfork W_u$ at $\big(\underline{w}(0),\underline{w}_z(0)\big)$, while the combination of a-transversality and (a,p)-transversality (which we call *strong transversality*) corresponds to $W_{cs}\pitchfork W_{cu}$ at $\big(\underline{w}(0),\underline{w}_z(0)\big)$.

Next consider the system obtained by augmenting (2.3) with the equation $s'=0$, and let W^a_{cs} (resp., W^a_{cu}) be the center-stable (resp., center-unstable) manifold of the rest point $(\underline{p}_+,0,0)$ (resp., $(\underline{p}_-,0,0)$) for the augmented system. Then the combination of a-transversality and (a,p,s)-transversality (which we call *transversality*) corresponds to $W^a_{cs}\pitchfork W^a_{cu}$ at $\big(\underline{w}(0),\underline{w}_z(0),0\big)$.

3. One could apply the implicit function theorem directly to the equation

$$\Psi(p,s,h,a)=0$$

using the rank conditions on $\nabla_a\Psi$ and $\nabla_{a,p}\Psi$ in Prop. 2.13. But then instead of obtaining a function $p_\alpha(p_\beta,s,h)$ as in (2.44), we'd obtain a function $p_\alpha(p_\beta,s,h,a_i)$, where a_i is one of the a components. This is not the form a generalized RH condition should have, since a_i does not correspond to any of the unknowns in the inviscid hyperbolic problem; so we use the extra boundary condition of $\tilde{\Psi}$ instead.

3 Linearized GRH derived from transmission conditions

In this section we derive more explicit characterizations of $T_q\mathcal{C}_\mathcal{B}$ and the linearized GRH conditions that are useful in the later low frequency analysis of the standard and modified Evans functions.

3.1 Linearized hyperbolic problem

Suppose we are given a shock manifold $\mathcal{C}$ as in Assumption 1.2 and a planar shock $\underline{q} = (\underline{p}_+, \underline{p}_-, 0, 0) \in \mathcal{C}$. To construct a curved nonconservative shock as a perturbation of $\underline{q}$, we solve the transmission problem

$$\begin{aligned} &\sum_{j=0}^{d-1} A_j(u)\partial_j u + \mathcal{A}_d(u, d\psi)\partial_d u = 0 \\ &\chi(u, d\psi) = 0 \text{ on } x_d = 0 \end{aligned} \tag{3.1}$$

where u is near $\underline{p}$, $d\psi$ is near 0, and $\chi(p, s, h)$ (2.44) is the defining function for $\mathcal{C}$ near $\underline{q}$.

To solve this nonlinear problem we need to solve linearized problems at $q = (p, s, h)$ near $\underline{q}$. The (fully) linearized problem at $q = (p_+, p_-, s, h)$ is

$$\begin{aligned} &(a) \sum_{j=0}^{d-1} A_j(p)\partial_j \dot{u} + \mathcal{A}_d(p, s, h)\partial_d \dot{u} = f \text{ on } \pm x_d \geq 0 \\ &(b) \chi'(p, s, h)(\dot{u}, d\dot{\psi}) = g \text{ on } x_d = 0. \end{aligned} \tag{3.2}$$

The p in (3.2)(a) should be understood as $p_\pm$, while that in (3.2)(b) should be understood as (p_+, p_-). Similar notation is sometimes used below. When $g = 0$, the boundary condition in (3.2) is the requirement

$$(\dot{u}, d\dot{\psi}) \in T_q\mathcal{C}. \tag{3.3}$$

In the low frequency analysis of viscous stability determinants in the case when $\mathcal{C} = \mathcal{C}_\mathcal{B}$, we'll sometimes require a more explicit form of the linearized boundary condition, namely,

$$\Gamma_{0,red}(q)(\dot{u}, d\dot{\psi}) = 0, \tag{3.4}$$

where $\Gamma_{0,red}(q)$ is defined in (3.44). We construct $\Gamma_{0,red}(q)$ by considering the full linearization with respect to w, s, and h of the profile transmission problem for $w(z)$, where $w = w_\pm$ on $\pm z \geq 0$.

$$\begin{aligned} &\mathcal{A}_d(w, s, h)w' = (1 + |h|^2)w'' \text{ in } \pm z \geq 0 \\ &\Gamma_1(w, w_z, s, h) := \begin{pmatrix} [w] \\ [w_z] \\ s + w_+ \cdot \underline{w}_z - \underline{w}_z \cdot \underline{w} \end{pmatrix} = 0 \text{ on } z = 0. \end{aligned} \tag{3.5}$$

3.2 Linearized parabolic problem and transversality

Let us write $W(z,q)$ for the profile on $\mathbb{R}$ given by (2.39); thus $\underline{w}(z) = W(z,\underline{q})$. The full linearization of (3.5) at $(W(z,q),s,h)$ is

$$(a)\mathcal{L}_0(z,q,\partial_z)\dot{w} := -(1+|h|^2)\dot{w}'' + \mathcal{A}_d(W,s,h)\dot{w}' + \partial_w\mathcal{A}_d(W,s,h)\dot{w}W' =$$

$$\dot{s}A_0(W)W' + \sum_{j=1}^{d-1}\dot{h}_jA_j(W)W' + 2\sum_{j=1}^{d-1}h_j\dot{h}_jW'' := \mathcal{L}_{0,1}(z,q)(\dot{s},\dot{h}) \tag{3.6}$$

$$(b)\Gamma_2(\dot{w},\dot{w}_z,\dot{s},\dot{h}) := \begin{pmatrix}[\dot{w}]\\ [\dot{w}_z]\\ \dot{s}+\dot{w}_+\cdot\underline{w}_z\end{pmatrix} = 0 \text{ on } z=0$$

Before studying (3.6) and in order to avoid later repetitions, it is desirable at this point to consider the full linearization of the interior parabolic problem in (1.24)

$$\sum_{j=0}^{d-1}A_j(u)\partial_ju + \mathcal{A}_d(u,d\psi)\partial_zu - \sum_{j=1}^{d}(\partial_j-\partial_j\psi\partial_z)^2u = 0 \text{ on } \pm z\geq 0 \tag{3.7}$$

about the exact solution given by a profile $W(z,q)$ and front $\psi = st+hy$. The Laplace-Fourier transform in (t,y) of that linearization is

$$\mathcal{L}(z,q,\zeta,\partial_z)\dot{u} - \dot{\psi}\mathcal{L}_1(z,q,\zeta) = f \text{ in } \pm z\geq 0. \tag{3.8}$$

Here $\zeta = (\tau,\gamma,\eta)$ where $\gamma>0$ and, with $W' = W_z(z,q)$, the operators $\mathcal{L}$ and $\mathcal{L}_1$ are given explicitly by

(3.9)

$$\mathcal{L}(z,q,\zeta,\partial_z)\dot{u} = -(1+|h|^2)\dot{u}_{zz} + \left(\mathcal{A}_d(W,s,h) + 2\sum_{j=1}^{d-1}h_ji\eta_j\right)\dot{u}_z + \partial_w\mathcal{A}_d(W,s,h)\dot{u}W' +$$

$$A_0(W)(i\tau+\gamma)\dot{u} + \sum_{j=1}^{d-1}A_j(W)i\eta_j\dot{u} + |\eta|^2\dot{u},$$

$$\mathcal{L}_1(z,q,\zeta) = A_0(W)W'(i\tau+\gamma) + \sum_{j=1}^{d-1}A_j(W)W'i\eta_j + 2\sum_{j=1}^{d-1}h_ji\eta_jW'' + |\eta|^2W'.$$

It is important in the sequel to know the relationship between the operators $\mathcal{L}_0$ and $\mathcal{L}_{0,1}$ appearing in (3.6) and the operators $\mathcal{L}$ and $\mathcal{L}_1$ in (3.9). Introducing polar coordinates

$$\zeta = \rho\hat{\zeta},\ \rho = |\zeta|,\ \hat{\zeta} = (\hat{\tau},\hat{\gamma},\hat{\eta}) \in S^d_+ = S^d\cap\{\hat{\gamma}>0\},\ \overline{S}^d_+ = S^d\cap\{\hat{\gamma}\geq 0\} \tag{3.10}$$

and writing $\mathcal{L}_1(z,q,\zeta) = \rho\check{\mathcal{L}}_1(z,q,\hat{\zeta},\rho)$ we clearly have

$$\mathcal{L}_0(z,q,\partial_z) = \mathcal{L}(z,q,0,\partial_z) \text{ and } \mathcal{L}_{0,1}(z,q)(i\hat{\tau}+\hat{\gamma},i\hat{\eta}) = \check{\mathcal{L}}_1(z,q,\hat{\zeta},0). \tag{3.11}$$

3.2.1 Conjugation to limiting and block diagonal systems

In order to understand the behavior of solutions to (3.6) and (3.8), it is helpful first to rewrite the partially linearized problem

$$\mathcal{L}(z,q,\zeta,\partial_z)\dot{u} = f \tag{3.12}$$

as a first order system and then conjugate it to simpler forms. Here we'll also establish notation that will be used in the rest of the paper. Recall $q=(p_+,p_-,s,h)$, where $p_\pm \in \omega_\pm$ and $|s,h|\leq \epsilon_\omega$ for $\omega_\pm$, ϵ_ω as in Prop. 2.2. Set

$$\mathcal{Q} := \omega_+ \times \omega_- \times \{(s,h) : |s,h| \leq \epsilon_\omega\}. \tag{3.13}$$

With $U := (\dot{u}, \dot{u}_z)^t$ we can rewrite (3.12) as a $2N\times 2N$ first order system

$$\partial_z U = G(z,q,\zeta)U + F \text{ on } \pm z\geq 0, \tag{3.14}$$

where

(3.15)

$$F=(0,-b(h)f)^t,\ \ b(h)=(1+|h|^2)^{-1},\ \ G=\begin{pmatrix}0 & I\\ G_{21} & G_{22}\end{pmatrix} \text{ with}$$

$$G_{21}(z,q,\zeta)\dot{u} = b(h)\left(\partial_w \mathcal{A}_d(W,s,h)\dot{u}W' + A_0(W)(i\tau+\gamma)\dot{u} + \sum_{j=1}^{d-1} A_j(W) i\eta_j \dot{u} + |\eta|^2\dot{u}\right)$$

$$G_{22}(z,q,\zeta) = b(h)\left(\mathcal{A}_d(W,s,h) + 2\sum_{j=1}^{d-1} h_j i\eta_j\right).$$

Recalling that $W(z,q)\to p_\pm$ as $z\to\pm\infty$, we define limiting systems $G_\pm(q,\zeta)$ by replacing W by $p_\pm$ and W' by 0 in (3.15):

$$G_\pm(q,\zeta) = \begin{pmatrix}0 & I\\ G^{21}_\pm(q,\zeta) & G^{22}_\pm(q,\zeta)\end{pmatrix}. \tag{3.16}$$

The following conjugation lemma is a version of Lemma 2.10 with parameters.

Lemma 3.1 ([MZ3], Lemma 2.6). *For all $q_0\in\mathcal{Q}$ and $\zeta_0 \in \overline{\mathbb{R}}^{1+d}_+ = \mathbb{R}^{d+1}\cap\{\gamma\geq 0\}$ there is a neighborhood Ω of (q_0,ζ_0) in $\mathcal{Q}\times\overline{\mathbb{R}}^{1+d}_+$ and there are matrices $Y_\pm$ defined and C^∞ on $\{\pm z\geq 0\}\times\Omega$ such that:*

i) $Y_\pm$ and $(Y_\pm)^{-1}$ are uniformly bounded and for $\delta>0$ as in (1.20) *there are positive constants C_α, $\delta'<\delta$ such that for $(q,\zeta)\in\Omega$:*

$$\left|\partial^\alpha_{z,q,\zeta}\big(Y_\pm(z,q,\zeta)-\mathrm{Id}\big)\right| \leq C_\alpha e^{-\delta_1|z|} \ \text{ on } \pm z\geq 0 \text{ for all } \alpha;$$

ii) Y_+ and Y_- satisfy

$$\partial_z Y_\pm(z,q,\zeta) = G(z,q,\zeta)Y_\pm(z,q,\zeta) - Y_\pm(z,q,\zeta)G_\pm(q,\zeta) \text{ on } \pm z\geq 0.$$

Observe that U satisfies (3.14) if and only if V defined by $U_\pm = Y_\pm V_\pm$ satisfies

$$\partial_z V = G_\pm(q,\zeta)V + Y^{-1}F \text{ on } \pm z \geq 0. \tag{3.17}$$

As in [GMWZ3] for $\rho = |\zeta|$ small we perform an additional conjugation of (3.17) to block diagonal (or "HP") form using

$$V_\pm = \Lambda_\pm(q,\zeta)\begin{pmatrix} u_{H\pm}, \\ u_{P\pm} \end{pmatrix} \tag{3.18}$$

where $\Lambda_\pm$ is C^∞ and

$$\Lambda_\pm(q,0) = \begin{pmatrix} I & (G^{22}_\pm)^{-1} \\ 0 & I \end{pmatrix}. \tag{3.19}$$

This transforms (3.17) to

$$\partial_z \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} = \begin{pmatrix} H_\pm(q,\zeta) & 0 \\ 0 & P_\pm(q,\zeta) \end{pmatrix}\begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} + \Lambda^{-1}Y^{-1}F \text{ on } \pm z \geq 0, \tag{3.20}$$

where

$$\begin{aligned} &P_\pm(q,\zeta) = G^{22}_\pm(q,\zeta) + O(\rho) \\ &H_\pm(q,\zeta) = -(G^{22}_\pm)^{-1}G^{21}_\pm + O(\rho^2) = \\ &\qquad -\mathcal{A}_d(p_\pm,s,h)^{-1}\left(A_0(p_\pm)(i\tau+\gamma) + \sum_{j=1}^{d-1} A_j(p_\pm)i\eta_j\right) + O(\rho^2). \end{aligned} \tag{3.21}$$

3.2.2 Other characterizations of transversality.

Next we give characterizations of transversality that are useful in formulating the reduced transmission conditions. The first step is to rewrite the fully linearized profile problem (3.6) in a form with modified transmission conditions and where $\mathcal{L}_{0,1}$ no longer appears.

Proposition 3.2. *There exist C^∞ functions $\mathcal{R}_\pm(z,q,\dot s,\dot h)$ valued in $\mathbb{R}^N$ and satisfying*

$$\begin{aligned} &(a)\ \mathcal{L}_0(z,q,\partial_z)\mathcal{R}(z,q,\dot s,\dot h) = \mathcal{L}_{0,1}(z,q)(\dot s,\dot h) \text{ on } \pm z \geq 0 \\ &(b)\ \mathcal{R}_\pm(z,q,\dot s,\dot h) \text{ is linear in } (\dot s,\dot h) \\ &(c)\ \underline{w}_z(0)\cdot\mathcal{R}_\pm(0,q,\dot s,\dot h) = -\dot s \\ &(d)\ \nabla_{\dot s,\dot h}\mathcal{R}_\pm = O(e^{-\delta|z|}) \text{ for some } \delta > 0. \end{aligned} \tag{3.22}$$

Proof. **1.** We first construct $\mathcal{R}_1(z,q,\dot s,\dot h)$ such that

$$\mathcal{L}_0(z,q,\partial_z)\mathcal{R}_1(z,q,\dot s,\dot h) = \mathcal{L}_{0,1}(z,q)(\dot s,\dot h) \text{ on } \pm z \geq 0, \tag{3.23}$$

by setting

$$\mathcal{R}_1(z,q,\dot s,\dot h) := \mathcal{R}_{10}(z,q)\dot s + \sum_{j=1}^{d-1}\mathcal{R}_{1j}(z,q)\dot h_j, \tag{3.24}$$

where the $\mathcal{R}_{1k}$ are C^∞, exponentially decaying as $z \to \pm\infty$, and satisfy

$$\begin{aligned}\mathcal{L}_0(z,q,\partial_z)\mathcal{R}_{10}^{\pm} &= A_0(W)W' \text{ on } \pm z \geq 0\\ \mathcal{L}_0(z,q,\partial_z)\mathcal{R}_{1j}^{\pm} &= A_j(W)W' + 2h_jW''.\end{aligned} \tag{3.25}$$

Functions $\mathcal{R}_{1k}$ as above may be constructed, for example, by rewriting the equations (3.25) as first order systems and then conjugating by $T_\pm(z,q,0)$, where

$$T_\pm(z,q,\zeta) := Y_\pm(z,q,\zeta)\Lambda_\pm(q,\zeta) \tag{3.26}$$

(recall (3.11), Lemma 3.1, and (3.18)). This yields easily solvable systems of the form

$$\partial_z \begin{pmatrix} u_{H\pm} \\ u_{P\pm}\end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & P_\pm(q,0)\end{pmatrix}\begin{pmatrix} u_{H\pm} \\ u_{P\pm}\end{pmatrix} + F_{k\pm} \tag{3.27}$$

with $F_{k\pm}$ exponentially decaying, where

$$\begin{pmatrix} \mathcal{R}_{1k} \\ \partial_z\mathcal{R}_{1k}\end{pmatrix}(z,q) = T_\pm(z,q,0)\begin{pmatrix} u_{H\pm} \\ u_{P\pm}\end{pmatrix}(z,q) \text{ on } \pm z \geq 0. \tag{3.28}$$

2. Since

$$\mathcal{L}_0(z,q,\partial_z)W_z(z,q) = 0, \tag{3.29}$$

we can arrange the transmission condition (3.22)(c) by setting

$$\mathcal{R}(z,q,\dot{s},\dot{h}) := \mathcal{R}_1(z,q,\dot{s},\dot{h}) - \left(\frac{\underline{w}_z(0)\cdot\mathcal{R}_1(0,q,\dot{s},\dot{h}) + \dot{s}}{\underline{w}_z(0)\cdot W_z(0,q)}\right)W_z(z,q). \tag{3.30}$$

□

Remark 3.3. Although $\mathcal{R}$ is initially defined only for $(\dot{s},\dot{h}) \in \mathbb{R}^d$, it is important for later applications to note that it extends immediately to $(\dot{s},\dot{h}) \in \mathbb{C}^d$ (for example, see (3.44) and (4.1)).

An immediate consequence of Proposition 3.2 is:

Corollary 3.4. *Define*

$$\dot{v} := \dot{w} - \mathcal{R}(z,q,\dot{s},\dot{h}). \tag{3.31}$$

Then $(\dot{w},\dot{s},\dot{h})$ satisfies (3.6) if and only if $(\dot{v},\dot{s},\dot{h})$ satisfies

$$\begin{aligned}&(a)\ \mathcal{L}_0(z,q,\partial_z)\dot{v} = 0 \text{ on } \pm z \geq 0\\ &(b)\ \Gamma_3(q)(\dot{v},\dot{v}_z,\dot{s},\dot{h}) := \begin{pmatrix} [\dot{v}] + [\mathcal{R}(z,q,\dot{s},\dot{h}] \\ [\dot{v}_z] + [\mathcal{R}_z(z,q,\dot{s},\dot{h})] \\ \dot{v}_+\cdot\underline{w}_z\end{pmatrix} = 0 \text{ on } z = 0.\end{aligned} \tag{3.32}$$

Let us write $T(z,q,\zeta)$ in (3.26) as (suppressing $\pm$)

$$T(z,q,\zeta)=\begin{pmatrix}T_{11} & T_{12}\\ T_{21} & T_{22}\end{pmatrix}(z,q,\zeta). \tag{3.33}$$

Remark 3.5. By conjugation to (3.27) with $F_\pm=0$ we see that solutions of (3.32)(a) can be written as

$$\dot v(z)=T_{11}(z,q,0)u_H+T_{12}(z,q,0)e^{zP(q,0)}u_P \text{ on } \pm z\geq 0, \tag{3.34}$$

where $u_{H\pm}\in\mathbb{R}^N$, $u_{P\pm}\in\mathbb{R}^N$, and $T_{11\pm}(z,q,0)\to I$ as $z\to\pm\infty$. A solution $\dot v$ is bounded if and only if $u_{P\pm}\in\mathbb{E}_\mp(P_\pm(q,0))$. Thus, the space of bounded solutions $\mathcal{S}(q)$ of (3.32)(a) has dimension $2N+(N+1-k)$ and the space $\mathcal{S}_0(q)$ of solutions tending to zero as $z\to\pm\infty$ has dimension $R_-+L_+=N+1-k$.

A small modification of the proof of Prop. 2.12 (replace Ψ by $\tilde\Psi$) now yields:

Proposition 3.6. *The property of a-transversality holds for $W(z,q)$ (Definition 2.14) if and only if there are no nontrivial solutions of* (3.32) *which tend to zero as $z\to\pm\infty$ when $(\dot s,\dot h)=0$. Similarly, (a,p)-transversality holds if and only if for all $g\in\mathbb{R}^{2N+1}$ the problem*

$$\begin{aligned}&(a)\ \mathcal{L}_0(z,q,\partial_z)\dot v=0 \ \text{ on } \pm z\geq 0\\ &(b)\ \Gamma_3(q)(\dot v,\dot v_z,0,0):=g \ \text{ on } z=0\end{aligned} \tag{3.35}$$

has a bounded solution.

We can use (3.34) to write the boundary condition (3.32)(b) as

$$\Gamma_{0,H}(q)u_H+\Gamma_{0,P}(q)u_P+\Gamma_{\mathcal{R}}(\dot s,\dot h)=\Gamma_3(q)(\dot v,\dot v_z,\dot s,\dot h)=0, \tag{3.36}$$

where

(3.37)

$$\Gamma_{0,H}(q)u_H=\begin{pmatrix}\begin{bmatrix}T_{11}(0,q,0)u_H\\ T_{21}(0,q,0)u_H\end{bmatrix}\\ (T_{11+}(0,q,0)u_{H+})\cdot\underline{w}_z(0)\end{pmatrix},\ \Gamma_{0,P}(q)u_P=\begin{pmatrix}\begin{bmatrix}T_{12}(0,q,0)u_P\\ T_{22}(0,q,0)u_P\end{bmatrix}\\ (T_{12+}(0,q,0)u_{P+})\cdot\underline{w}_z(0)\end{pmatrix},$$

$$\text{and } \Gamma_{\mathcal{R}}(\dot s,\dot h):=\begin{pmatrix}[\mathcal{R}(0,q,\dot s,\dot h]\\ [\mathcal{R}_z(0,q,\dot s,\dot h)]\\ 0\end{pmatrix}.$$

With (3.34) and (3.36) we obtain directly the following rephrasing of Prop. 3.6:

Proposition 3.7. *With $P_{0\pm}(q):=P(q,0)=b(h)\mathcal{A}_d(p_\pm,s,h)$, a-transversality holds for $W(z,q)$ if and only if*

$$\ker\Gamma_{0,P}(q)\cap(\mathbb{E}_-(P_{0+}(q))\times\mathbb{E}_+(P_{0-}(q)))=\{0\}, \tag{3.38}$$

while (a,p)-transversality means that the rank of the matrix

$$(\Gamma_{0,H}(q),\Gamma_{0,P}(q),\Gamma_{\mathcal{R}}):\mathbb{R}^{2N}\times(\mathbb{E}_-(P_{0+}(q))\times\mathbb{E}_+(P_{0-}(q)))\times\mathbb{R}^d\to\mathbb{R}^{2N+1} \tag{3.39}$$

is $2N+1$ when $(\dot s,\dot h)=0$.

Equivalent characterizations of a-transversality and (a,p)-transversality are obtained by replacing each space appearing in (3.38) *or* (3.39) *by its complexification.*

In making the last assertion of the Proposition we used the observations made in Notation 2.1 together with the extension of $\mathcal{R}$ to $(\dot s,\dot h)\in\mathbb{C}^d$.

3.3 Reduced transmission conditions and $T_q\mathcal{C}_\mathcal{B}$

The reduced boundary operator constructed in this section will play a key role in the later stability analyses.

If a-transversality holds there is a decomposition

$$\mathbb{C}^{2N+1} = \mathbb{F}_{H,\mathcal{R}}(q) \oplus \mathbb{F}_P(q), \tag{3.40}$$

where

$$\mathbb{F}_P(q) = \Gamma_{0,P}(q)\left(\mathbb{E}_-(P_{0+}(q)) \times \mathbb{E}_+(P_{0-}(q))\right) \tag{3.41}$$

has dimension $N+1-k$ and $\mathbb{F}_{H,\mathcal{R}}(q)$ is an arbitrary complementary subspace (necessarily of dimension $N+k$). Denote by $\pi_{H,\mathcal{R}}(q)$ and $\pi_P(q)$ the projections associated to this splitting.

Remark 3.8 (More careful choice of $\mathbb{F}_{H,\mathcal{R}}(q)$). Henceforth, we'll work with a choice of $\mathbb{F}_{H,\mathcal{R}}(q) \subset \mathbb{C}^{2N+1}$ made as follows. Using the remarks made in Notation 2.1 and the fact that $\Gamma_{0,P}(q)$ is a real matrix, we first choose an $N+k$ dimensional subspace $\mathbb{F}_{H,\mathcal{R}}(q) \subset \mathbb{R}^{2N+1}$ such that

$$\mathbb{R}^{2N+1} = \mathbb{F}_{H,\mathcal{R}}(q) \oplus \mathbb{F}_P(q). \tag{3.42}$$

We then take $\mathbb{F}_{H,\mathcal{R}}(q)$ in (3.40) to be the complexification of $\mathbb{F}_{H,\mathcal{R}}(q) \subset \mathbb{R}^{2N+1}$.

For $\dot{v}$ as in (3.34) we can eliminate u_P from the boundary conditions (3.36). That is to say, $(u_H, u_P, \dot{s}, \dot{h})$ satisfies (3.36) if and only if

$$\Gamma_{0,red}(q)(u_H, \dot{s}, \dot{h}) = 0, \text{ and } \quad u_P = R_P(q)(u_H, \dot{s}, \dot{h}) \tag{3.43}$$

where

$$\Gamma_{0,red}(q)(u_H, \dot{s}, \dot{h}) := \pi_{H,\mathcal{R}}(q)\left(\Gamma_{0,H}(q)u_H + \Gamma_\mathcal{R}(\dot{s}, \dot{h})\right) \tag{3.44}$$

and

$$R_P(q)(u_H, \dot{s}, \dot{h}) = -\Gamma_{0,P}^{-1}(q)\pi_P(q)\left(\Gamma_{0,H}(q)u_H + \Gamma_\mathcal{R}(\dot{s}, \dot{h})\right). \tag{3.45}$$

Remark 3.9. 1. Using Remark 3.8 we see that $\Gamma_{0,red}(q)$ can be regarded as a map defined between complex spaces

$$\Gamma_{0,red}(q) : \mathbb{C}^{2N} \times \mathbb{C}^d \to \mathbb{F}_{H,\mathcal{R}}(q) \subset \mathbb{C}^{2N+1} \tag{3.46}$$

or as a map between real spaces

$$\Gamma_{0,red}(q) : \mathbb{R}^{2N} \times \mathbb{R}^d \to \mathbb{F}_{H,\mathcal{R}}(q) \subset \mathbb{R}^{2N+1}. \tag{3.47}$$

The choice should be clear from the context. For example, in statements like (3.49) below, we take $\Gamma_{0,red}(q)$ as in (3.47).

2. In view of Proposition 3.7, if a-transversality holds for $W(z,q)$, then (a,p)-transversality means that the map

$$\mathbb{C}^{2N} \ni u_H \to \Gamma_{0,red}(q)(u_H, 0, 0) \in \mathbb{C}^{2N+1} \tag{3.48}$$

has rank $N+k$. The same holds for the map (3.47).

The next step is to show that transversality of $W(z,q)$ implies

$$\ker \Gamma_{0,red}(q) = T_q\mathcal{C}_{\mathcal{B}}. \tag{3.49}$$

For this we need

Proposition 3.10. *For $q \in \mathcal{C}_{\mathcal{B}}$ consider the set $\mathbb{S}(q)$ of solutions $(\dot w_+, \dot w_-, \dot s, \dot h)$ of the fully linearized interior equation on $\pm z \geq 0$, (3.6)(a), for which $\dot w_\pm$ are bounded as $z \to \pm\infty$. We have*

$$\dim\ \mathbb{S}(q) = 2N + d + (N+1-k) \tag{3.50}$$

and

$$\mathbb{S}(q) = \left\{ \begin{pmatrix} \phi'_+(z,q,a;\dot p_+,\dot s,\dot h,\dot a_+) \\ \phi'_-(z,q,a;\dot p_-,\dot s,\dot h,\dot a_-) \\ \dot s \\ \dot h \end{pmatrix} : (\dot p, \dot s, \dot h, \dot a) \in \mathbb{R}^{2N} \times \mathbb{R}^d \times \mathbb{E}_-(G_d(\underline{p},0,0)) \right\}, \tag{3.51}$$

where, for example,

$$\begin{aligned} &(a)\phi'_+(z,q,a;\dot p_+,\dot s,\dot h,\dot a_+) := \nabla_{p_+,s,h,a_+}\phi_+(z,p_+,s,h,a_+)(\dot p_+,\dot s,\dot h,\dot a_+) \ \text{and} \\ &(b) \lim_{z\to+\infty} \phi'_+(z,q,a;\dot p_+,\dot s,\dot h,\dot a_+) = \dot p_+, \end{aligned} \tag{3.52}$$

and $a = a(p_\beta, s, h)$ is as in (2.43).

Proof. Corollary 2.6 shows that the map

$$(\dot p,\dot s,\dot h,\dot a) \to \begin{pmatrix} \phi'_+(z,q,a;\dot p_+,\dot s,\dot h,\dot a_+) \\ \phi'_-(z,q,a;\dot p_-,\dot s,\dot h,\dot a_-) \\ \dot s \\ \dot h \end{pmatrix} \tag{3.53}$$

is injective. The functions $\phi_\pm(z,p_\pm,s,h,a_\pm)$ satisfy the interior (nonlinear) profile equation in (3.5), and differentiation with $\nabla_{p_\pm,s,h,a_\pm}$ shows that the column vector on the right in (3.53) gives a bounded solution of (3.6)(a). On the other hand any bounded solution $\dot w$ of (3.6)(a) can be written $\dot w = \dot v + \mathcal{R}(z,q,\dot s,\dot h)$, where $\dot v$ is in the $2N+(N+1-k)$ dimensional space of bounded solutions to the problem with $(\dot s,\dot h)=0$. This implies (3.50) (first) and then (3.51).

□

As a corollary we obtain the following analogues of Propositions 3.6 and 3.7.

Corollary 3.11. *(1) The condition*

$$\text{rank}\nabla_{a,p,s}\tilde\Psi(p,s,h,a) = 2N+1 \tag{3.54}$$

holds if and only if for all $g \in \mathbb{R}^{2N+1}$ the problem

$$\begin{aligned} &\mathcal{L}_0(z,q,\partial_z)\dot w = \mathcal{L}_{0,1}(z,q)(\dot s,0) \ \text{on} \ \pm z \geq 0 \\ &\Gamma_2(\dot w,\dot w_z,\dot s,0) = g \ \text{on} \ z=0 \ \ (\Gamma_2 \text{ as in } (3.6)) \end{aligned} \tag{3.55}$$

has a solution $(\dot w, \dot s)$ *with* $\dot w$ *bounded.*

(2) The condition (3.54) *means that the rank of the matrix*

$$(3.56) \qquad (\Gamma_{0,H}(q), \Gamma_{0,P}(q), \Gamma_{\mathcal{R}}) : \mathbb{C}^{2N} \times (\mathbb{E}_-(P_{0+}(q)) \times \mathbb{E}_+(P_{0-}(q))) \times \mathbb{C}^d \to \mathbb{C}^{2N+1}$$

is $2N+1$ *when* $(\dot s, \dot h) = (\dot s, 0)$.

(3) If a-transversality holds for $W(z,q)$*, then (a,p,s)-transversality means that the map*

$$(3.57) \qquad \mathbb{C}^N \times \mathbb{C} \ni (u_H, s) \to \Gamma_{0,red}(q)(u_H, s, 0) \in \mathbb{C}^{2N+1}$$

has rank $N+k$.

Proof. Let $\phi' = \phi'(z,p,s,h,a;\dot p,\dot s,\dot h,\dot a)$ be given by the first two components of the column vector in (3.51). Part (1) then follows directly from Prop. 3.10 and the observation that

$$(3.58) \qquad \Gamma_2(\phi', \phi'_z, \dot s, \dot h) = \nabla_{(p,s,h,a)} \tilde\Psi(p,s,h,a)(\dot p, \dot s, \dot h, \dot a).$$

Let $\Gamma_3(q)$ be as in (3.32) and note that for

$$(3.59) \qquad \dot v = \dot w - \mathcal{R}(z, q, \dot s, \dot h)$$

as in (3.31), we have

$$(3.60) \qquad \Gamma_2(\dot w, \dot w_z, \dot s, \dot h) = g \Leftrightarrow \Gamma_3(q)(\dot v, \dot v_z, \dot s, \dot h) = g.$$

After rewriting $\Gamma_3(q)$ as in (3.36), we deduce part (2) from part (1). Part (3) then follows directly from part (2) and the definition of $\Gamma_{0,red}(q)$.

□

Proposition 3.12. *(1) Assume* $W(z,q)$ *is strongly transversal. Then we have*

(3.61)
$(a) \ker \Gamma_{0,red}(q) = T_q\mathcal{C}_{\mathcal{B}}$
$(b) T_q\mathcal{C}_{\mathcal{B}} =$
$\{(\dot p_+, \dot p_-, \dot s, \dot h) :$ *there exists a solution* $(\dot w_+, \dot w_-, \dot s, \dot h)$ *of* (3.6) *with* $\lim_{z\to\pm\infty} \dot w_\pm = \dot p_\pm\}$.

(2) The same conclusions hold if we assume just that $W(z,q)$ *is transversal.*

Proof. **1.** We show that both sets in (3.61)(a) are equal to the set on the right in (3.61)(b). Assume that $W(z,q)$ is strongly transversal; the proof in the other case is essentially the same.

2. Suppose $(u_H, \dot s, \dot h) \in \ker \Gamma_{0,red}(q)$. By (3.43)-(3.45) there exists

$$(3.62) \qquad u_P \in \mathbb{E}_-(P_{0+}(q)) \times \mathbb{E}_+(P_{0-}(q))$$

such that $\dot v_\pm$ as in (3.34) satisfies (3.32) with

$$(3.63) \qquad \lim_{z\to\pm\infty} \dot v_\pm = u_{H\pm}.$$

But then

$$\dot w = \dot v + \mathcal{R}(z, q, \dot s, \dot h) \tag{3.64}$$

satisfies (3.6) with

$$\lim_{z\to\pm\infty} \dot w_\pm = u_{H\pm}. \tag{3.65}$$

Conversely, if $(u_H, \dot s, \dot h)$ is such that there exists $\dot w$ satisfying (3.6) and (3.65), then $\dot v$ as in (3.64) satisfies (3.32) and (3.63). So there exists u_P as in (3.62) such that $\dot v$ is given by (3.34) and (3.36) holds. Apply $\pi_{H,\mathcal{R}}(q)$ to (3.36) to obtain $(u_H, \dot s, \dot h) \in \ker \Gamma_{0,red}(q)$.

3. Recall from (2.44) that $\mathcal{C}_\mathcal{B}$ is given as the graph of

$$p_\alpha = p_\alpha(p_\beta, s, h), \tag{3.66}$$

so its tangent space at (p, s, h) is given by the graph of

$$\dot p_\alpha = \nabla_{(p_\beta,s,h)} p_\alpha(p_\beta, s, h)(\dot p_\beta, \dot s, \dot h). \tag{3.67}$$

For $\tilde\Psi(p, s, h, a)$ as in (2.34), the equation $\tilde\Psi = 0$ also yielded the relation

$$a = a(p_\beta, s, h), \tag{3.68}$$

whose linearization at (p_β, s, h) is

$$\dot a = \nabla_{(p_\beta,s,h)} a(p_\beta, s, h)(\dot p_\beta, \dot s, \dot h). \tag{3.69}$$

We can also differentiate the equation $\tilde\Psi(p, s, h, a) = 0$ in all variables to obtain the linearized form

$$\nabla_{(p,s,h,a)} \tilde\Psi(p, s, h, a)(\dot p, \dot s, \dot h, \dot a) = 0, \tag{3.70}$$

Observe that since the rank conditions hold

$$(\dot p, \dot s, \dot h, \dot a) \text{ satisfies (3.70)} \Leftrightarrow \text{ both (3.67) and (3.69) hold.} \tag{3.71}$$

Suppose now that $(\dot p, \dot s, \dot h) \in T_q\mathcal{C}_\mathcal{B}$. Then (3.67) holds. If we take $\dot a$ as in (3.69), then (3.70) holds and implies that for these choices of $(\dot p, \dot s, \dot h, \dot a)$ the column vector in (3.51) satisfies (3.6). From (3.52)(b) we see that $(\dot p, \dot s, \dot h)$ is an element of the set on the right in (3.61)(b).

Conversely, suppose $(\dot p, \dot s, \dot h)$ is such that there exists a solution $(\dot w_+, \dot w_-, \dot s, \dot h)$ as on the right in (3.61)(b). By Prop. 3.10 this solution must have the form of the column vector in (3.51) for some $\dot a$. Since this solution satisfies the boundary conditions in (3.6), this means that (3.70) holds, which implies (3.67).

□

Corollary 3.13. *Assume $\underline{w}(z)$ is either transversal or strongly transversal. For $q \in \mathcal{C}_\mathcal{B}$ near $\underline{q}$ the linearized hyperbolic transmission problem at q ((3.2) with $f = 0$, $g = 0$) can now be written*

$$\begin{aligned} &\sum_{j=0}^{d-1} A_j(p)\partial_j \dot u + \mathcal{A}_d(p, s, h)\partial_d \dot u = 0 \\ &\Gamma_{0,red}(q)(\dot u, d\dot\psi) = 0 \text{ on } x_d = 0. \end{aligned} \tag{3.72}$$

4 Stability determinants

4.1 Lopatinski determinants

In this section we define and study the relationship between two stability determinants, the Lopatinski determinant, $D_{Lop}(q,\hat{\zeta})$, and the modified Lopatinski determinant, $D_{Lop,m}(q,\hat{\zeta})$, for the linearized hyperbolic problem (3.2). Later we'll see how these arise in low frequency expansions of, respectively, the standard and modified Evans functions. But for now we study them in the context of inviscid $\mathcal{C}$ shocks, where $\mathcal{C}$ is a shock manifold as in Assumption 1.2

4.1.1 $\mathcal{C}$ shocks

Dropping dots and some hats, we write the Laplace-Fourier transform of (3.2), with $f=0$, $g=0$, as

$$\text{(4.1)}\qquad \begin{aligned} &(a)\ A_0(p)(i\hat{\tau}+\hat{\gamma})u+\sum_{j=1}^{d-1}A_j(p)i\hat{\eta}_j u+\mathcal{A}_d(p,s,h)\partial_d u=0 \text{ in } \pm x_d\geq 0\\ &(b)\ \hat{\Gamma}_\chi(q,\hat{\zeta})(u_+,u_-,\psi)=0, \end{aligned}$$

where $\hat{\Gamma}_\chi(q,\zeta):\mathbb{C}^{2N+1}\to\mathbb{C}^{N+k}$ is defined by

$$\text{(4.2)}\qquad \hat{\Gamma}_\chi(q,\hat{\zeta})(u_+,u_-,\psi)=\chi'_{p_+}(q)u_++\chi'_{p_-}(q)u_-+\chi'_s(q)(i\hat{\tau}+\hat{\gamma})\psi+\chi'_h(q)i\hat{\eta}\psi.$$

Definition 4.1. *1. For $\psi\in\mathbb{C}$, $u_\pm\in\mathbb{C}^N$, and $\hat{\zeta}\in S^d_+=S^d\cap\{\hat{\gamma}>0\}$ we set*

$$\text{(4.3)}\qquad \begin{aligned} &H_{0\pm}(q,\hat{\zeta})=-\mathcal{A}_d(p_\pm,s,h)^{-1}\left(A_0(p_\pm)(i\hat{\tau}+\hat{\gamma})+\sum_{j=1}^{d-1}A_j(p_\pm)i\hat{\eta}_j\right)\\ &\mathbb{E}_-(H_0(q,\hat{\zeta})):=\mathbb{E}_-(H_{0+}(q,\hat{\zeta}))\times\mathbb{E}_+(H_{0-}(q,\hat{\zeta})), \end{aligned}$$

where, as usual, $\mathbb{E}_-(H_{0+}(q,\hat{\zeta}))$ denotes the generalized eigenspace of $H_{0+}(q,\zeta)$ associated to eigenvalues μ with $\Re\mu<0$.

2. The Lopatinski determinant *for the problem* (4.1) *is the $(N+k)\times(N+k)$ determinant*

$$\text{(4.4)}\qquad D_{Lop}(q,\hat{\zeta}):=\det\left(\chi'_p(q)\mathbb{E}_-(H_0(q,\hat{\zeta})),\chi'_s(q)(i\hat{\tau}+\hat{\gamma})+\chi'_h(q)i\hat{\eta}\right).$$

The first $N+k-1$ columns of the matrix in (4.4) *are computed using an orthonormal basis of $\mathbb{E}_-(H_0(q,\hat{\zeta}))$.*

We'll say that the uniform Lopatinski condition *holds at $q\in\mathcal{C}$ when*

$$\text{(4.5)}\qquad \exists c>0 \text{ such that } |D_{Lop}(q,\hat{\zeta})|\geq c \text{ for all } \hat{\zeta}\in S^d_+.$$

Definition 4.2. *1. When E and F are subspaces of $\mathbb{C}^D$ with $\dim E+\dim F=D$, $\det(E,F)$ denotes the determinant formed by taking orthonormal bases in E and F. Up to a sign this determinant is independent of the choice of bases. Determinants of nonsquare matrices are defined to be* 0.

2. We define the modified Lopatinski determinant *for the hyperbolic problem* (4.1)

$$D_{Lop,m}(q,\hat{\zeta}) = \det\left(\mathbb{E}_-(H_0(q,\hat{\zeta}))\times\mathbb{C}, \ker\hat{\Gamma}_\chi(q,\hat{\zeta})\right). \tag{4.6}$$

The modified uniform Lopatinski condition *holds at $q\in\mathcal{C}$ when there is a $c>0$ such that $|D_{Lop,m}(q,\hat{\zeta})|\geq c$ for all $\hat{\zeta}\in S^d_+$.*

Remark 4.3. *1.* Hyperbolicity (H1) implies that for all $\hat{\zeta}\in S^d_+$

$$\dim\mathbb{E}_-(H_{0+}(q,\hat{\zeta})) = N - R_-,\ \ \dim\mathbb{E}_+(H_{0-}(q,\hat{\zeta})) = N - L_+ \tag{4.7}$$

for R_-, L_+ as in Defn. 2.3. Thus, the determinant in (4.6) is a $(2N+1)\times(2N+1)$ determinant when $\dim\ker\hat{\Gamma}_\chi(q,\hat{\zeta}) = N+1-k$.

2. The spaces $\mathbb{E}_\mp(H_{0\pm}(q,\hat{\zeta}))$ define C^∞ vector bundles over S^d_+.

3. The uniform Lopatinski condition allows one to construct Kreiss symmetrizers and prove maximal estimates for the linearized inviscid problem (3.2); see section 7.

4. Both determinants have been defined only for $\hat{\zeta}\in S^d_+$. The function $D_{Lop,m}(q,\cdot)$ has a continuous extension to any subset of $\overline{S}^d_+$ where $\mathbb{E}_-(H_0(q,\hat{\zeta}))$ is continuous and $\hat{\Gamma}_\chi(q,\hat{\zeta})$ maintains full rank, while $D_{Lop}(q,\cdot)$ has a continuous extension to any subset of $\overline{S}^d_+$ where $\mathbb{E}_-(H_0(q,\hat{\zeta}))$ is continuous.

Suppose that χ_1 and χ_2 are local defining functions (Defn. 1.4) for $\mathcal{C}$ near q. Since

$$\ker\chi_1'(q) = T_q\mathcal{C} = \ker\chi_2'(q) \Rightarrow \ker\hat{\Gamma}_{\chi_1}(q,\hat{\zeta}) = \ker\hat{\Gamma}_{\chi_2}(q,\hat{\zeta}), \tag{4.8}$$

we see that $D_{Lop,m}(q,\hat{\zeta})$, and hence the validity of the modified uniform Lopatinski condition, is independent of the choice of local defining function. The determinant $D_{Lop}(q,\hat{\zeta})$ is clearly not independent of the choice of local defining function, but we will show that the uniform Lopatinski condition is. In the proof we use the following general fact about determinants.

Proposition 4.4. *Let M be a $p\times q$ matrix, $p\neq q$, E a $q\times p$ matrix, and denote by $\ker M$ a matrix whose columns form an orthonormal basis of the kernel of M. Then we have*

$$\det ME = c(M)\det\begin{pmatrix}E & \ker M\end{pmatrix}, \tag{4.9}$$

where $c(M)\neq 0$ if M has full rank and is otherwise defined to be 0.

If $p>q$ or if $p\leq q$ and M is not full rank, the matrix $\begin{pmatrix}E & \ker M\end{pmatrix}$ on the right is not square and its determinant is then defined to be zero.

Proof. **1.** When $p>q$ or when $p\leq q$ and M is not full rank, both determinants are zero.

2. Assume then that $p\leq q$ and M has full rank p, so the determinant on the right in (4.9) is $q\times q$. In this case MM^* is invertible and

$$\begin{pmatrix}M\\ (\ker M)^*\end{pmatrix}^{-1} = \begin{pmatrix}M^*(MM^*)^{-1} & \ker M\end{pmatrix}. \tag{4.10}$$

Note that $(\ker M)^*$ is a $(q-p)\times q$ matrix.

3. We have

$$\begin{pmatrix} M \\ (\ker M)^* \end{pmatrix} \begin{pmatrix} E & \ker M \end{pmatrix} = \begin{pmatrix} ME & 0 \\ (\ker M)^* E & I_{(q-p)\times(q-p)} \end{pmatrix}, \tag{4.11}$$

so (4.9) holds with

$$c(M) := \det \begin{pmatrix} M \\ (\ker M)^* \end{pmatrix}, \tag{4.12}$$

which is nonzero when M has full rank by part 2. □

Proposition 4.5. *(a) The uniform Lopatinski condition is independent of the choice of local defining function used to compute $D_{Lop}(q,\hat{\zeta})$.*

(b) The uniform Lopatinski condition at q implies the modified uniform Lopatinski condition at q.

Proof. **1.** Let χ_1 and χ_2 be local defining functions for $\mathcal{C}$ near q, and let D_{Lop,χ_i} denote the corresponding determinants (4.4). Suppose there exists $c > 0$ such that

$$|D_{Lop,\chi_1}(q,\hat{\zeta})| \geq c \text{ for } \hat{\zeta} \in S^d_+. \tag{4.13}$$

Then $\hat{\Gamma}_{\chi_1}(q,\hat{\zeta})$ (and thus also $\hat{\Gamma}_{\chi_2}(q,\hat{\zeta})$ by (4.8)) has full rank $N+k$ for $\hat{\zeta} \in S^d_+$. Applying Prop. 4.4 with

$$M := \hat{\Gamma}_{\chi_i}(q,\hat{\zeta}) \text{ and } E := \mathbb{E}_-(H_0(q,\hat{\zeta})) \times \mathbb{C}, \tag{4.14}$$

we obtain

$$D_{Lop,\chi_i}(q,\hat{\zeta}) = c_i(q,\hat{\zeta}) D_{Lop,m}(q,\hat{\zeta}),\ i = 1,2 \tag{4.15}$$

where $c_i(q,\hat{\zeta}) \neq 0$ and is continuous near $(q,\hat{\zeta})$. (More precisely, E is a $(2N+1)\times(N+k)$ matrix whose columns form a basis of the given space.)

2. To complete the proof of part (a) it suffices to show that $\hat{\Gamma}_{\chi_i}(q,\hat{\zeta})$ cannot drop rank at a point $\hat{\zeta}_0$ with $\hat{\gamma}_0 = 0$, for then $c_i(q,\hat{\zeta})$ must be continuous and nonvanishing on $\overline{S}^d_+$. When $\mathbb{E}_-(H_0(q,\hat{\zeta}))$ extends continuously to $\overline{S}^d_+$, so does $D_{Lop,\chi_1}(q,\cdot)$, and (4.13) implies that $\hat{\Gamma}_{\chi_1}(q,\cdot)$ has full rank on $\overline{S}^d_+$.

In the general case the Lopatinski determinants are not defined for $\hat{\gamma} = 0$, but we can substitute compactness of Grassmannians for compactness of $\overline{S}^d_+$. If $\hat{\Gamma}_{\chi_1}(q,\hat{\zeta})$ drops rank at $\hat{\zeta}_0$, then

$$\det(\hat{\Gamma}_{\chi_1}(q,\hat{\zeta}_0)E_0) = 0 \tag{4.16}$$

for any E_0 in the set of limit points of $\mathbb{E}_-(H_0(q,\hat{\zeta}_j)) \times \mathbb{C}$ as $\hat{\zeta}_j \to \hat{\zeta}_0$ with $\hat{\gamma}_j > 0$. But then we must have

$$D_{Lop,\chi_1}(q,\hat{\zeta}_j) \to 0$$

for some sequence $\hat{\zeta}_j$ with $\hat{\gamma}_j > 0$, a contradiction.

3. Part (b) follows directly from (4.15) and the fact that $c_1(q,\hat{\zeta})$ is continuous and nonvanishing on $\overline{S}^d_+$. □

Remark 4.6. (a) A similar argument fails for the reverse implication in part (b) above, because although $\hat{\Gamma}_\chi(q,\hat{\zeta}_j) \to \hat{\Gamma}_\chi(q,\hat{\zeta}_0)$, we don't have

$$\ker \hat{\Gamma}_\chi(q,\hat{\zeta}_j) \to \ker \hat{\Gamma}_\chi(q,\hat{\zeta}_0)$$

when $\hat{\Gamma}_\chi(q,\cdot)$ drops rank at $\hat{\zeta}_0$.

(b) Observe that if χ is a local defining for $\mathcal{C}$ near q and if $\chi'_{p,s}(q)$ (or, alternatively, $\chi'_p(q)$) has full rank $N+k$, then the same is true for any other defining function near q. This is relevant to the next Proposition.

Proposition 4.7. *Let χ be a local defining function for $\mathcal{C}$ near q.*

(a) Suppose $\chi'_p(q)$ has full rank $N+k$. Then

$$D_{Lop}(q,\hat{\zeta}) = c(q,\hat{\zeta}) D_{Lop,m}(q,\hat{\zeta}),\ i = 1,2 \tag{4.17}$$

with $c(q,\hat{\zeta})$ continuous and nonvanishing on $\overline{S}^d_+$. Consequently, the modified uniform Lopatinski condition at q implies the uniform Lopatinski condition at q.

(b)Suppose $\chi'_{p,s}(q)$ has full rank $N+k$ and $d=1$. Then (4.17) *holds with $c(q,\hat{\zeta})$ continuous and nonvanishing on $\overline{S}^d_+$. Again, the modified uniform Lopatinski condition at q implies the uniform Lopatinski condition at q.*

Proof. **1.** If $\chi'_p(q)$ has full rank, then $\hat{\Gamma}_\chi(q,\hat{\zeta})$ maintains full rank even when restricted to the subspace $\psi = 0$; hence it maintains full rank for $\hat{\zeta} \in \overline{S}^d_+$. Thus, (4.17) holds with $c(q,\hat{\zeta})$ continuous and nonvanishing on $\overline{S}^d_+$.

2. Since $\chi'_{p,s}(q)$ has full rank, when $d=1$ it follows that $\hat{\Gamma}_\chi(q,\hat{\tau},\hat{\gamma})$ must have full rank for $\hat{\zeta} \in \overline{S}^d_+$. The result then follows as in part (a).

□

Proposition 4.8. *Suppose the uniform Lopatinski condition holds at $q \in \mathcal{C}$.*

(a) Then $\chi'_{p,s}(q)$ has full rank $N+k$ for any local defining function χ.

(b) If $d \geq 2$, then $\chi'_p(q)$ has full rank $N+k$.

Proof. **1. (a)** This follows immediately from (4.5) by taking $\hat{\eta} = 0$.

2. (b) If $\chi'_p(q)$ does not have full rank, we claim

$$\dim \ker \hat{\Gamma}_\chi(q,\hat{\zeta}) \geq N+2-k \text{ for some } \hat{\zeta} \in \overline{S}^d_+. \tag{4.18}$$

By part (a) we can choose coordinates $(p_\alpha, p_\beta) \in \mathbb{R}^{N-1+k} \times \mathbb{R}^{N+1-k}$ such that $\chi'_{p_\alpha,s}(q)$ is nonsingular; write $u = (u_\alpha, u_\beta) \in \mathbb{C}^{2N}$. If $\chi'_p(q)$ does not have full rank, it must have rank $N-1+k$, so the map

$$u \to \chi'_p(q)u = \chi'_{p_\alpha}(q)u_\alpha + \chi'_{p_\beta}(q)u_\beta \tag{4.19}$$

has an $N+1-k$ dimensional kernel. This yields an $N+1-k$ dimensional subspace $S(q) \subset \ker \hat{\Gamma}_\chi(q,\hat{\zeta})$ independent of $\hat{\zeta}$, namely,

$$\begin{aligned} S(q) = \{(u,\psi) = (u,0) : u \in \ker \chi'_p(q)\} = \\ \{(u_\alpha,u_\beta,0) : (u_\alpha,0) = -(\chi'_{p_\alpha,s}(q))^{-1}\chi'_{p_\beta}(q)u_\beta,\ u_\beta \in \mathbb{C}^{N+1-k}\}. \end{aligned} \tag{4.20}$$

3. To finish we show that for certain $\hat{\zeta}$, there are elements of $\ker \hat{\Gamma}_\chi(q,\hat{\zeta})$ that have nonzero ψ components.

Consider the d vectors in $\mathbb{R}^N$:

$$\chi_s'(q), \chi_{h_i}'(q), i = 1, \dots, d-1. \tag{4.21}$$

If these vectors are linearly dependent over $\mathbb{R}$, there clearly exist $\hat{\tau}$, $\hat{\eta}$ such that for $\hat{\zeta} = (\hat{\tau}, 0, \hat{\eta})$,

$$(0,0,1) \in \ker \hat{\Gamma}_\chi(q,\hat{\zeta}). \tag{4.22}$$

On the other hand if the vectors (4.21) are linearly independent over $\mathbb{R}$, and thus over $\mathbb{C}$, the range of the map

$$(\hat{\zeta}, \psi) \to \chi_s'(q)(i\hat{\tau} + \hat{\gamma})\psi + \chi_h'(q) i\hat{\eta}\psi \tag{4.23}$$

must have nontrivial intersection in $\mathbb{C}^{N+k}$ with the $N-1+k$ dimensional range of the map

$$(u_\alpha, u_\beta) \to \chi_{p_\alpha}'(q)u_\alpha + \chi_{p_\beta}'(q)u_\beta. \tag{4.24}$$

This is because $d \geq 2$. In view of (4.2) this means that in any case, for some $\hat{\zeta}$, there are elements of $\ker \hat{\Gamma}_\chi(q,\hat{\zeta})$ that have nonzero ψ components. Since $S(q) \subset \ker \hat{\Gamma}_\chi(q,\hat{\zeta})$ as well, this implies (4.18).

4. If the set of bad directions $\hat{\zeta}$ where (4.18) holds includes a point $\hat{\zeta}_0 \in S^d_+$, then $D_{Lop,\chi}(q,\hat{\zeta}_0) = 0$ since $\hat{\Gamma}_\chi(q,\hat{\zeta}_0)$ does not have full rank. So suppose all bad directions $\hat{\zeta}_0$ satisfy $\hat{\gamma}_0 = 0$. Use compactness of Grassmannians as before to obtain an $N+k$ dimensional limit point E_0 of $\mathbb{E}_-(H_0(q,\hat{\zeta}_j)) \times \mathbb{C}$ as $\hat{\zeta}_j \to \hat{\zeta}_0$ with $\hat{\gamma}_j > 0$. We then obtain a sequence

$$D_{Lop,\chi}(q,\hat{\zeta}_j) \to 0$$

just as in the proof of Proposition 4.5. □

4.1.2 $\mathcal{C}_\mathcal{B}$ shocks

Suppose now that $\mathcal{C} = \mathcal{C}_\mathcal{B}$ and that $W(z,q)$ is a viscous profile corresponding to the planar shock $q \in \mathcal{C}_\mathcal{B}$. We have seen that for any local defining function χ, if $W(z,q)$ is transversal, then $\chi_{p,s}'(q)$ has full rank $N+k$, and if $W(z,q)$ is strongly transversal, then $\chi_p'(q)$ has full rank. Thus, we obtain the following immediate corollaries of Propositions 4.5, 4.7, and 4.8.

Corollary 4.9. *Assume $W(z,q)$ is transversal.*

(a) When $d = 1$, the uniform Lopatinski condition holds at q if and only if the modified uniform Lopatinski condition holds at q.

(b) For $d \geq 1$ the uniform Lopatinski condition at q implies the modified uniform Lopatinski condition at q.

(c) Suppose $W(z,q)$ is strongly transversal. Then the modified uniform Lopatinski condition at q implies the uniform Lopatinski condition at q.

Corollary 4.10. *When $d \geq 2$ and $W(z,q)$ is transversal, if the uniform Lopatinski condition holds at q, then $W(z,q)$ is strongly transversal.*

These results are useful in the analysis of the viscous stability determinants or Evans functions defined in the following sections; see Proposition 5.15.

Remark 4.11. (a) For $\psi \in \mathbb{C}$, $u_\pm \in \mathbb{C}^N$, and $\hat{\zeta} \in \overline{S}^d_+$, we define

$$\hat{\Gamma}_{0,red}(q,\hat{\zeta})(u_+,u_-,\psi) := \Gamma_{0,red}(q)(u,(i\hat{\tau}+\hat{\gamma})\psi, i\hat{\eta}\psi). \tag{4.25}$$

If χ is a local defining function for $\mathcal{C}_{\mathcal{B}}$ near q, then Proposition 3.12 implies

$$\ker \hat{\Gamma}_{0,red}(q,\hat{\zeta}) = \ker \hat{\Gamma}_{\chi}(q,\hat{\zeta}) \text{ for all } \hat{\zeta} \in \overline{S}^d_+. \tag{4.26}$$

Thus, we can rewrite $D_{Lop,m}$ as

$$D_{Lop,m}(q,\hat{\zeta}) = \det\left(\mathbb{E}_-(H_0(q,\hat{\zeta})) \times \mathbb{C}, \ker \hat{\Gamma}_{0,red}(q,\hat{\zeta})\right). \tag{4.27}$$

This form of $D_{Lop,m}$ appears naturally in the low frequency expansion of the modified Evans function (see, e.g., Cor. 5.13).

(b) Since

$$\hat{\Gamma}_{0,red}(q,\hat{\zeta})(u,\psi) = \pi_{H,\mathcal{R}}(q)\left(\Gamma_{0,H}(q)u + \begin{pmatrix} [\mathcal{R}(0,q,i\hat{\tau}+\hat{\gamma},i\hat{\eta})]\psi \\ [\mathcal{R}_z(0,q,i\hat{\tau}+\hat{\gamma},i\hat{\eta})]\psi \\ 0 \end{pmatrix}\right), \tag{4.28}$$

by considering $(u_+,u_-,\psi) = (0,0,1)$, we see that the uniform Lopatinski condition implies

$$\begin{pmatrix} [\mathcal{R}(0,q,i\hat{\tau}+\hat{\gamma},i\hat{\eta})] \\ [\mathcal{R}_z(0,q,i\hat{\tau}+\hat{\gamma},i\hat{\eta})] \end{pmatrix} \neq 0 \text{ for all } \hat{\zeta} \in \overline{S}^d_+. \tag{4.29}$$

This is immediate for $\hat{\zeta} \in S^d_+$. It is true for $\hat{\zeta} \in \overline{S}^d_+$ as well, since $\hat{\Gamma}_{0,red}(q,\hat{\zeta})$ cannot drop rank at points with $\hat{\gamma} = 0$ by (4.26) and the argument used in the proof of Prop. 4.5.

4.2 The standard Evans function

In this section and the next we consider $q \in \mathcal{C}_{\mathcal{B}}$ and define the Evans functions that turn out to govern such nonlinear stability questions as the small viscosity limit of curved viscous shocks (i.e., convergence of viscous to inviscid shocks as viscosity goes to zero) and the long time stability of planar viscous shocks.

For the partially linearized operator $\mathcal{L}$ as in (3.9) consider

$$\begin{aligned} &\mathcal{L}(z,q,\zeta,\partial_z)u = f \text{ on } \pm z \geq 0 \\ &[u] = 0, [u_z] = 0 \text{ on } z = 0. \end{aligned} \tag{4.30}$$

With $U := (u, u_z)$ we can as in (3.14) rewrite (4.30) when $f = 0$ as a $2N \times 2N$ first-order problem:

$$\text{(4.31)} \qquad \begin{aligned} &(a)\ \partial_z U = G(z, q, \zeta)U \text{ on } \pm z \geq 0, \\ &(b)\ \Gamma_s(U_+, U_-) = 0 \text{ on } z = 0, \end{aligned}$$

where $\Gamma_s : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ is given by

$$\text{(4.32)} \qquad \Gamma_s(U_+, U_-) := \begin{pmatrix} [u(0)] \\ [\partial_z u(0)] \end{pmatrix}.$$

For $\hat{\zeta} \in \overline{S}^d_+$, $\rho > 0$, let $E_\pm(q, \hat{\zeta}, \rho)$ denote the set of initial data, $U_\pm(0)$, of bounded solutions of (4.31)(a) on $\pm z \geq 0$. Below we show that the spaces $E_\pm(q, \hat{\zeta}, \rho)$ are C^∞ in $(q, \hat{\zeta}, \rho)$ and satisfy

$$\text{(4.33)} \qquad \dim E_\pm(q, \hat{\zeta}, \rho) = N \text{ for } \hat{\zeta} \in \overline{S}^d_+,\ \rho > 0.$$

Definition 4.12. *1. The* standard Evans function *is defined for $\hat{\zeta} \in \overline{S}^d_+$ and $\rho > 0$ by the $4N \times 4N$ determinant*

$$\text{(4.34)} \qquad D_s(q, \hat{\zeta}, \rho) = \det\left(E_-(q, \hat{\zeta}, \rho) \times E_+(q, \hat{\zeta}, \rho), \ker \Gamma_s\right).$$

2. The profile $W(z, q)$ satisfies the standard low frequency Evans condition at q *if and only if for some positive constants c and ρ_0:*

$$\text{(4.35)} \qquad |D_s(q, \hat{\zeta}, \rho)| \geq c\rho \textit{ for } \hat{\zeta} \in \overline{S}^d_+ \textit{ and } 0 < \rho \leq \rho_0.$$

3. The profile $W(z, q)$ satisfies the standard uniform Evans condition at q *if and only if in addition to (4.35) we have $D_s(q, \hat{\zeta}, \rho) \neq 0$ for $\hat{\zeta} \in \overline{S}^d_+$ and $\rho > 0$.*

Defining $V_\pm$ by $U_\pm = Y_\pm V_\pm$ as in (3.17), we transform (4.31) to

$$\text{(4.36)} \qquad \partial_z V = G_\pm(q, \zeta)V \text{ on } \pm z \geq 0,\ \tilde{\Gamma}(q, \hat{\zeta}, \rho)(V_+, V_-) = 0 \text{ on } z = 0,$$

where $\tilde{\Gamma}(q, \hat{\zeta}, \rho)(V_+, V_-) := \Gamma_s(Y_+V_+, Y_-V_-)$. The argument of [MZ3], Lemma 2.5, shows that for $\hat{\zeta} \in \overline{S}^d_+$, $\rho > 0$, each of $G_\pm(q, \zeta)$ has N eigenvalues counted with their multiplicities in $\Re\mu > 0$ and N eigenvalues in $\Re\mu < 0$. Together with the properties of $Y_\pm(0, q, \zeta)$, this implies (4.33) and the smooth dependence of $E_\pm(q, \hat{\zeta}, \rho)$ on $(q, \hat{\zeta}, \rho)$.

Next we give an alternative form of $D_s(q, \hat{\zeta}, \rho)$ for low frequencies. For ρ small conjugation of (4.36) to HP form (3.20) using

$$\text{(4.37)} \qquad V_\pm = \Lambda_\pm(q, \zeta) \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix}$$

transforms it to

$$\text{(4.38)} \qquad \begin{aligned} &\partial_z \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} = \begin{pmatrix} H_\pm(q, \zeta) & 0 \\ 0 & P_\pm(q, \zeta) \end{pmatrix} \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} \\ &\tilde{\Gamma}_H(q, \zeta)u_H + \tilde{\Gamma}_P(q, \zeta)u_P = 0 \text{ on } z = 0, \end{aligned}$$

where

$$\tilde\Gamma_H(q,\zeta)u_H=\begin{bmatrix}T_{11}(0,q,\zeta)u_H\\T_{21}(0,q,\zeta)u_H\end{bmatrix},\quad \tilde\Gamma_P(q,\zeta)u_P=\begin{bmatrix}T_{12}(0,q,\zeta)u_P\\T_{22}(0,q,\zeta)u_P\end{bmatrix} \tag{4.39}$$

(compare (3.37)).

Remark 4.13. *1.* We note that for $q=(p_+,p_-,s,h)$, $\underline{q}=(\underline{p}_+,\underline{p}_-,0,0)$, $P_{0\pm}(q)$ as in (3.7), and $H_{0\pm}(q,\hat\zeta)$ as in (4.3), we have

$$\begin{aligned}&(a)\ H_\pm(q,\zeta)=\rho\check H_\pm(q,\hat\zeta,\rho),\ \check H_\pm(q,\hat\zeta,\rho)=H_{0\pm}(q,\hat\zeta)+O(\rho)\\&(b)\ P_\pm(q,\zeta)=P_{0\pm}(q)+O(\rho).\end{aligned} \tag{4.40}$$

2. For $\rho>0$ small let us set

$$\begin{aligned}&(a)\mathbb{E}_-(H(q,\zeta)):=\mathbb{E}_-(H_+(q,\zeta))\times\mathbb{E}_+(H_-(q,\zeta))\subset\mathbb{C}^{2N}\\&(b)\mathbb{E}_-(P(q,\zeta)):=\mathbb{E}_-(P_+(q,\zeta))\times\mathbb{E}_+(P_-(q,\zeta))\subset\mathbb{C}^{2N}.\end{aligned} \tag{4.41}$$

3. For $\check H_\pm(q,\hat\zeta,\rho)$ as in (4.40), $\hat\zeta\in\overline{S}^d_+$, and $\rho>0$ small we clearly have

$$\mathbb{E}_-(H(q,\zeta))=\mathbb{E}_-(\check H(q,\hat\zeta,\rho)). \tag{4.42}$$

Proposition 4.14. *1. The spaces appearing in* (4.41)*(a) have dimensions $N-1+k$, $N-R_-$, and $N-L_+$, respectively.*

2. The spaces appearing in (4.41)*(b) have dimensions $N+1-k$, R_-, and L_+, respectively.*

3. Analogously, one can define spaces $\mathbb{E}_+(H(q,\zeta))$ and $\mathbb{E}_+(P(q,\zeta))$ of dimensions $N+1-k$ and $N-1+k$ respectively.

4. The spaces in (4.41) *are C^∞ in $(q,\hat\zeta,\rho)$ for $\hat\zeta\in\overline{S}^d_+$ and $\rho>0$ small.*

Proof. The dimensions of $\mathbb{E}_\mp(P_\pm(q,\zeta))$ follow directly from Defn. 2.3 and the fact that

$$P_\pm(q,\zeta)=b(h)\mathcal{A}_d(p_\pm,s,h)+O(\rho)\ \ (3.21). \tag{4.43}$$

The dimensions of $\mathbb{E}_\mp(H_\pm(q,\zeta))$ then follow immediately from

$$\dim\mathbb{E}_-(G_+(q,\zeta))=N,\ \dim\mathbb{E}_+(G_-(q,\zeta))=N. \tag{4.44}$$

The C^∞ dependence of the spaces on $(q,\hat\zeta,\rho)$ follows from the absence of pure imaginary eigenvalues of $H_\pm(q,\zeta)$ and $P_\pm(q,\zeta)$ for $\hat\zeta\in\overline{S}^d_+$, $\rho>0$.

□

Remark 4.15. Using the properties of the conjugators $T_\pm$ and (4.38), we see that up to a C^∞ factor bounded away from 0, the *standard Evans function* is given for $\hat\zeta\in\overline{S}^d_+$ and $\rho>0$ small by the $4N\times4N$ determinant:

$$\tilde D_s(q,\hat\zeta,\rho):=\det\Big(\mathbb{E}_-(H(q,\zeta))\times\mathbb{E}_-(P(q,\zeta)),\ker\tilde\Gamma_{H,P}(q,\zeta)\Big), \tag{4.45}$$

where $\tilde{\Gamma}_{H,P} : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \to \mathbb{C}^{2N}$ is given by

$$\tilde{\Gamma}_{H,P}(q,\zeta)(u_H, u_P) := \tilde{\Gamma}_H(q,\zeta)u_H + \tilde{\Gamma}_P(q,\zeta)u_P. \tag{4.46}$$

The operators

$$T_\pm(z,q,\zeta) := Y_\pm(z,q,\zeta)\Lambda_\pm(q,\zeta) \tag{4.47}$$

map solutions of (4.38) to solutions of (4.31). For $\rho > 0$ choose smooth bases

$$\{u^\pm_{H,j}(q,\hat{\zeta},\rho)\}, \ \ \{u^\pm_{P,k}(q,\zeta)\} \tag{4.48}$$

of $\mathbb{E}^-(H(q,\zeta))$ and $\mathbb{E}^-(P(q,\zeta))$, respectively, and set

$$\begin{aligned} S^\pm_j(z,q,\hat{\zeta},\rho) &= \begin{pmatrix} s^\pm_j \\ \partial_z s^\pm_j \end{pmatrix} = T_\pm(z,q,\zeta)\begin{pmatrix} e^{zH_\pm(q,\zeta)}u^\pm_{H,j}(q,\hat{\zeta},\rho) \\ 0 \end{pmatrix} \\ F^\pm_k(z,q,\zeta) &= \begin{pmatrix} f^\pm_j \\ \partial_z f^\pm_j \end{pmatrix} = T_\pm(z,q,\zeta)\begin{pmatrix} 0 \\ e^{zP_\pm(q,\zeta)}u^\pm_{P,j}(q,\zeta) \end{pmatrix}. \end{aligned} \tag{4.49}$$

The set of functions $\{S^+_j, j = 1,\dots,N-R_-\} \cup \{F^+_k, k = 1,\dots,R_-\}$ is a basis for the space of solutions to (4.31)(a) which decay to zero, when $\rho > 0$, as $z \to +\infty$. We refer to the S^+_j, F^+_k as *slow modes* and *fast modes* respectively. Similar statements apply to the S^-_j, F^-_k, where now $j = 1,\dots,N-L_+$ and $k = 1,\dots,L_+$.

Remark 4.16. *1.* Define the $2N \times 2N$ determinant (suppressing evaluation at $(0,q,\hat{\zeta},\rho)$)

$$\mathbb{D}_s(q,\hat{\zeta},\rho) := \det\begin{pmatrix} S^+_j & S^-_k & F^+_l & F^-_m \end{pmatrix}. \tag{4.50}$$

Performing a few row and column operations shows that $\mathbb{D}_s$ is equal, up to a sign, to the $4N \times 4N$ determinant

$$\det\begin{pmatrix} S^+_j & F^+_l & 0 & 0 & \tilde{e}_n & \tilde{f}_n \\ 0 & 0 & S^-_k & F^-_m & \tilde{e}_n & \tilde{f}_n \end{pmatrix}, \tag{4.51}$$

where, with $\{e_n, n = 1,\dots,N\}$ a basis for $\mathbb{C}^N$, we've set $\tilde{e}_n = \begin{pmatrix} e_n \\ 0 \end{pmatrix}$, $\tilde{f}_n = \begin{pmatrix} 0 \\ e_n \end{pmatrix}$. But the determinant in (4.51) is just $D_s(q,\hat{\zeta},\rho)$ (4.34), so the low frequency standard Evans condition can equivalently be expressed in terms of $\mathbb{D}_s$.

2. For $\hat{\gamma} > 0$ the functions $u^\pm_{H,j}(q,\hat{\zeta},\rho)$ can be chosen to extend smoothly to $[0,\rho_0)$, since $H_{0,\pm}(q,\hat{\zeta})$ (4.40) has no pure imaginary eigenvalues when $\hat{\gamma} > 0$. Hence the same is true for the Evans functions D_s and $\mathbb{D}_s$. Note also that the $u^\pm_{H,j}(q,\hat{\zeta},0)$ span $\mathbb{E}_\mp(H_{0\pm}(q,\hat{\zeta})$.

3. a-transversality is equivalent to the condition that the kernel of

$$\tilde{\Gamma}_P(q,0) : \mathbb{E}_-(P(q,0)) \to \mathbb{C}^{2N} \tag{4.52}$$

be of dimension one. We know that $(c_+(q), c_-(q))$ is in the kernel, where

$$\begin{pmatrix} 0 \\ c_\pm(q) \end{pmatrix} := T_\pm^{-1}(0,q,0) \begin{pmatrix} W_z(0,q) \\ W_{zz}(0,q) \end{pmatrix}, \tag{4.53}$$

so $D_s(q,\hat\zeta,\rho)$ must vanish to at least first order at $\rho = 0$ when $\hat\gamma > 0$. When the low frequency Evans condition holds, D_s vanishes to precisely first order at $\rho = 0$, so weak transversality holds.

4.3 The modified Evans function

The translational degeneracy in the partially linearized problem represented by the nontrivial kernel of (4.52) is a serious obstacle to proving robust L^2 estimates. Thus, we are led as in [GMWZ3] to consider the fully linearized parabolic problem with an extra transmission condition

$$\begin{aligned} &\mathcal{L}(z,q,\zeta,\partial_z)u - \psi\mathcal{L}_1(z,q,\zeta) = f \\ &[u] = 0,\ [u_z] = 0,\ \underline{w}_z(0)\cdot u^+ + c_0(\zeta)\psi = 0. \end{aligned} \tag{4.54}$$

The operators the operators $\mathcal{L}$ and $\mathcal{L}_1$ are given explicitly in (3.9) and $c_0(\zeta) = i\tau + \gamma + |\eta|^2$. The choice of the third transmission condition here is related to the choice in (3.5) and (3.6); again, it serves to remove the translational degeneracy. In this section we define a modified Evans function for (4.54) which turns out to be bounded away from zero for $\rho > 0$ small when the standard low frequency Evans condition (4.35) holds.

Setting

$$\mathcal{L}_1(z,q,\zeta) = \rho\check{\mathcal{L}}_1(z,q,\hat\zeta,\rho),\ \ c_0(\zeta) = \rho\check c_0(\hat\zeta,\rho),\ \ \phi = \rho\psi \tag{4.55}$$

we can rewrite (4.54):

$$\begin{aligned} &\mathcal{L}(z,q,\zeta,\partial_z)u - \phi\check{\mathcal{L}}_1(z,q,\hat\zeta,\rho) = f \text{ on } \pm z \geq 0, \\ &[u] = 0,\ [u_z] = 0,\ \underline{w}_z(0)\cdot u^+ + \check c_0(\hat\zeta,\rho)\phi = 0. \end{aligned} \tag{4.56}$$

Let $\widetilde{\mathbb{E}}(q,\hat\zeta,\rho)$ denote the space of triples $(U_0^-, U_0^+, \phi) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N} \times \mathbb{C}$ with $U_0^\pm = (u_0^\pm, v_0^\pm)$, such that the solutions $u^\pm$ of

$$\mathcal{L}u^\pm - \phi\check{\mathcal{L}}_1 = 0 \text{ on } \pm z \geq 0, \qquad u^\pm(0) = u_0^\pm, \quad \partial_z u^\pm(0) = v_0^\pm$$

are bounded at infinity. Let $\ker\widetilde\Gamma(q,\hat\zeta,\rho)$ denote the set of $(U_0^-, U_0^+, \phi) \in \mathbb{C}^{2N} \times \mathbb{C}^{2N} \times \mathbb{C}$ such that

$$U_0^- = U_0^+, \qquad \underline{w}_z(0)\cdot u^+(0) + \check c_0(\hat\zeta,\rho)\phi = 0. \tag{4.57}$$

Definition 4.17. *The* modified Evans function *is the* $(4N+1)\times(4N+1)$ *determinant*

$$\widetilde D(q,\hat\zeta,\rho) = \det\left(\widetilde{\mathbb{E}}(q,\hat\zeta,\rho), \ker\widetilde\Gamma(q,\hat\zeta,\rho)\right). \tag{4.58}$$

Parallel to what we did in Proposition 3.2 and Corollary 3.4 for the linearized profile equations, in the next Proposition we recast (4.56) in an equivalent form where the operator $\check{\mathcal{L}}_1$ no longer appears. For this we need a good extension of $W_z(z,q)$ to nonzero frequencies.

We recall that

$$\mathcal{W}_\pm(z,q,0) := (W_z(z,q), W_{zz}(z,q)) \text{ on } \pm z \geq 0 \tag{4.59}$$

satisfies

$$\partial_z \mathcal{W} = G(z,q,0)\mathcal{W} \text{ on } \pm z \geq 0,\ \Gamma_s(\mathcal{W}_+, \mathcal{W}_-) = 0 \text{ on } z = 0. \tag{4.60}$$

We can smoothly extend $\mathcal{W}_\pm$ to $|\zeta|$ small as solutions of

$$\partial_z \mathcal{W} = G(z,q,\zeta)\mathcal{W} \text{ on } \pm z \geq 0 \tag{4.61}$$

by setting, for $T_\pm$ as in (4.47),

$$\mathcal{W}_\pm(z,q,\zeta) = T_\pm(z,q,\zeta) \begin{pmatrix} 0 \\ e^{zP_\pm(q,\zeta)}\pi_\pm(q,\zeta)c_\pm(q) \end{pmatrix}. \tag{4.62}$$

Here $\pi_\pm(q,\zeta)$ are, respectively, the projections of $\mathbb{C}^N$ onto $\mathbb{E}_-(P_+(q,\zeta))$, $\mathbb{E}_+(P_-(q,\zeta))$ (along $\mathbb{E}_+(P_+(q,\zeta))$, $\mathbb{E}_-(P_-(q,\zeta))$ respectively), and $c_\pm(q)$ are elements of $\mathbb{E}_\mp(P_\pm(q,0))$ defined by (4.62) at $\zeta = 0$.

Proposition 4.18. *For $\hat\zeta \in \overline{S}^d_+$ and $0 \leq \rho \leq \rho_0$ small, there exist C^∞ functions $\boldsymbol{R}_\pm(z,q,\zeta)$ on $\pm z \geq 0$, exponentially decaying to zero as $z \to \pm\infty$, and satisfying:*

$$\begin{cases} \mathcal{L}(z,q,\zeta,\partial_z)\boldsymbol{R} = \mathcal{L}_1(z,q,\zeta) \quad \text{on } \pm z \geq 0, \\ \underline{w}_z(0) \cdot \boldsymbol{R}_\pm(0,q,\zeta) = -c_0(\zeta) := -(i\tau + \gamma + |\eta|^2), \quad \boldsymbol{R}^\pm(z,q,0) = 0. \end{cases} \tag{4.63}$$

Define $\check{\boldsymbol{R}}_\pm(z,q,\hat\zeta,\rho)$ by

$$\boldsymbol{R}(z,q,\zeta) = \rho\check{\boldsymbol{R}}_\pm(z,q,\hat\zeta,\rho). \tag{4.64}$$

The functions $\boldsymbol{R}_\pm(z,q,\zeta)$ can be constructed so that

$$\mathcal{R}_\pm(z,q,i\hat\tau + \hat\gamma, i\hat\eta) := \check{\boldsymbol{R}}_\pm(z,q,\hat\zeta,0) \tag{4.65}$$

is linear in $(i\hat\tau + \hat\gamma, i\hat\eta)$. It then makes sense to consider $\mathcal{R}_\pm(z,q,\dot s,\dot h)$ for $(\dot s,\dot h) \in \mathbb{R}^d$, and these functions have all the properties of the functions constructed in Proposition 3.2. In particular, we have

$$\underline{w}_z(0) \cdot \mathcal{R}_\pm(0,q,i\hat\tau + \hat\gamma, i\hat\eta) = \underline{w}_z(0) \cdot \check{\boldsymbol{R}}_\pm(0,q,\hat\zeta,0) = -(i\hat\tau + \hat\gamma). \tag{4.66}$$

Proof. The proof is quite similar to that of Proposition 3.2. In particular, the first N-dimensional component of $\mathcal{W}_\pm(z,q,\zeta)$ (4.62) plays the role of $W_z(z,q)$ in the earlier proof. We construct

$$\boldsymbol{R}(z,q,\zeta) = S_0(z,q,\zeta)(i\tau + \gamma) + \sum_{j=1}^{d-1} S_j(z,q,\zeta)i\eta_j, \tag{4.67}$$

where the $S_j^\pm$ are C^∞ functions chosen to satisfy:

$$(4.68)\qquad \begin{aligned} &\mathcal{L}(z,q,\zeta,\partial_z)S_0^\pm = A_0(W)W' \text{ on } \pm z \geq 0\\ &\mathcal{L}(z,q,\zeta,\partial_z)S_j^\pm = A_j(W)W' + 2h_j W'' - i\eta_j W',\ j=1,\dots,d-1.\\ &\underline{w}_z(0)\cdot S_0^\pm(0,q,\zeta) = -1,\ \ \underline{w}_z(0)\cdot S_j^\pm(0,q,\zeta) = i\eta_j. \end{aligned}$$

The interior forcing terms are exponentially decaying, so exponentially decaying functions S_j with these properties are readily constructed by using the conjugators $T_\pm$ to reduce to HP form (4.38) (see [GMWZ3], Lemma 3.14 for details). The functions

$$(4.69)\qquad \mathcal{R}(z,q,\dot{s},\dot{h}) = S_0(z,q,0)\dot{s} + \sum_{j=1}^{d-1} S_j(z,q,0)\dot{h}$$

then have all the properties of the functions constructed in Proposition 3.2. □

Remark 4.19. Henceforth, we'll use the functions $\mathcal{R}(z,q,\dot{s},\dot{h})$ as in (4.69) in place of the functions constructed in Proposition 3.2.

With $\check{\boldsymbol{R}}^\pm$ as in (4.64), for $0 \leq \rho \leq \rho_0$ the problem (4.56) is equivalent to

$$(4.70)\qquad \begin{cases} u^\pm = v^\pm + \phi\check{\boldsymbol{R}}^\pm\\ \mathcal{L}(z,q,\zeta,\partial_z)v^\pm = f^\pm \quad \text{on } \pm z \geq 0\,,\\ [v(0)] + \phi[\check{\boldsymbol{R}}(0)] = 0 \quad [\partial_z v(0)] + \phi[\partial_z\check{\boldsymbol{R}}(0)] = 0,\\ \underline{w}_z(0)\cdot v^+(0) = 0\,. \end{cases}$$

As before let $\mathbb{E}_\pm(q,\hat{\zeta},\rho)$ be the set of $V^\pm = (v_0^\pm, v_1^\pm) \in \mathbb{C}^{2N}$ such that the solutions of

$$(4.71)\qquad \mathcal{L}v^\pm = 0 \text{ on } \pm z \geq 0,\ v^\pm(0) = v_0^\pm,\ \partial_z v^\pm(0) = v_1^\pm$$

are bounded at $\pm\infty$. Then,

$$\widetilde{\mathbb{E}}(q,\hat{\zeta},\rho) = \mathcal{J}\big(\mathbb{E}_- \times \mathbb{E}_+ \times \mathbb{C}\big)$$

where

$$\mathcal{J}(q,\hat{\zeta},\rho):\quad (V^-,V^+,\phi) \mapsto (V^- + \phi\mathrm{R}^-, V^+ + \phi\mathrm{R}^+, \phi)$$

with

$$\mathrm{R}^\pm(\hat{\zeta},\rho) = {}^t\big(\check{\boldsymbol{R}}^\pm(0), \partial_z\check{\boldsymbol{R}}^\pm(0)\big)\,.$$

Moreover, $\ker\widetilde{\Gamma} = \mathcal{J}\mathbb{G}'$ with

$$\mathbb{G}'(q,\hat{\zeta},\rho) = \big\{(V^-,V^+,\phi) \in \mathbb{C}^{2N}\times\mathbb{C}^{2N}\times\mathbb{C} : V^+ - V^- = \phi(\mathrm{R}^- - \mathrm{R}^+),\ \ell\cdot V^+ = 0\big\}$$

where $\ell = {}^t(\underline{w}_z(0), 0)$. Therefore, the Evans function of the problem (4.70)

$$(4.72)\qquad D_m(q,\hat{\zeta},\rho) := \det\Big(\mathbb{E}_- \times \mathbb{E}_+ \times \mathbb{C}, \mathbb{G}'\Big),\ 0 < \rho \leq \rho_0$$

satisfies:

$$(4.73)\qquad \frac{1}{C}|\widetilde{D}(q,\hat{\zeta},\rho)| \leq |D_m(q,\hat{\zeta},\rho)| \leq C|\widetilde{D}(q,\hat{\zeta},\rho)| \text{ for } 0 < \rho \leq \rho_0.$$

Definition 4.20. *1. The profile $W(z,q)$ satisfies the* modified low frequency Evans condition at q *when there exist positive constants c and ρ_0 such that*

$$|\tilde{D}(q,\hat{\zeta},\rho)| \geq c \text{ for } 0 < \rho \leq \rho_0,\ \hat{\zeta} \in \overline{S}^d_+. \tag{4.74}$$

Equivalently, one can replace $\tilde{D}$ by D_m in (4.74)*.*

2. The profile $W(z,q)$ satisfies the modified uniform Evans condition at q *when in addition to* (4.74) *we have $\tilde{D}(q,\hat{\zeta},\rho) \neq 0$ for $\hat{\zeta} \in \overline{S}^d_+$, $\rho > 0$.*

3. Henceforth, it will be convenient to define $D_m(q,\hat{\zeta},\rho) := \tilde{D}(q,\hat{\zeta},\rho)$ in $\rho > \rho_0$ and to drop the notation $\tilde{D}$.

Remark 4.21. Observe that no transversality assumptions are needed to define the standard and modified Evans functions.

4.4 Inviscid and viscous continuity

In view of the noncompactness of S^d_+ the question arises as to whether or not the stability conditions defined in the last few sections must necessarily hold near $\underline{q}$ if they hold at $\underline{q}$. To address this and also for later use we give the following definition:

Definition 4.22. *(a) We say* inviscid continuity *holds at $q_0 \in \mathcal{C}$ when the vector bundle $\mathbb{E}_-(H_0(q,\hat{\zeta}))$ has a continuous extension from S^d_+ to $\overline{S}^d_+$ for q near q_0.*

(b) We say viscous continuity *holds at $q_0 \in \mathcal{C}$ when the vector bundle $\mathbb{E}_-(\check{H}(q,\hat{\zeta},\rho))$ has a continuous extension from $\overline{S}^d_+ \times (0,\infty)$ to $\overline{S}^d_+ \times [0,\infty)$ for q near q_0.*

Remark 4.23. *1.* Inviscid continuity holds for both fast and slow shocks in inviscid MHD and more generally, for Friedrichs symmetrizable systems for which all characteristic roots of the linearized operator are either geometrically regular or totally nonglancing (Theorem 5.6, [MZ2]). More generally still, inviscid continuity is a necessary condition for the existence of a smooth K-family of inviscid symmetrizers.

2. The above structural conditions on the hyperbolic part are not enough to guarantee viscous continuity; for example, viscous continuity can fail when there is viscous coupling between incoming and outgoing crossing eigenvalues (Proposition 6.5, [GMWZ5]). The existence of a smooth K-family of viscous symmetrizers implies viscous continuity. Viscous continuity holds for fast (i.e., extreme) shocks in viscous MHD, but not for slow shocks.

3. Since $H_0(q,\hat{\zeta}) = \check{H}(q,\hat{\zeta},0)$, viscous continuity implies inviscid continuity.

4. Viscous continuity holds when the hyperbolic problem has characteristics of constant multiplicity [MZ1].

5. When $d = 1$ variable multiplicities are impossible, so we always have viscous continuity.

Remark 4.24. 1. If both the uniform Lopatinski condition and inviscid continuity hold at $\underline{q}$, then it is clear by compactness of $\overline{S}^d_+$ and continuity that the uniform Lopatinski condition holds for q near $\underline{q}$. Similarly, if the low frequency standard Evans condition and viscous continuity hold at $\underline{q}$, then the standard low frequency Evans condition holds for q near $\underline{q}$.

2. The determinants $D_{Lop}(q,\hat\zeta)$ and $D_{Lop,m}(q,\zeta)$ were defined for q belonging to some shock manifold $\mathcal{C}$ (or $\mathcal{C}_{\mathcal{B}}$). The same definition makes sense for all q in a small $\mathbb{R}^{2N+d}$ neighborhood of $\mathcal{C}$. Similarly, if $q=(p_+,p_-,s,h)\in\mathcal{C}_{\mathcal{B}}$ and $(u_+',u_-',s',h')\in\mathbb{R}^{2N+d}$ is sufficiently small, the definitions of $D_s(q,\hat\zeta,\rho)$ and $D_m(q,\hat\zeta,\rho)$ make sense if the linearizations are computed at

$$(W_\pm(z,q)+u_\pm',s+s',h+h') \tag{4.75}$$

instead of at $(W(z,q),s,h)$. This extension is important for obtaining estimates in the linearized problems that arise in the iteration schemes used to prove the nonlinear stability theorems.

5 Low frequency analysis of the Evans functions

First, we present an important technical lemma that is needed for the low frequency analysis of the standard Evans function; in particular, it is used in the proofs of Theorems 5.2 and 5.14.

For $\mathcal{W}_\pm(z,q,\zeta)$ as constructed in 4.62, let us define the variations of the profile at $\rho=0$:

$$\mathcal{Z}_\pm(z,q,\hat\zeta,0):=\partial_\rho|_{\rho=0}\mathcal{W}_\pm(z,q,\zeta),\quad \mathcal{Z}_\pm=(\mathcal{Z}^1_\pm,\mathcal{Z}^2_\pm)\in\mathbb{C}^{2N}. \tag{5.1}$$

The lemma permits us to replace $\mathcal{Z}^1_\pm(z,q,\hat\zeta,0)$ by $-\check{\mathbf{R}}_\pm(z,q,\hat\zeta,0)$ in some computations.

Lemma 5.1 (Variation of the extended W_z at $\rho=0$). *(1) For $\mathcal{Z}^1_\pm$ as in* (5.1) *and $\check{\boldsymbol{R}}$ as in* (4.70) *we have*

$$\mathcal{Z}^1_\pm(z,q,\hat\zeta,0)+\check{\boldsymbol{R}}_\pm(z,q,\hat\zeta,0)=T_{12\pm}(z,q,0)e^{zP_\pm(q,0)}u_{P,z,\pm}(q,\hat\zeta) \tag{5.2}$$

for some $u_{P,z\pm}(q,\hat\zeta)\in\mathbb{E}_\mp(P_\pm(q,0))$.

(2) For $\mathcal{Z}^1_\pm$ as in (5.1) *and $\phi_\pm$ as in Prop. 3.10 we have*

$$\mathcal{Z}^1_\pm(z,q,\hat\zeta,0)+\nabla_{s,h}\phi_\pm(z,p_\pm,s,h,a_\pm)(i\hat\tau+\hat\gamma,i\hat\eta)=\nabla_{a\pm}\phi_\pm(z,p_\pm,s,h,a_\pm)\dot a_{z\pm}(q,\hat\zeta) \tag{5.3}$$

for some $\dot a_{z\pm}(q,\hat\zeta)\in\mathbb{E}_\mp(P_\pm(q,0))$.

Proof. **1.** Noting that $\check{\mathbf{R}}_\pm(z,q,\hat\zeta,0)=\partial_\rho|_{\rho=0}\mathbf{R}_\pm(z,q,\zeta)$, we apply $\partial_\rho|_{\rho=0}$ to the equation (recall (4.63))

$$\mathcal{L}(z,q,\zeta,\partial_z)\mathbf{R}_\pm=\mathcal{L}_1(z,q,\zeta) \tag{5.4}$$

to find (for $\mathcal{L}_0(z,q,\partial_z)=\mathcal{L}(z,q,0,\partial_z)$ as in (3.6))

$$\mathcal{L}_0(z,q,\partial_z)\check{\mathbf{R}}_\pm(z,q,\hat\zeta,0)=A_0(W)W'(i\hat\tau+\hat\gamma)+\sum_{j=1}^{d-1}A_j(W)W'i\hat\eta_j+2\sum_{j=1}^{d-1}h_ji\hat\eta_jW''. \tag{5.5}$$

2. Applying $\partial_\rho|_{\rho=0}$ to

$$\mathcal{L}(z,q,\zeta,\partial_z)\mathcal{W}_\pm(z,q,\zeta)=0 \tag{5.6}$$

and using (4.59), we find $-\mathcal{Z}^1_\pm(z,\underline{q},\hat\zeta,0)$ satisfies the same equation:

$$(5.7)\quad -\mathcal{L}_0(z,q,\partial_z)\mathcal{Z}^1_\pm(z,q,\hat\zeta,0)=A_0(W)W'(i\hat\tau+\hat\gamma)+\sum_{j=1}^{d-1}A_j(W)W'i\hat\eta_j+2\sum_{j=1}^{d-1}h_j i\hat\eta_j W''.$$

For each $\hat\zeta$ the sum $\mathcal{Z}^1_\pm(z,q,\hat\zeta,0)+\check{\mathbf{R}}_\pm(z,q,\hat\zeta,0)$ is thus a solution of $\mathcal{L}_0(z,q,\partial_z)\dot v=0$ which decays exponentially to zero as $z\to\pm\infty$; hence it must be given by the right side of (5.2) for some $u_{P,z,\pm}(q,\hat\zeta)\in\mathbb{E}_\mp(P_\pm(q,0))$ (recall (3.34)).

3. Similarly, arguing as in the proof of Prop. 3.10, we see that the image of

$$\nabla_{s,h}\phi_\pm(z,p_\pm,s,h,a_\pm)(i\hat\tau+\hat\gamma,i\hat\eta)$$

under $\mathcal{L}_0(z,q,\partial_z)$ is again equal to the right side of (5.7). So Proposition 3.10 implies (5.3). □

5.1 Nonconservative Zumbrun-Serre Theorem

In the next Theorem we show that the standard low frequency Evans condition (4.35) implies transversality and the uniform Lopatinski condition. Moreover, if one assumes viscous continuity (Defn. 4.22), then the converse holds. Here we'll use the functions $\tilde\Psi(p,s,h,a)$ defined in (2.34) as well as Proposition 2.8. In particular, we'll use the fact that transversality of $\underline{w}$ allows one to define χ and the manifold $\mathcal{C}_\mathcal{B}$ given by $\chi(p,s,h)=0$.

Theorem 5.2. *(a). Consider the shock profile $\underline{w}(z)=W(z,\underline{q})$, where $\underline{q}=(\underline{p},0,0)$, and suppose the low frequency standard Evans condition holds at $\underline{q}$. That is, suppose there are positive constants c and ρ_0 such that*

$$(5.8)\qquad |\mathbb{D}_s(\underline{q},\hat\zeta,\rho)|:=|\det\begin{pmatrix}S^+_j & S^-_k & F^+_l & F^-_m\end{pmatrix}|\geq c\rho$$

for all $\hat\zeta\in\overline{S}^d_+$ and $0<\rho\leq\rho_0$. Then $\underline{w}(z)$ is transversal (Defn. 2.14) and the uniform Lopatinski condition (4.5) holds at $\underline{q}$. In fact, for $\hat\gamma>0$

$$(5.9)\qquad \mathbb{D}_s(\underline{q},\hat\zeta,\rho)=\rho\beta(\underline{q})D_{Lop}(\underline{q},\hat\zeta)+O_{\hat\gamma}(\rho^2),$$

where $O_{\hat\gamma}(\rho^2)\leq C_{\hat\gamma}\rho^2$ and $\beta(\underline{q})$ is a constant whose nonvanishing is equivalent to a-transversality of $\underline{w}$.

(b). Assume $\underline{w}(z)$ is transversal. If $\mathbb{E}^-(\check H(\underline{q},\hat\zeta,\rho))$ has a continuous extension from $\overline{S}^d_+\times(0,\rho_0)$ to $\overline{S}^d_+\times[0,\rho_0)$ for some $\rho_0>0$, then

$$(5.10)\qquad \mathbb{D}_s(\underline{q},\hat\zeta,\rho)=\rho\beta(\underline{q})D_{Lop}(\underline{q},\hat\zeta)+o(\rho),$$

with an error term that is uniform for $\hat\zeta\in\overline{S}^d_+$. Thus, if the uniform Lopatinski condition also holds, then the low frequency standard Evans condition holds.

Proof. **1. Part (a): Reduce to $\partial_\rho\mathbb{D}_s(\underline{q},\hat\zeta,0)$.** We can suppose $F^+_1(z,\underline{q},\zeta)$ and $F^-_{L_+}(z,\underline{q},\zeta)$ are given respectively by $\mathcal{W}_+(z,\underline{q},\zeta)$ and $\mathcal{W}_-(z,\underline{q},\zeta)$ (4.62), so that

$$(5.11)\qquad f^+_1(z,\underline{q},0)=\underline{w}_z(z)\text{ on }z\geq0;\ f^-_{L+}(z,\underline{q},0)=\underline{w}_z(z)\text{ on }z\leq0.$$

Set $Z_\pm(z,\underline{q},\hat\zeta,0)=(z_\pm,\partial_z z_\pm)$, where

(5.12) $$z_+(z,\underline{q},\hat\zeta,0)=\partial_\rho|_{\rho=0}f_1^+(z,\underline{q},0);\ z_-(z,\underline{q},\hat\zeta,0)=\partial_\rho|_{\rho=0}f_{L+}^-(z,\underline{q},0).$$

Next, fix $\hat\gamma>0$ and subtract the F_1^+ column from the F_{L+}^- column to obtain

(5.13)
$$\begin{aligned}\mathbb{D}_s(\underline{q},\hat\zeta,\rho)&=\rho\det\Big(S_j^+(0,\underline{q},\hat\zeta,0)\quad S_k^-\quad F_l^+(0,\underline{q},0)\quad F_m^-\quad (Z_--Z_+)(0,\underline{q},\hat\zeta,0)\Big)+O_{\hat\gamma}(\rho^2),\\ &:=\rho\mathbb{D}_s^*(\underline{q},\hat\zeta)+O_{\hat\gamma}(\rho^2).\end{aligned}$$

where now $m=1,\dots,L_+-1$. It remains to analyze $\mathbb{D}_s^*(\underline{q},\hat\zeta)$. Observe that the assumption (5.8) implies

(5.14) $$|\mathbb{D}_s^*(\underline{q},\hat\zeta)|\geq c \text{ for all } \hat\zeta\in S_+^d.$$

2. Rewrite in terms of Ψ. Recall

(5.15) $$\Psi(p,s,h,a)=\begin{pmatrix}\phi_+-\phi_-\\ \phi_{+,z}-\phi_{-,z}\end{pmatrix}(0,p,s,h,a)$$

For $\mathcal{L}_0(z,q,\partial_z)$ as in (3.6) we have

(5.16)
$$\begin{aligned}&\mathcal{L}_0(z,\underline{q},\partial_z)s_j^+(z,\underline{q},\hat\zeta,0)=0 \text{ and } \lim_{z\to+\infty}s_j^+(z,\underline{q},\hat\zeta,0)=u_{H,j}^+(\underline{q},\hat\zeta,0)\in\mathbb{E}_-(H_{0+}(\underline{q},\hat\zeta)),\\ &\mathcal{L}_0(z,\underline{q},\partial_z)f_l^+(z,\underline{q},0)=0 \text{ and } \lim_{z\to+\infty}f_l^+(z,\underline{q},0)=0,\end{aligned}$$

so Proposition 3.10 implies

(5.17)
$$\begin{aligned}&s_j^+(z,\underline{q},\hat\zeta,0)=\nabla_{p+}\phi_+(z,\underline{p}_+,0,0,\underline{a}_+)u_{H,j}^+(\underline{q},\hat\zeta,0)+\nabla_{a+}\phi_+(z,\underline{p}_+,0,0,\underline{a}_+)\dot a_{j+}(\underline{q},\hat\zeta)\\ &f_l^+(z,\underline{q},0)=\nabla_{a+}\phi_+(z,\underline{p}_+,0,0,\underline{a}_+)\dot a_{l+}(\underline{q})\end{aligned}$$

for some $\dot a_{j+}(\underline{q},\hat\zeta)$ and $\dot a_{l+}(\underline{q})$ in $\mathbb{E}_-(P_{0+}(\underline{q}))$. Analogous statements apply to the s_k^- and f_m^-.

In addition, Lemma 5.1 implies

(5.18) $$z_+(z,\underline{q},\hat\zeta,0)+\nabla_{s,h}\phi_+(z,\underline{p}_+,0,0,\underline{a}_+)(i\hat\tau+\hat\gamma,i\hat\eta)=\nabla_{a+}\phi_+(z,\underline{p}_+,0,0,\underline{a}_+)\dot a_{z+}(\underline{q},\hat\zeta)$$

for some $\dot a_{z+}(\underline{q},\hat\zeta)\in\mathbb{E}_-(P_{0+}(\underline{q}))$. Again, there is an analogue for z_-.

We claim that, up to a sign,

(5.19)
$$\mathbb{D}_s^*(\underline{q},\hat\zeta)=$$
$$\det\Big(\nabla_{p+}\Psi(\underline{q},\underline{a})u_{H,j}^+\quad\nabla_{p-}\Psi(\underline{q},\underline{a})u_{H,k}^-\quad\nabla_{a+}\Psi(\underline{q},\underline{a})\dot a_{l+}\quad\nabla_{a-}\Psi(\underline{q},\underline{a})\dot a_{m-}\quad\nabla_{s,h}\Psi(\underline{q},\underline{a})(i\hat\tau+\hat\gamma,i\hat\eta)\Big).$$

Here we have used (5.17), (5.18) and, for example, the observation that

$$(5.20)\qquad \nabla_{a+}\begin{pmatrix}\phi_+\\ \phi_{+,z}\end{pmatrix}(0,\underline{p}_+,0,0,\underline{a}_+)=\nabla_{a+}\Psi(\underline{q},\underline{a}).$$

Observe that (5.14) implies the linear independence of the $N-k$ columns of (5.19) indexed by l and m (which is equivalent to a-transversality of $\underline{w}$). In deriving (5.19) we have used those N columns to eliminate the evaluations at $z=0$ of fast decaying terms like the one on the right in (5.18).

Taking $\hat{\eta}=0$ in (5.19) and using (5.14) again, we conclude $\text{rank}\nabla_{a,p,s}\Psi(\underline{p},0,0,\underline{a})=2N$, and thus (a,p,s)-transversality holds for $\underline{w}$.

3. Row and column operations. Next we determine row and column operations that reduce (5.19) to the required block form. We show that after some preparation, these turn out to be the same operations needed to compute $\chi'(\underline{p},0,0)$ starting from $\tilde{\Psi}'(\underline{p},0,0,\underline{a})$.

Consider first the $(2N+1)\times(N+1-k)$ matrix

$$(5.21)\qquad \begin{pmatrix}\nabla_{a+}\tilde{\Psi}(\underline{q},\underline{a})\dot{a}_{l+}(\underline{q}) & \nabla_{a-}\tilde{\Psi}(\underline{q},\underline{a})\dot{a}_{m-}(\underline{q})\end{pmatrix}=\begin{pmatrix}\underline{w}_z(0) & * & * & -\underline{w}_z(0)\\ \underline{w}_{zz}(0) & * & * & -\underline{w}_{zz}(0)\\ \underline{w}_z(0)\cdot\underline{w}_z(0) & * & 0 & 0\end{pmatrix}$$

(recall (5.11)). By a linear change of coordinates in $\mathbb{E}_-(P_0(\underline{q}))$ we arrange so that the matrix on the left in (5.21) is simply

$$(5.22)\qquad \begin{pmatrix}\nabla_{a+}\tilde{\Psi}(\underline{q},\underline{a}) & \nabla_{a-}\tilde{\Psi}(\underline{q},\underline{a})\end{pmatrix}.$$

The $2N\times(N-k)$ submatrix of (5.19) given by the columns indexed by l and m then becomes

$$(5.23)\qquad \Psi_{\tilde{a}}(\underline{q},\underline{a}):=\begin{pmatrix}\nabla_{a_+}\Psi(\underline{q},\underline{a}) & \nabla_{\tilde{a}_-}\Psi(\underline{q},\underline{a})\end{pmatrix},\ \text{where } \tilde{a}_-=(a_{1-},\dots,a_{(L_+-1)-}).$$

Now, a-transversality implies that the equation $\tilde{\Psi}(p,s,h,a)=0$ defines a function $a(p,s,h)$ near $(\underline{p},0,0)$ such that for some choices $\tilde{\Psi}^1$ (resp. $\tilde{\Psi}^2$) of $N+1-k$ (resp. $N+k$) components of $\tilde{\Psi}$ we have

$$(5.24)\qquad \begin{pmatrix}\tilde{\Psi}^1(p,s,h,a(p,s,h))\\ \tilde{\Psi}^2(p,s,h,a(p,s,h))\end{pmatrix}=\begin{pmatrix}0\\ \chi(p,s,h)\end{pmatrix}.$$

In view of (5.21) we can (and do) take the last component of $\tilde{\Psi}$ to be the last component of $\tilde{\Psi}^1$.

Now differentiate both sides of (5.24) at $(\underline{p},0,0)$ to obtain:

$$(5.25)\qquad \begin{pmatrix}0\\ \chi_{p,s,h}\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ \chi_p & \chi_s & \chi_h\end{pmatrix}=\begin{pmatrix}\tilde{\Psi}^1_{p,s,h}+\tilde{\Psi}^1_a(a_p,a_s,a_h)\\ \tilde{\Psi}^2_{p,s,h}+\tilde{\Psi}^2_a(a_p,a_s,a_h)\end{pmatrix}.$$

If we start with the $(2N+1)\times(2N+d+N+1-k)$ matrix $\tilde{\Psi}_{p,s,h,a}(\underline{p},0,0,\underline{a})$, then (5.25) provides column operations that transform this matrix (after row switches) to

$$(5.26)\qquad \begin{pmatrix}0 & 0 & 0 & \tilde{\Psi}^1_a\\ \chi_p & \chi_s & \chi_h & \tilde{\Psi}^2_a\end{pmatrix}.$$

Inspection of (5.21) shows that the transformation to (5.26) can (mostly) be performed without the help of the last column of $\tilde\Psi_{p,s,h,a}(\underline{p},0,0,\underline{a})$. More precisely, if we start with the $2N \times (2N+d+N-k)$ submatrix $\Psi_{p,s,h,\tilde a}(\underline{p},0,0,\underline{a})$, there are column operations that reduce it after some row switches to the submatrix of (5.26) given by

$$\begin{pmatrix} 0 & 0 & 0 & \Psi^1_{\tilde a} \\ \chi_p & \chi_s & \chi_h & \Psi^2_{\tilde a} \end{pmatrix}, \tag{5.27}$$

where $\Psi^2_{\tilde a} = \tilde\Psi^2_{\tilde a}$ and $\Psi^1_{\tilde a}$ is the upper left $(N-k)\times(N-k)$ block of $\tilde\Psi^1_a$.

Since (after the earlier change of coordinates in $\mathbb{E}_-(P_0(\underline{q}))$) $\Psi_{\tilde a}$ appears as a submatrix of (5.19), we conclude that by adding appropriate combinations, which now involve coefficients that depend on $\hat\zeta$, of the columns in (5.23) to the other columns of (5.19), we can reduce (5.19) to the form

$$\begin{pmatrix} 0 & 0 & \Psi^1_{\tilde a} & 0 \\ \chi_{p+}(\underline{q})u^+_{H,j} & \chi_{p-}(\underline{q})u^-_{H,k} & \Psi^2_{\tilde a} & \chi_{s,h}(\underline{q})(i\hat\tau+\hat\gamma, i\hat\eta) \end{pmatrix} \tag{5.28}$$

after row switches. With (5.13) this completes the proof of (5.9), and shows that $\beta(\underline{q})$ is a nonvanishing multiple of the $(N-k)\times(N-k)$ determinant $\det \Psi^1_{\tilde a}(\underline{p},0,0,\underline{a})$.

To see that $\beta(\underline{q}) \neq 0$, note that are obvious column operations that we can perform on (5.22) to reduce it to the form

$$\begin{pmatrix} \underline{w}_z(0) & * & * & 0 \\ \underline{w}_{zz}(0) & * & * & 0 \\ 0 & 0 & 0 & \underline{w}_z(0)\cdot\underline{w}_z(0) \end{pmatrix}, \tag{5.29}$$

and these have no effect on the $2N \times (N-k)$ submatrix (5.23). Thus, the same column operations reduce $\tilde\Psi^1_a(\underline{p},0,0,\underline{a})$, whose determinant is nonzero by a-transversality, to

$$\begin{pmatrix} \Psi^1_{\tilde a} & 0 \\ 0 & \underline{w}_z(0)\cdot\underline{w}_z(0) \end{pmatrix}. \tag{5.30}$$

4. Part (b). The hypothesis on $\underline{w}$ allows us to define χ and write down D_{Lop}. The $u^\pm_{H,j}(\underline{q},\hat\zeta,\rho)$ in (4.49) now extend continuously to $\overline{S}^d_+ \times [0,\rho_0)$, so (5.13) holds with $O_{\hat\gamma}(\rho^2)$ replaced by an error $o(\rho)$ that is uniform for $\hat\zeta \in \overline{S}^d_+$. Repetition of parts 2 and 3 of this proof yields (5.10).

□

Combining Theorem 5.2 and Corollary 4.10 we obtain the immediate corollary:

Corollary 5.3. *When $d \geq 2$, the low frequency standard Evans condition at $\underline{q}$ implies strong transversality of $\underline{w}$:*

$$\begin{aligned} &\mathrm{rank}\nabla_a\tilde\Psi(\underline{p},0,0,\underline{a}) = N+1-k \\ &\mathrm{rank}\nabla_{a,p}\tilde\Psi(\underline{p},0,0,\underline{a}) = 2N+1. \end{aligned} \tag{5.31}$$

5.2 Low frequency analysis of the modified Evans function

First we rewrite the transmission problem from (4.70) (with $f^{\pm} = 0$) as a $2N \times 2N$ first-order system on $\mathbb{R}$. With $V = (v, v_z)$ we obtain:

$$\partial_z V = G(z,q,\zeta)V \text{ on } \pm z \geq 0,\ \Gamma(q,\hat{\zeta},\rho)(V_+,V_-,\phi) = 0 \text{ on } z = 0. \tag{5.32}$$

Conjugation to HP form using

$$V_{\pm} = T_{\pm}(z,q,\zeta)\begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} \text{ recall (3.26)} \tag{5.33}$$

transforms (5.32) to

$$\begin{aligned} &\partial_z \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} = \begin{pmatrix} H_{\pm}(q,\zeta) & 0 \\ 0 & P_{\pm}(q,\zeta) \end{pmatrix} \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} \\ &\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho)(u_H,u_P,\phi) := \Gamma_H(q,\zeta)u_H + \Gamma_P(q,\zeta)u_P + \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},\rho)\phi = 0 \text{ on } z = 0, \end{aligned} \tag{5.34}$$

where

(5.35)

$$\Gamma_H(q,\zeta)u_H = \begin{pmatrix} \begin{bmatrix} T_{11}(0,q,\zeta)u_H \\ T_{21}(0,q,\zeta)u_H \end{bmatrix} \\ (T_{11+}(0,q,\zeta)u_{H+}) \cdot \underline{w}_z(0) \end{pmatrix},\ \Gamma_P(q,\zeta)u_P = \begin{pmatrix} \begin{bmatrix} T_{12}(0,q,\zeta)u_P \\ T_{22}(0,q,\zeta)u_P \end{bmatrix} \\ (T_{12+}(0,q,\zeta)u_{P+}) \cdot \underline{w}_z(0) \end{pmatrix},$$

$$\text{and } \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},\rho)\phi = \begin{pmatrix} [\check{\mathbf{R}}(0,q,\hat{\zeta},\rho)] \\ [\check{\mathbf{R}}_z(0,q,\hat{\zeta},\rho)] \\ 0 \end{pmatrix}\phi.$$

Remark 5.4. For $\Gamma_{\mathcal{R}}$ as in (3.37) we have in view of (4.65)

$$\Gamma_{\mathcal{R}}(i\hat{\tau}+\hat{\gamma},\hat{\eta}) = \begin{pmatrix} [\mathcal{R}(0,q,i\hat{\tau}+\hat{\gamma},i\hat{\eta})] \\ [\mathcal{R}_z(0,q,i\hat{\tau}+\hat{\gamma},i\hat{\eta})] \\ 0 \end{pmatrix} = \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},0). \tag{5.36}$$

For $\Gamma_{0,H}(q)$, $\Gamma_{0,P}(q)$ as in (3.36) we have

$$\Gamma_{0,H}(q) = \Gamma_H(q,0),\ \ \Gamma_{0,P}(q) = \Gamma_P(q,0), \tag{5.37}$$

where the operators on the right in (5.37) are as in (5.34).

The following Lemma is used, for example, in the proofs of Proposition 5.6 and Theorem 5.9.

Lemma 5.5 ([Me1], Lemma 6.2.4). *Consider a subspace $E \subset \mathbb{C}^D$ with $\dim E = D_+$, and let Γ be a $D_+ \times D$ matrix such that $\dim E + \dim \ker \Gamma = D$. If*

$$|\det(E, \ker\Gamma)| \geq c > 0, \tag{5.38}$$

then

$$|e| \leq C|\Gamma e| \textit{ for all } e \in E, \tag{5.39}$$

where $C = c^{-1}|\Gamma^*(\Gamma\Gamma^*)^{-1}|$.

Conversely, if (5.39) holds, then (5.38) is satisfied with $c = (C|\Gamma|)^{-D_+}$.

Proposition 5.6. *If $W(z,q)$ satisfies the modified low frequency Evans condition at $q \in \mathcal{C}$, then $W(z,q)$ is transversal.*

Proof. **1. a-transversality**. For $\rho > 0$ let $\mathbb{E}_-(H(q,\zeta))$ and $\mathbb{E}_-(P(q,\zeta))$ be as defined in (4.41). Note that the boundary operator $\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho)$ in (5.34) maintains full rank on $\mathbb{C}^{4N+1}$ for ρ small since $\Gamma(q,\hat{\zeta},\rho)$ in (5.32) does. Thus, by Lemma 5.5 the modified low frequency Evans condition holds if and only if there exist positive constants C and ρ_0 such that for $\zeta \in \overline{R}^{d+1}_+$ with $0 < |\zeta| \leq \rho_0$

$$
\begin{aligned}
&|u_H| + |u_P| + |\phi| \leq C \left|\Gamma_H(q,\zeta)u_H + \Gamma_P(q,\zeta)u_P + \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},\rho)\phi\right| \\
&\quad \text{for } u_H \in \mathbb{E}_-(H(q,\zeta)),\ \ u_P \in \mathbb{E}_-(P(q,\zeta)),\ \ \phi \in \mathbb{C}.
\end{aligned} \tag{5.40}
$$

Take $u_H = 0$, $\phi = 0$ and use the smoothness of $\mathbb{E}_\mp(P_\pm(q,\zeta))$ at $\zeta = 0$ to conclude

$$
|u_P| \leq C|\Gamma_P(q,0)u_P| \text{ on } \mathbb{E}^-(P(q,0)). \tag{5.41}
$$

In view of Prop. 3.7 and (5.37), this implies a-transversality.

2. (a,p,s)-transversality. Fix $\hat{\zeta} = (\hat{\tau},\hat{\gamma},0)$ and set $\zeta = \rho\hat{\zeta}$. The estimate (5.40) implies that for all $h \in \mathbb{C}^{2N+1}$ and all $0 < \rho \leq \rho_0$ there is a $(u_H(q,\zeta), u_P(q,\zeta), \phi)$ in

$$
\mathbb{E}_-(H(q,\zeta)) \times \mathbb{E}_-(P(q,\zeta)) \times \mathbb{C} \subset \mathbb{C}^{4N+1}
$$

such that

$$
\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho)(u_H,u_P,\phi) = h \text{ and } |(u_H(q,\zeta),u_P(q,\zeta),\phi)| \leq C|h|.
$$

Thus, letting $\rho \to 0$ and using compactness and continuity, we obtain an element $(u_H^*, u_P^*, \phi^*) \in \mathbb{C}^{2N} \times \mathbb{E}_-(P(q,0)) \times \mathbb{C}$ such that

$$
\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},0)(u_H^*, u_P^*, \phi^*) = h.
$$

In view of the choice of $\hat{\zeta}$ and Corollary 3.11, this implies (a,p,s)-transversality.

□

5.2.1 Reduced modified Evans condition

The reduced modified Evans function, $D_{red}(q,\hat{\zeta},\rho)$ defined in this section is needed for the block decompositions of the standard and modified Evans functions. Its definition requires only a-transversality; when viscous continuity and strong transversality hold, D_{red} provides a continuous extension of $D_{Lop}(q,\hat{\zeta})$ to $\rho > 0$.

Suppose $W(z,q)$ is a-transversal. Then the map

$$
\Gamma_P(q,0) : \mathbb{E}_-(P(q,0)) \to \mathbb{F}_P(q) \tag{5.42}
$$

is an isomorphism which extends by continuity to a neighborhood of $\zeta = 0$ (recall the notation (3.41), (4.41)). Thus we get a smooth extension of the decomposition (3.40):

$$
\mathbb{C}^{2N+1} = \mathbb{F}_{H,\mathcal{R}}(q) \oplus \mathbb{F}_P(q,\zeta),\ \ \mathbb{F}_P(q,\zeta) := \Gamma_P(q,\zeta)\mathbb{E}_-(P(q,\zeta)). \tag{5.43}
$$

Denote by $\pi_{H,\mathcal{R}}(q,\zeta)$ and $\pi_P(q,\zeta)$ the associated projections, and define the *reduced (transformed) boundary operator* by

$$\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)(u_H,\phi) := \pi_{H,\mathcal{R}}(q,\zeta)\left(\Gamma_H(q,\zeta)u_H + \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},\rho)\phi\right) \tag{5.44}$$

and the *reduced (transformed) transmission problem*

$$\partial_z u_H - H(q,\zeta)u_H = f_H \text{ on } \pm z \geq 0, \quad \hat{\Gamma}_{red}(q,\hat{\zeta},\rho)(u_H,\phi) = g \text{ on } z = 0. \tag{5.45}$$

The *reduced modified Evans function* is

$$D_{red}(q,\hat{\zeta},\rho) = \det\left(\mathbb{E}_-(H(q,\zeta))\times\mathbb{C}, \ker\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)\right). \tag{5.46}$$

Definition 5.7. *The reduced modified Evans condition at q is satisfied when there exist positive constants c and ρ_0 such that*

$$|D_{red}(q,\hat{\zeta},\rho)| \geq c \textit{ for } \hat{\zeta}\in\overline{S}^d_+,\ 0<\rho\leq\rho_0. \tag{5.47}$$

Remark 5.8. *1.* Using (5.36) we see that for $\hat{\gamma}\geq 0$

$$\hat{\Gamma}_{0,red}(q,\hat{\zeta}) \text{ (as in (4.25))} \ = \hat{\Gamma}_{red}(q,\hat{\zeta},0) \text{ (as in (5.44))}. \tag{5.48}$$

2. In general the estimate

$$\begin{gathered}|u_H| + |\phi| \leq C|\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)(u_H,\phi)| \\ \text{for } (u_H,\phi)\in\mathbb{E}_-(H(q,\zeta))\times\mathbb{C} \text{ and } \hat{\zeta}\in\overline{S}^d_+,\ 0<\rho\leq\rho_0.\end{gathered} \tag{5.49}$$

implies (5.47). By Lemma 5.5 the converse holds when the norm of

$$\left(\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)\hat{\Gamma}^*_{red}(q,\hat{\zeta},\rho)\right)^{-1} \tag{5.50}$$

is uniformly bounded for $\hat{\zeta}\in\overline{S}^d_+$ and $0<\rho\leq\rho_0$. When $d=1$ this is the case, for example, when $W(z,q)$ is transversal. When $d\geq 2$, this holds when $W(z,q)$ is strongly transversal. In each case the transversality hypothesis implies $\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)$ has full rank for $\hat{\zeta}\in\overline{S}^d_+,\ 0\leq\rho\leq\rho_0$.

3. For $\check{H}_\pm(q,\hat{\zeta},\rho)$ as in (4.40), $\hat{\zeta}\in\overline{S}^d_+$, and $\rho>0$ we clearly have

$$D_{red}(q,\hat{\zeta},\rho) = \det\left(\mathbb{E}_-(\check{H}(q,\hat{\zeta},\rho))\times\mathbb{C}, \ker\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)\right). \tag{5.51}$$

4. The function $D_{red}(q,\cdot,\cdot)$ extends continuously to any subset of $\overline{S}^d_+\times[0,\rho_0]$ where $\mathbb{E}_-(\check{H}(q,\hat{\zeta},\rho))$ is continuous and $\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)$ maintains full rank. When such a subset includes a point $(\hat{\zeta},0)$ with $\hat{\zeta}\in S^d_+$, we have

$$D_{Lop,m}(q,\hat{\zeta}) = D_{red}(q,\hat{\zeta},0). \tag{5.52}$$

(recall (4.40)(a)).

Note that $\mathbb{E}_-(\check{H}_\pm(q,\hat{\zeta},\rho))$, which was defined on $\overline{S}^d_+\times(0,\rho_0]$, always has a smooth extension to $S^d_+\times[0,\rho_0]$.

Theorem 5.9. *(a) If $W(z,q)$ satisfies the modified low frequency Evans condition at q (Defn. 4.20), then $W(z,q)$ is transversal and the reduced modified Evans condition (Defn. 5.7) holds.*

(b) The converse holds for $d=1$. When $d\geq 2$, the converse holds if transversality of $W(z,q)$ is replaced by strong transversality.

Proof. **1.** By Prop. 5.6 the modified low frequency Evans condition implies transversality.

Using the splitting (5.43) we see that the estimate (5.40) is equivalent to (suppress (q,ζ) dependence):

$$\begin{aligned}|u_H|+|u_P|+|\phi|\leq C\left(|\hat{\Gamma}_{red}(u_H,\phi)|+|\Gamma_P u_P+\pi_P(\Gamma_H u_H+\Gamma_{\check{\mathbf{R}}}\phi)|\right)\\ \text{for } u_H\in\mathbb{E}_-(H(q,\zeta)),\ \ u_P\in\mathbb{E}_-(P(q,\zeta)),\ \ \phi\in\mathbb{C}.\end{aligned}\tag{5.53}$$

Since Γ_P is surjective from $\mathbb{E}_-(P(q,\zeta))$ onto $\mathbb{F}_P(q,\zeta)$, for all $u_H\in\mathbb{E}_-(H(q,\zeta))$ and $\phi\in\mathbb{C}$ there is $u_P\in\mathbb{E}_-(P(q,\zeta))$ such that

$$\Gamma_P u_P=-\pi_P(\Gamma_H u_H+\Gamma_{\check{\mathbf{R}}}\phi),$$

so (5.53) implies (5.49).

2. Conversely, if the profile has the stated transversality property, the estimate (5.41) holds and remains valid by continuity for ζ in a neighborhood of zero. Using Remark 5.8 part 2, we deduce the estimate (5.49) and this implies (5.53). To see this note

$$\begin{aligned}|u_P|\leq C|\Gamma_P u_P|&\leq|\Gamma_P u_P+\pi_P(\Gamma_H u_H+\Gamma_{\check{\mathbf{R}}}\phi)|+|\pi_P(\Gamma_H u_H+\Gamma_{\check{\mathbf{R}}}\phi)|\\ &\leq|\Gamma_P u_P+\pi_P(\Gamma_H u_H+\Gamma_{\check{\mathbf{R}}}\phi)|+C|\hat{\Gamma}_{red}(u_H,\phi)|.\end{aligned}\tag{5.54}$$

□

It is now a simple matter to prove a Zumbrun-Serre type result for the modified Evans function associated to a nonconservative shock.

Theorem 5.10. *1. If the modified low frequency Evans condition holds, then the profile $W(z,q)$ is transversal and the hyperbolic problem* (3.72) *satisfies the modified uniform Lopatinski condition.*

2. Conversely, when $d=1$ suppose $W(z,q)$ is transversal and that the hyperbolic problem (3.72) *satisfies the modified uniform Lopatinski condition. Then the modified low frequency Evans condition holds.*

When $d\geq 2$ assume strong transversality of $W(z,q)$, the modified uniform Lopatinski condition, and continuous extendability of the vector bundle $\mathbb{E}_-(\check{H}(q,\hat{\zeta},\rho))$ to $\overline{S}^d_+\times[0,\rho_0]$. Then the modified low frequency Evans condition holds.

Proof. **1.** Assuming the modified low frequency Evans condition, from Theorem 5.9 and its proof, we deduce transversality of $W(z,q)$ and the estimate (5.49). If $\hat{\gamma}>0$ every term in (5.49) is continuous up to $\rho=0$ (recall Remark 5.8, part 4), so estimate (5.49) implies,

$$\begin{aligned}|u_H|+|\phi|\leq C|\hat{\Gamma}_{red}(q,\hat{\zeta},0)(u_H,\phi)|\\ \text{for } (u_H,\phi)\in\mathbb{E}_-(\check{H}(q,\hat{\zeta},0))\times\mathbb{C}\text{ and }\hat{\zeta}\in S^d_+.\end{aligned}\tag{5.55}$$

From (4.40)(a) and (5.48), we see that (5.55) implies the modified uniform Lopatinski condition.

2. By Remark 5.8 part 4, continuity of $\mathbb{E}_-(\check{H}_\pm(q,\hat{\zeta},\rho))$ on $\overline{S}^d_+ \times [0,\rho_0]$ (which is automatic when $d=1$) and the given transversality hypotheses imply that the reduced modified Evans function $D_{red}(q,\hat{\zeta},\rho)$ has a continuous extension to $\overline{S}^d_+ \times [0,\rho_1]$ for some $\rho_1 > 0$. The modified uniform Lopatinski condition and (5.52) imply

$$|D_{red}(q,\hat{\zeta},\rho)| \geq c > 0 \tag{5.56}$$

for $\hat{\zeta} \in S^d_+$ and $\rho = 0$. By continuity this extends first to $\hat{\zeta} \in \overline{S}^d_+$ and next to $\rho \in [0,\rho_1]$ for some $\rho_1 > 0$. Thus, the reduced modified Evans condition holds. An application of Theorem 5.9 now yields the converse. □

5.3 Block decomposition of the modified Evans function

A block decomposition was not needed to prove the Zumbrun-Serre result for the modified Evans function, Theorem 5.10, but such decompositions are useful for understanding the relation between the modified and standard Evans functions. In fact, together with Theorem 5.2, they yield a proof that the low frequency standard Evans condition implies the low frequency modified Evans condition (Proposition 5.15), a proof that requires neither continuity of decaying eigenspaces (Defn. 4.22) nor constant multiplicities of hyperbolic characteristics. This fact allows our results to be applied, for example, to viscous MHD, even in the more difficult case of slow shocks.

In this subsection and the next we obtain block decompositions assuming strong transversality of the profile $W(z,q)$. The modified Evans function is $D_m(q,\hat{\zeta},\rho)$ as in (4.72). We'll use the notation

$$\Gamma_H(q,\zeta),\ \Gamma_P(q,\zeta),\ \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},\rho) \tag{5.57}$$

introduced in (5.34), as well as the notation

$$\hat{\Gamma}_{red}(q,\hat{\zeta},\rho),\ D_{red}(q,\hat{\zeta},\rho),\ \text{and } \mathbb{E}_-(H(q,\zeta)),\ \mathbb{E}_-(P(q,\zeta)) \tag{5.58}$$

as in (5.44), (5.46), and (4.41) respectively.

For small ρ we can use the conjugators $T_\pm(z,q,\zeta)$ (4.47) to rewrite D_m up to a nonvanishing factor as

$$D_m(q,\hat{\zeta},\rho) = \det\Big(\mathbb{E}_-(H(q,\zeta)) \times \mathbb{E}_-(P(q,\zeta)) \times \mathbb{C}, \ker\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho)\Big), \tag{5.59}$$

where

$$\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho) : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \times \mathbb{C} \to \mathbb{C}^{2N+1} \tag{5.60}$$

is given as in (5.34) by

$$\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho)(u_H,u_P,\phi) := \Gamma_H(q,\zeta)u_H + \Gamma_P(q,\zeta)u_P + \Gamma_{\check{\mathbf{R}}}(q,\hat{\zeta},\rho)\phi. \tag{5.61}$$

Clearly, the map in (5.60) is surjective, so

$$\dim\ker\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat{\zeta},\rho) = 2N. \tag{5.62}$$

Proposition 5.11. *(a) Assume the profile $W(z,q)$ is strongly transversal. Then for ρ_0 small enough, $0<\rho\leq\rho_0$, and $\hat\zeta\in\overline{S}^d_+$, the modified Evans determinant satisfies*

$$D_m(q,\hat\zeta,\rho)=\beta(q,\zeta)D_{red}(q,\hat\zeta,\rho), \tag{5.63}$$

where $\beta(q,\cdot)$ is a nonvanishing smooth function on a neighborhood of the origin.

(b) When $d=1$ the same result holds if we just assume that $W(z,q)$ is transversal.

Proof. As in the proof of [Me1], Theorem 6.4.1 (which gives an analogous result for Dirichlet boundary layers), the key is to decompose the kernel of $\Gamma_{H,P,\check{\mathbf{R}}}$ as in (5.60) in a suitable way.

1. Define the map $\Pi_{H,\phi}$ by

$$\Pi_{H,\phi}(u_H,u_P,\phi)=(u_H,\phi), \tag{5.64}$$

and for ρ small set

$$\hat{\mathcal{C}}(q,\hat\zeta,\rho):=\Pi_{H,\phi}\left(\ker\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat\zeta,\rho)\cap(\mathbb{C}^{2N}\times\mathbb{E}_-(P(q,\zeta))\times\mathbb{C})\right). \tag{5.65}$$

Observe that

$$\hat{\mathcal{C}}(q,\hat\zeta,\rho)=\ker\hat\Gamma_{red}(q,\hat\zeta,\rho), \tag{5.66}$$

where

$$\hat\Gamma_{red}(q,\hat\zeta,\rho):\mathbb{C}^{2N}\times\mathbb{C}\to\mathbb{F}_{H,\mathcal{R}}(q)\text{ (recall (5.43))}. \tag{5.67}$$

Moreover, we have

$$\begin{gathered}(u_H,u_P,\phi)\in\ker\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat\zeta,\rho)\cap(\mathbb{C}^{2N}\times\mathbb{E}_-(P(q,\zeta))\times\mathbb{C})\Leftrightarrow\\(u_H,\phi)\in\hat{\mathcal{C}}(q,\hat\zeta,\rho)\text{ and }u_P=K(q,\hat\zeta,\rho)(u_H,\phi),\end{gathered} \tag{5.68}$$

where

$$K(q,\hat\zeta,\rho)(u_H,\phi):=-\Gamma_P(q,\zeta)^{-1}\pi_P(q,\zeta)\left(\Gamma_H(q,\zeta)u_H++\Gamma_{\check{\mathbf{R}}}(q,\hat\zeta,\rho)\phi\right). \tag{5.69}$$

This gives the parametrization

$$\begin{gathered}\mathbb{K}(q,\hat\zeta,\rho):=\ker\Gamma_{H,P,\check{\mathbf{R}}}(q,\hat\zeta,\rho)\cap(\mathbb{C}^{2N}\times\mathbb{E}_-(P(q,\zeta))\times\mathbb{C})=\\\{(u_H,K(q,\hat\zeta,\rho)(u_H,\phi),\phi):(u_H,\phi)\in\hat{\mathcal{C}}(q,\hat\zeta,\rho)\}.\end{gathered} \tag{5.70}$$

2. (a,p)-transversality implies that the intersection in (5.68) is transversal for all $\hat\zeta$ and hence of dimension $N+1-k$ when $\rho=0$ (use Prop. 3.7). Thus, $\hat{\mathcal{C}}(q,\hat\zeta,0)$ has dimension $N+1-k$ and by continuity, these properties persist for $\rho>0$ small.

When $d=1$, Corollary 3.11 and (4.65) imply the same can be said about $\hat{\mathcal{C}}$ when (a,p)-transversality is replaced by the weaker assumption of (a,p,s)-transversality.

3. Note that

$$\mathbb{C}^{2N}=\mathbb{E}_-(P(q,\zeta))\oplus\mathbb{E}_+(P(q,\zeta)), \tag{5.71}$$

and consider the map

$$\omega(q,\hat\zeta,\rho) : (u_H, u_{P-}+u_{P+},\phi) \to u_{P+} \tag{5.72}$$

from $\ker \Gamma_{H,P,\check{\mathbf{R}}}(q,\hat\zeta,\rho) \subset \mathbb{C}^{2N}\times\mathbb{C}^{2N}\times\mathbb{C}$ to $\mathbb{E}_+(P(q,\zeta))$. The kernel of ω is the subspace of dimension $N+1-k$ given by (5.70), so ω is surjective. Thus, there is a map

$$K'(q,\hat\zeta,\rho) : \mathbb{E}_+(P(q,\zeta)) \to \ker \Gamma_{H,P,\check{\mathbf{R}}}(q,\hat\zeta,\rho) \tag{5.73}$$

such that $\omega K' = I$. Setting $\mathbb{K}'(q,\hat\zeta,\rho) := K'(q,\hat\zeta,\rho)\mathbb{E}_+(P(q,\zeta))$ we have

$$\ker \Gamma_{H,P,\check{\mathbf{R}}}(q,\hat\zeta,\rho) = \mathbb{K}(q,\hat\zeta,\rho)\oplus\mathbb{K}'(q,\hat\zeta,\rho). \tag{5.74}$$

4. We are now ready to obtain the block decomposition of $D_m(q,\hat\zeta,\rho)$. Suppressing evaluation at $(q,\hat\zeta,\rho)$, we choose bases

(5.75)
$$\{u_{H,j}\}_{j=1,\dots,N-1+k},\ \{u_{P,k}\}_{k=1,\dots,N+1-k},\ \{(v_{H,l},\phi_l)\}_{l=1,\dots,N+1-k},\ \{w_{P,m}\}_{m=1,\dots,N-1+k}$$

of $\mathbb{E}_-(H)$, $\mathbb{E}_-(P)$, $\hat{\mathcal{C}}$, and $\mathbb{E}_+(P)$, respectively. Using (5.74) and writing $K' = (K'_H, K'_P, K'_\phi)$, we can compute D_m as the $(4N+1)\times(4N+1)$ determinant

$$D_m(q,\hat\zeta,\rho) = \det\begin{pmatrix} u_{H,j} & 0 & 0 & v_{H,l} & K'_H(w_{P,m}) \\ 0 & u_{P,k} & 0 & K(v_{H,l},\phi_l) & K'_P(w_{P,m}) \\ 0 & 0 & 1 & \phi_l & K'_\phi(w_{P,m})\end{pmatrix} \tag{5.76}$$

The terms $K(v_{H,l},\phi_l)$ lie in the span of the $u_{P,k}$, so we can eliminate them in the determinant. Switching rows and columns, this shows that, up to a sign,

$$\begin{aligned} D_m(q,\hat\zeta,\rho) &= \det\begin{pmatrix} u_{H,j} & 0 & v_{H,l} \\ 0 & 1 & \phi_l\end{pmatrix} \det\begin{pmatrix} u_{P,k} & K'_P(w_{P,m})\end{pmatrix} \\ &= D_{red}(q,\hat\zeta,\rho)\det\begin{pmatrix} u_{P,k} & w_{P,m}\end{pmatrix}, \end{aligned} \tag{5.77}$$

where we've used $\omega K' = I$ for the last equality. This gives (5.63) with

$$\beta(q,\zeta) = \det\begin{pmatrix} u_{P,k} & w_{P,m}\end{pmatrix} \tag{5.78}$$

up to a sign.

□

Remark 5.12. Note that for bases $\{u_{H,j}\}_{j=1,\dots,N-1+k}$, $\{(v_{H,l},\phi_l)\}_{l=1,\dots,N+1-k}$ of $\mathbb{E}_-(H(q,\zeta))$ and $\hat{\mathcal{C}}(q,\hat\zeta,\rho)$ as above, we have

$$D_{red}(q,\hat\zeta,\rho) = \det\begin{pmatrix} u_{H,j} & v_{H,l}\end{pmatrix},$$

so under the assumptions of Proposition 5.11:

$$D_m(q,\hat\zeta,\rho) = \det\begin{pmatrix} u_{H,j} & v_{H,l}\end{pmatrix}\beta(q,\zeta) \tag{5.79}$$

for $\hat\zeta \in \overline{S}^d_+$ and $0<\rho\le\rho_0$.

Using Remark 5.8 part 4, we obtain the following immediate corollary generalizing Prop. 3.13 of [GMWZ3] to the nonconservative case:

Corollary 5.13. *Assume $W(z,q)$ is strongly transversal and that the vector bundle $\mathbb{E}_-(\check{H}(q,\hat{\zeta},\rho))$ has a continuous extension to $\overline{S}^d_+ \times [0,\rho_0]$. Then, up to a sign,*

$$D_m(q,\hat{\zeta},0) = \beta(q,0)D_{Lop,m}(q,\hat{\zeta}), \tag{5.80}$$

where $\beta(q,0) \neq 0$ is given by (5.78).

When $d=1$, strong tranversality can be replaced by transversality in the above statement.

5.4 Block decomposition of the standard Evans function

The standard Evans function $D_s(q,\hat{\zeta},\rho)$ and the standard low frequency Evans condition were defined in Definition 4.12. The main result of this subsection is the following theorem, which we need in order to show that the low frequency standard Evans assumption implies the low frequency modified Evans assumption. We shall work now with the alternative form $\tilde{D}_s(q,\hat{\zeta},\rho)$ defined in Remark 4.15, and we recall the notations Γ_s (4.32), $\tilde{\Gamma}_{H,P}$ (4.46).

Theorem 5.14. *Assume the profile $W(z,q)$ is strongly transversal. Then, up to a sign,*

$$\tilde{D}_s(q,\hat{\zeta},\rho) = \rho\beta(q,\zeta)D_{red}(q,\hat{\zeta},\rho) + O(\rho^2), \tag{5.81}$$

where β is given by (5.78) and, for some $\rho_0 > 0$, the error is uniform for $\hat{\zeta} \in \overline{S}^d_+$, $0 < \rho \leq \rho_0$. So in particular we have

$$\tilde{D}_s(q,\hat{\zeta},\rho) = \rho D_m(q,\hat{\zeta},\rho) + O(\rho^2) \tag{5.82}$$

with the same kind of error.

Observe that (5.82) follows immediately from (5.81) and Proposition 5.11. Before giving the proof of (5.81), we need some preparation. The kernel of

$$\tilde{\Gamma}_{H,P}(q,\zeta) : \mathbb{C}^{2N} \times \mathbb{C}^{2N} \to \mathbb{C}^{2N} \tag{5.83}$$

has dimension $2N$, but in decomposing the kernel we now have to contend with the fact that $\tilde{\Gamma}_P(q,\zeta)$ degenerates on a one-dimensional subspace of $\mathbb{E}_-(P(q,\zeta))$ as $\zeta \to 0$.

First, recall from (4.62) the extensions of $(W_z(z,q), W_{zz}(z,q))$ to nonzero frequencies:

$$\begin{aligned}&(a)\ \mathcal{W}_\pm(z,q,\zeta) = T_\pm(z,q,\zeta)\begin{pmatrix}0\\ e^{zP_\pm(q,\zeta)}\pi_\pm(q,\zeta)c_\pm(q)\end{pmatrix}\\ &(b)\ \Gamma_s(\mathcal{W}_+,\mathcal{W}_-)(0,q,0) = 0.\end{aligned} \tag{5.84}$$

For $c_\pm(q)$ as in (5.84) let

$$\mathbb{E}^-_s(P(q,0)) := \text{span}\begin{pmatrix}c_+(q)\\ c_-(q)\end{pmatrix} \subset \mathbb{C}^{2N}\ \ (s \text{ for singular}) \tag{5.85}$$

and let $\mathbb{E}^-_n(P(q,0))$ (n for nonsingular) be any complementary subspace satisfying

$$\mathbb{E}_-(P(q,0)) = \mathbb{E}^-_n(P(q,0)) \oplus \mathbb{E}^-_s(P(q,0)) \subset \mathbb{C}^{2N} \tag{5.86}$$

with basis

$$\begin{pmatrix} d_{+,i}(q) \\ d_{-,i}(q) \end{pmatrix},\ i = 1, \dots, N-k. \tag{5.87}$$

Setting

$$\nu_{P,s}(q,\zeta) := \begin{pmatrix} \pi_+(q,\zeta)c_+(q) \\ \pi_-(q,\zeta)c_-(q) \end{pmatrix} \text{ and } \nu_{P,n,i}(q,\zeta) := \begin{pmatrix} \pi_+(q,\zeta)d_{+,i}(q) \\ \pi_-(q,\zeta)d_{-,i}(q) \end{pmatrix}, i = 1, \dots, N-k, \tag{5.88}$$

we smoothly extend the decomposition (5.86) to small $|\zeta|$:

$$\mathbb{E}_-(P(q,\zeta)) = \mathbb{E}^-_n(P(q,\zeta)) \oplus \mathbb{E}^-_s(P(q,\zeta)) \subset \mathbb{C}^{2N}, \tag{5.89}$$

where

$$\mathbb{E}^-_n(P(q,\zeta)) := \text{span}\left(\nu_{P,n,i}(q,\zeta),\ i = 1, \dots, N-k\right) \text{ and } \mathbb{E}^-_s(P(q,\zeta)) := \text{span}\left(\nu_{P,s}(q,\zeta)\right). \tag{5.90}$$

Assume now that $W(z,q)$ is a-transversal. $\tilde{\Gamma}_P(q,0)$ vanishes by (5.84)(b) on $\mathbb{E}^-_s(P(q,0))$, but is nonsingular (by a-transversality) on $\mathbb{E}^-_n(P(q,0))$. So we can write

$$\mathbb{C}^{2N} = \mathbb{F}_n(q) \oplus \mathbb{F}_{H,s}(q), \tag{5.91}$$

where

$$\mathbb{F}_n := \tilde{\Gamma}_P(q,0)\mathbb{E}^-_n(P(q,0)) \tag{5.92}$$

and $F_{H,s}(q)$ is any complementary ($N+k$ dimensional) subspace. The decomposition extends smoothly to $|\zeta|$ small:

$$\mathbb{C}^{2N} = \mathbb{F}_n(q,\zeta) \oplus \mathbb{F}_{H,s}(q), \qquad \mathbb{F}_n(q,\zeta) := \tilde{\Gamma}_P(q,\zeta)\mathbb{E}^-_n(P(q,\zeta)), \tag{5.93}$$

and we let $\pi_n(q,\zeta)$, $\pi_{H,s}(q,\zeta)$ be the associated projections.

We can now define a new reduced boundary operator, $\Gamma_*(q,\zeta)$, that will serve as a replacement for $\hat{\Gamma}_{red}(q,\hat{\zeta},\rho)$ in the proof of Theorem 5.14. Using the decomposition (5.89) to write (with obvious notation)

$$u_P = u_{P,n} + u_{P,s}, \qquad u_P \in \mathbb{E}_-(P(q,\zeta)), \tag{5.94}$$

we define

$$\Gamma_*(q,\zeta) : \mathbb{C}^{2N} \times \mathbb{E}^-_s(P(q,\zeta)) \to \mathbb{F}_{H,s}(q) \quad \text{and } \mathcal{C}^*(q,\zeta) \subset \mathbb{C}^{2N} \times \mathbb{E}^-_s(P(q,\zeta)) \tag{5.95}$$

by

$$\Gamma_*(q,\zeta)(u_H,u_{P,s}) := \pi_{H,s}(q,\zeta)\left(\tilde\Gamma_H(q,\zeta)u_H + \tilde\Gamma_P(q,\zeta)u_{P,s}\right), \qquad \mathcal{C}^*(q,\zeta) = \ker\Gamma_*(q,\zeta).$$

Then, parallel to (5.70), we have the parametrization

$$\begin{aligned}\mathbb{K}_*(q,\zeta) &:= \ker\tilde\Gamma_{H,P}(q,\zeta)\cap\left(\mathbb{C}^{2N}\times\mathbb{E}_-(P(q,\zeta))\right)\\ &= \{(u_H, K_*(q,\zeta)(u_H,u_{P,s}) + u_{P,s}) : (u_H,u_{P,s})\in\mathcal{C}^*(q,\zeta)\},\end{aligned} \tag{5.96}$$

where

$$K_*(q,\zeta)(u_H,u_{P,s}) := -\tilde\Gamma_P(q,\zeta)^{-1}\pi_n(q,\zeta)\left(\tilde\Gamma_H(q,\zeta)u_H + \tilde\Gamma_P(q,\zeta)u_{P,s}\right) \in \mathbb{E}_n^-(P(q,\zeta)). \tag{5.97}$$

A key step in the proof of Theorem 5.14 will be to set up a correspondence between $\mathcal{C}^*(q,\zeta)$ and $\hat{\mathcal{C}}(q,\hat\zeta,\rho)$. With this preparation we can now give the proof of Theorem 5.14.

Proof of Theorem 5.14.

First we decompose $\ker\tilde\Gamma_{H,P}(q,\zeta)$ in a suitable way.

1. Strong transversality implies that for $\zeta = 0$ the intersection $\mathbb{K}_*(q,\zeta)$ (5.96) is transversal and thus of dimension $N+1-k$; hence, $\mathcal{C}^*(q,0)$ has dimension $N+1-k$. By continuity these properties persist for $\rho > 0$ small.

2. Again using (5.71), we consider the map

$$\omega_*(q,\zeta) : (u_H, u_{P-} + u_{P+}) \to u_{P+} \tag{5.98}$$

from $\ker\tilde\Gamma_{H,P}(q,\zeta)\subset\mathbb{C}^{2N}\times\mathbb{C}^{2N}$ to $\mathbb{E}_+(P(q,\zeta))$. The kernel of $\omega_*(q,\zeta)$ is the subspace of $\ker\tilde\Gamma_{H,P}(q,\zeta)$ of dimension $N+1-k$ given by $\mathbb{K}_*(q,\zeta)$ (5.96), so ω_* is surjective. Thus, there is a map

$$K'_*(q,\zeta) : \mathbb{E}_+(P(q,\zeta)) \to \ker\tilde\Gamma_{H,P}(q,\zeta) \tag{5.99}$$

such that $\omega_* K'_* = I$. Setting $\mathbb{K}'_*(q,\zeta) := K'_*(q,\zeta)\mathbb{E}_+(P(q,\zeta))$ we have

$$\ker\tilde\Gamma_{H,P}(q,\zeta) = \mathbb{K}_*(q,\zeta)\oplus\mathbb{K}'_*(q,\zeta). \tag{5.100}$$

3. Suppressing evaluation at $(q,\hat\zeta,\rho)$ and recalling (4.45), we choose bases

$$\{u_{H,j}\}_{j=1,\dots,N-1+k},\ \{u_{P,k}\}_{k=1,\dots,N+1-k},\ \{(\nu_{H,l},\nu_{P,s,l})\}_{l=1,\dots,N+1-k},\ \{w_{P,m}\}_{m=1,\dots,N-1+k} \tag{5.101}$$

of $\mathbb{E}_-(\check H)$, $\mathbb{E}_-(P)$, $\mathcal{C}^*$, and $\mathbb{E}_+(P)$, respectively. Using (5.100) and writing $K'_* = (K'_{*,H}, K'_{*,P})$, we can compute $\tilde D_s$ as the $4N\times 4N$ determinant

$$\tilde D_s(q,\hat\zeta,\rho) = \det\begin{pmatrix} u_{H,j} & 0 & \nu_{H,l} & K'_{*,H}(w_{P,m})\\ 0 & u_{P,k} & \nu_{P,s,l} + K_*(\nu_{H,l},\nu_{P,s,l}) & K'_{*,P}(w_{P,m})\end{pmatrix}. \tag{5.102}$$

The terms $\nu_{P,s,l} + K_*(\nu_{H,l}, \nu_{P,s,l})$ lie in the span of the $u_{P,k}$, so we can eliminate them in the determinant. Switching rows and columns, this shows that, up to a sign,

$$\tilde{D}_s(q,\hat{\zeta},\rho) = \det \begin{pmatrix} u_{H,j} & \nu_{H,l} & 0 & K'_{*,H}(w_{P,m}) \\ 0 & 0 & u_{P,k} & K'_{*,P}(w_{P,m}) \end{pmatrix} = \det \begin{pmatrix} u_{H,j} & \nu_{H,l} \end{pmatrix} \beta(q,\zeta), \tag{5.103}$$

where we've used $\omega_* K'_* = I$ for the last equality and β is as in (5.78). To finish we will show

$$\det \begin{pmatrix} u_{H,j} & \nu_{H,l} \end{pmatrix} (q,\hat{\zeta},\rho) = \rho D_{red}(q,\hat{\zeta},\rho) + O(\rho^2). \tag{5.104}$$

4. To set up a correspondence between $\mathcal{C}^*$ and $\hat{\mathcal{C}}$ it is helpful to write out the boundary conditions more explicitly. We have

$$\begin{aligned} &(u_H, u_{P,s}) \in \mathcal{C}^*(q,\zeta) \Leftrightarrow \exists\, u_{P,n} \in \mathbb{E}_n^-(P(q,\zeta)) \text{ such that} \\ &\begin{bmatrix} T_{11}u_H \\ T_{21}u_H \end{bmatrix} + \begin{bmatrix} T_{12}u_{P,s} \\ T_{22}u_{P,s} \end{bmatrix} + \begin{bmatrix} T_{12}u_{P,n} \\ T_{22}u_{P,n} \end{bmatrix} = 0. \end{aligned} \tag{5.105}$$

On the other hand

$$\begin{aligned} &(u_H, \phi) \in \hat{\mathcal{C}}(q,\hat{\zeta},\rho) \Leftrightarrow \exists\, u_P \in \mathbb{E}_-(P(q,\zeta)) \text{ such that} \\ (a)\quad &\begin{bmatrix} T_{11}u_H \\ T_{21}u_H \end{bmatrix} + \begin{bmatrix} T_{21}u_P \\ T_{22}u_P \end{bmatrix} + \begin{bmatrix} \phi\check{\mathbf{R}}(0,q,\hat{\zeta},\rho) \\ \phi\check{\mathbf{R}}_z(0,q,\hat{\zeta},\rho) \end{bmatrix} = 0 \text{ and} \\ (b)\quad &(T_{11+}u_{H+}) \cdot \underline{w}_z(0) + (T_{12+}u_{P+}) \cdot \underline{w}_z(0) = 0. \end{aligned} \tag{5.106}$$

To proceed further we make a more explicit choice of basis of $\mathcal{C}^*(q,\zeta)$. Strong transversality allows us to choose near $\zeta = 0$ a smooth basis of $\mathcal{C}^*(q,\zeta)$ of the form

$$\left\{ \begin{pmatrix} \nu_{H,l}(q,\zeta) \\ 0 \end{pmatrix}, l = 1,\dots,N-k \right\} \cup \left\{ \begin{pmatrix} \nu_H(q,\zeta) \\ \nu_{P,s}(q,\zeta) \end{pmatrix} \right\}, \tag{5.107}$$

where $\nu_{P,s}(q,0) = (c_+(q), c_-(q))$. Moreover, with

$$\nu_{P,n}(q,\zeta) := K_*(q,\zeta)(\nu_H, \nu_{P,s}) \text{ and } c(q) := (c_+(q), c_-(q)), \tag{5.108}$$

we have

$$\underline{V}(q,\zeta) := \begin{pmatrix} \nu_H(q,\zeta) \\ \nu_{P,s}(q,\zeta) + \nu_{P,n}(q,\zeta) \end{pmatrix} \in \mathbb{K}_*(q,\zeta),\ \begin{pmatrix} \nu_H(q,0) \\ \nu_{P,s}(q,0) \end{pmatrix} = \begin{pmatrix} 0 \\ c(q) \end{pmatrix},\ \nu_{P,n}(q,0) = 0. \tag{5.109}$$

5. Starting with the basis of $\mathcal{C}^*(q,\zeta)$ given by (5.107), we next derive from it a basis for $\hat{\mathcal{C}}(q,\hat{\zeta},\rho)$.

Given $(\nu_{H,l}, 0)$ as in (5.107), there exists $\nu_{P,n,l} \in \mathbb{E}_n^-(P(q,\zeta))$ such that part (a) of (5.106) holds with $u_H = \nu_{H,l}$, $u_P = \nu_{P,n,l}$, and $\phi = 0$. Using (5.84), (5.109) and setting

$$\begin{pmatrix} v_{H,l}(q,\zeta) \\ v_{P,l}(q,\zeta) \end{pmatrix} := \begin{pmatrix} \nu_{H,l}(q,\zeta) \\ \nu_{P,n,l}(q,\zeta) \end{pmatrix} + \alpha(q,\zeta)\underline{V}(q,\zeta),\ l = 1,\dots,N-k, \tag{5.110}$$

for an appropriate smooth scalar $\alpha(q,\zeta)$, we see that both (a) and (b) of (5.106) are satisfied with $u_H = v_{H,l}$, $u_P = v_{P,l}$, $\phi = 0$. Thus,

$$(v_{H,l}(q,\zeta),0),\ l=1,\dots,N-k \tag{5.111}$$

are linearly independent elements of $\hat{\mathcal{C}}(q,\hat\zeta,\rho)$ for $\rho \geq 0$ small.

To obtain the last basis element of $\hat{\mathcal{C}}(q,\hat\zeta,0)$ we apply $\partial_\rho|_{\rho=0}$ to the equation

$$\begin{bmatrix} T_{11}\nu_H \\ T_{21}\nu_H \end{bmatrix} + \begin{bmatrix} T_{12}\nu_{P,s} \\ T_{22}\nu_{P,s} \end{bmatrix} + \begin{bmatrix} T_{12}\nu_{P,n} \\ T_{22}\nu_{P,n} \end{bmatrix} = 0. \tag{5.112}$$

With

$$\nu_H^\sharp(q,\hat\zeta) := \partial_\rho|_{\rho=0}\nu_H(q,\zeta),\ \ \nu_{P,n}^\sharp(q,\hat\zeta) := \partial_\rho|_{\rho=0}\nu_{P,n}(q,\zeta) \tag{5.113}$$

we obtain using (5.109) and Lemma 5.1

$$\begin{bmatrix} T_{11}\nu_H^\sharp \\ T_{21}\nu_H^\sharp \end{bmatrix} - \begin{bmatrix} \check{\mathbf{R}}(0,q,\hat\zeta,0) \\ \check{\mathbf{R}}_z(0,q,\hat\zeta,0) \end{bmatrix} + \begin{bmatrix} T_{12}u_{P,z} \\ T_{22}u_{P,z} \end{bmatrix} + \begin{bmatrix} T_{12}\nu_{P,n}^\sharp \\ T_{22}\nu_{P,n}^\sharp \end{bmatrix} = 0, \tag{5.114}$$

where $u_{P,z,\pm}$ was defined in (5.2).

Observe that since we can use the basis of $\mathbb{E}_n^-(P(q,\zeta))$ given in (5.90) to write

$$\nu_{P,n}(q,\zeta) = \sum_{i=1}^{N-k} c_i(q,\zeta)\nu_{P,n,i}(q,\zeta) \tag{5.115}$$

with $c_i(q,0)=0$ for all i (by (5.109)), we have $\nu_{P,n}^\sharp(q,\hat\zeta) \in \mathbb{E}_n^-(P(q,0))$.

In view of (5.114), by setting

$$\begin{pmatrix} v_{H,N+1-k}(q,\hat\zeta) \\ v_{P,N+1-k}(q,\hat\zeta) \end{pmatrix} := \begin{pmatrix} \nu_H^\sharp(q,\hat\zeta) \\ u_{P,z}(q,\hat\zeta) + \nu_{P,n}^\sharp(q,\hat\zeta) \end{pmatrix} + \beta(q,\hat\zeta)\underline{V}(q,0) \tag{5.116}$$

for an appropriate scalar $\beta(q,\hat\zeta)$, we can arrange so that both (a) and (b) of (5.106) are satisfied at $\rho=0$ with $u_H = v_{H,N+1-k}$, $u_P = v_{P,N+1-k}$, $\phi=-1$. Thus, $(\nu_H^\#(q,\hat\zeta),-1)^t \in \hat{\mathcal{C}}(q,\hat\zeta,0)$. Choosing a smooth extension of this element,

$$\begin{pmatrix} \nu_{H,e}^\#(q,\hat\zeta,\rho) \\ -1+O(\rho) \end{pmatrix} \in \hat{\mathcal{C}}(q,\hat\zeta,\rho),$$

we see that a basis of $\hat{\mathcal{C}}(q,\hat\zeta,\rho)$ for ρ small is given by

$$\left\{ \begin{pmatrix} v_{H,l}(q,\zeta) \\ 0 \end{pmatrix}, l=1,\dots,N-k \right\} \cup \left\{ \begin{pmatrix} \nu_{H,e}^\sharp(q,\hat\zeta,\rho) \\ -1+O(\rho) \end{pmatrix} \right\}, \tag{5.117}$$

where $v_{H,l}$, $l=1,\dots,N-k$ are as in (5.111).

6. We can now finish the proof by showing (5.104). We work with the basis (5.107) of $\mathcal{C}^*(q,\zeta)$ and the associated basis (5.117) of $\hat{\mathcal{C}}(q,\hat{\zeta},\rho)$. From (5.110) we obtain

$$\nu_{H,l'}(q,\zeta) = v_{H,l'}(q,\zeta) + O(\rho),\ l' = 1,\dots,N-k, \tag{5.118}$$

and from (5.109), (5.113) we have

$$\nu_{H,N+1-k}(q,\zeta) := \nu_H(q,\zeta) = \rho\nu_H^\sharp(q,\hat{\zeta}) + O(\rho^2). \tag{5.119}$$

Using (5.118), (5.119), and Remark 5.12, we find

$$\begin{aligned} \det\left(u_{H,j} \quad \nu_{H,l}\right)(q,\hat{\zeta},\rho) &= \det\Big(u_{H,j} \quad v_{H,l'}(q,\zeta) + O(\rho) \quad \rho\nu^\sharp_{H,e}(q,\hat{\zeta},\rho) + O(\rho^2)\Big) \\ &= \rho D_{red}(q,\hat{\zeta},\rho) + O(\rho^2). \end{aligned} \tag{5.120}$$

□

5.5 Summary of low frequency results

The following theorem, which ties together the results of sections 5 and 6, is our main low frequency stability result.

Theorem 5.15. *(a) The low frequency standard Evans condition* (4.35) *implies the low frequency modified Evans condition* (4.74)*. The converse holds when $d = 1$.*

(b) Assume the profile $W(z,q)$ is strongly transversal. Then the low frequency standard Evans condition is equivalent to the low frequency modified Evans condition.

(c) Assume the profile $W(z,q)$ is strongly transversal and that viscous continuity holds. Then for ρ_0 small enough, $0 < \rho \le \rho_0$, and $\hat{\zeta} \in \overline{S}^d_+$, we have

$$\begin{aligned} &(a)\ D_m(q,\hat{\zeta},\rho) = \beta(q,0)D_{Lop}(q,\hat{\zeta}) + o(1) \\ &(b)\ \tilde{D}_s(q,\hat{\zeta},\rho) = \rho\beta(q,0)D_{Lop}(q,\hat{\zeta}) + o(\rho), \end{aligned} \tag{5.121}$$

where $\beta(q,0)$ is given by (5.78) *and the errors are uniform for $(\hat{\zeta},\rho) \in \overline{S}^d_+ \times [0,\rho_0]$. Consequently, under these assumptions the modified and standard low frequency Evans conditions are both equivalent to the uniform Lopatinski condition for the inviscid hyperbolic problem.*

(d) The low frequency standard Evans condition implies transversality and the uniform Lopatinski condition. When $d \ge 2$, the low frequency standard Evans condition implies strong transversality.

(e) The low frequency modified Evans condition implies transversality and the modified uniform Lopatinski condition.

Proof. **1. Parts (b) and (c).** Part (b) follows immediately from Theorem 5.14. Part (c) follows from the block decompositions of the modified and standard Evans functions, Remark 5.8, part 4, and Proposition 4.7.

2. Part (a). When $d \ge 2$, it follows from Theorem 5.2 and Corollary 4.10 that the low frequency standard Evans condition implies strong transversality. So we can apply (b) to deduce the low frequency modified Evans condition. When $d = 1$, the low frequency standard Evans condition implies transversality and the uniform Lopatinski condition (Theorem

5.2). Hence, the low frequency modified Evans condition follows from part 2 of Theorem 5.10.

The low frequency modified Evans condition implies the modified uniform Lopatinski condition and transversality (Thm. 5.10). When $d = 1$ we can apply Prop. 4.9 to deduce that the uniform Lopatinski condition holds, so the standard low frequency Evans condition follows from part (b) of Theorem 5.2.

3. Part (d). This is contained in the statement of Theorem 5.2 and Corollary 5.3.

4. Part (e). This is contained in Theorem 5.10.

□

Remark 5.16. 1. The standard Evans condition is the one that is easier to verify numerically [B, HZ] or analytically [PZ, FS], while the modified Evans condition is the one needed for the rigorous study of small viscosity limits. Thus the implication (a) in Theorem 5.15 is especially important. The implication (d) allows us to construct curved inviscid $\mathcal{C}_{\mathcal{B}}$ shocks near a given planar shock when the low frequency standard Evans condition is satisfied (as long as there exists a K-family of smooth inviscid symmetrizers). The next section shows that implication (d) also permits the construction of high order approximate solutions to the nonlinear parabolic transmission problem 6.1.

2. When viscous continuity holds, the implication in part (a) of Theorem 5.15 can be proved without the use of the block decompositions given by Proposition 5.11 and Theorem 5.14. In fact, it follows immediately from Theorem 5.2, Corollary 5.3, and Theorem 5.10. This type of argument was used in [GMWZ3] in the conservative setting.

6 Approximate viscous shocks

In this section we construct high order approximate solutions to the nonlinear small viscosity transmission problem:

(6.1)

$$(a)\ \mathcal{E}(u, d\psi) := \sum_{j=0}^{d-1} A_j(u)\partial_j u + \mathcal{A}_d(u, d\psi)\partial_d u - \epsilon \sum_{j=1}^{d} (\partial_j - \partial_j\psi\partial_d)^2 u = 0 \text{ on } \pm x_d \geq 0$$

$$(b)\ [u] = 0,\ [\partial_d u] = 0,$$

$$(c)\ \partial_t\psi - \epsilon\triangle_y\psi + \ell(t, y) \cdot u = \partial_t\psi^0 - \epsilon\triangle_y\psi^0 + \ell(t, y) \cdot \mathcal{U}^0(t, y, 0, 0),$$

where $t = x_0$, $y = (x_1, \ldots, x_{d-1})$. Here we suppose that we are given an inviscid $\mathcal{C}_{\mathcal{B}}$-shock (U^0, ψ^0) on $[-T_0, T_0] \times \mathbb{R}^d_{y,x_d}$ satisfying

$$\sum_{j=0}^{d-1} A_j(U^0)\partial_j U^0 + \mathcal{A}_d(U^0, d\psi^0)\partial_d U^0 = 0 \text{ on } \pm x_d \geq 0 \tag{6.2}$$
$$(U^0_+(t, y, 0), U^0_-(t, y, 0), d\psi^0(t, y)) \in \mathcal{C}_{\mathcal{B}}$$

In (6.1) we have set

$$\ell(t, y) := W_z(0, q(t, y)),\ \text{where } q(t, y) := (U^0_+(t, y, 0), U^0_-(t, y, 0), d\psi^0(t, y)) \text{ and} \tag{6.3}$$

$W(z,q)$ is the profile associated to q. Moreover, we suppose that the inviscid shock satisfies the hypotheses of Theorem 1.16, with the modifications that the standard uniform Evans condition is replaced by the standard low frequency Evans condition (Definition 4.12), and the existence of a K-family of smooth viscous symmetrizers is replaced by the existence of a K-family of smooth *inviscid* symmetrizers.

The function $\mathcal{U}^0$ appearing in (6.1)(c) can be written as $\mathcal{U}^0(t,y,x_d,\frac{x_d}{\epsilon})$ where

$$\mathcal{U}^0(t,y,x_d,z) := U^0(t,y,x_d) + \left(W(z,q(t,y)) - U^0(t,y,0)\right), \tag{6.4}$$

We seek an approximate solution (u^a, ψ^a) of the form

$$\psi^a = \psi^0(t,y) + \epsilon\psi^1(t,y) + \cdots + \epsilon^M \psi^M(t,y), \tag{6.5}$$

$$u^a = \left(\mathcal{U}^0(t,y,x_d,z) + \epsilon\mathcal{U}^1(t,y,x_d,z) + \cdots + \epsilon^M \mathcal{U}^M(t,y,x_d,z)\right)|_{z=\frac{x_d}{\epsilon}}, \tag{6.6}$$

where

$$\mathcal{U}^j(t,y,x_d,z) = U^j(t,y,x_d) + V^j(t,y,z). \tag{6.7}$$

Here V^0 is already determined and is given by

$$V^0(t,y,z) = W(z,q(t,y)) - U^0(t,y,0). \tag{6.8}$$

The $V^j_\pm(t,y,z)$ are boundary layer profiles constructed to be exponentially decreasing to 0 as $z \to \pm\infty$. For the moment we just assume enough regularity so that all the operations involved in the construction make sense. A precise statement is given in Prop. 6.3.

6.1 Profile equations

We substitute (6.5), (6.6) into (6.1) and write the result as

$$\sum_{-1}^{M} \epsilon^j \mathcal{F}^j(t,y,x_d,z)|_{z=\frac{x_d}{\epsilon}} + \epsilon^M R^{\epsilon,M}(t,y,x_d), \tag{6.9}$$

where we separate $\mathcal{F}^j$ into slow and fast parts

$$\mathcal{F}^j(t,y,x_d,z) = F^j(t,y,x_d) + G^j(t,y,z), \tag{6.10}$$

and the G^j decrease exponentially to 0 as $z \to \pm\infty$.

The interior profile equations are obtained by setting the F^j, G^j equal to zero. In the following expressions for $G^j(t,y,z)$, the functions $U^j(t,y,x_d)$ and their derivatives are evaluated at $(t,y,0)$. Let $\mathcal{L}_0(z,q,\partial_z)$ and $\mathcal{L}_{0,1}(z,q)$ be the operators defined in (3.6) and set

$$L_0 v := \sum_{j=0}^{d-1} A_j(U^0)\partial_j v + \mathcal{A}_d(U^0, d\psi^0)\partial_d v. \tag{6.11}$$

We have

$$F^{-1}(t,y,x_d)=0 \tag{6.12}$$
$$G^{-1}(t,y,z)=-(1+|d\psi^0|^2)\partial_z^2\mathcal{U}^0+\mathcal{A}_d(\mathcal{U}^0,d\psi^0)\partial_z\mathcal{U}^0,$$

$$F^0(t,y,x_d)=L_0U^0 \tag{6.13}$$
$$G^0(t,y,z)=\mathcal{L}_0(z,q(t,y),\partial_z)\mathcal{U}^1-\mathcal{L}_{0,1}(z,q(t,y))d\psi^1-Q^0(t,y,z),$$

where Q^0 decays exponentially as $z\to\pm\infty$ and depends only on $(U^0,V^0,d\psi^0)$. For $j\geq 1$ we have

$$F^j(t,y,x_d)=L_0U^j-P^{j-1}(t,y,x_d) \tag{6.14}$$
$$G^j(t,y,z)=\mathcal{L}_0(z,q(t,y),\partial_z)\mathcal{U}^{j+1}-\mathcal{L}_{0,1}(z,q(t,y))d\psi^{j+1}-Q^j(t,y,z),$$

where Q^j decays exponentially as $z\to\pm\infty$ and P^j, Q^j depend only on $(U^k,V^k,d\psi^k)$ for $k\leq j$.

Similarly, we obtain the boundary profile equations in which (t,y,x_d,z) is evaluated at $(t,y,0,0)$:

$$\begin{aligned}&(a)\ [\mathcal{U}^0]=0\\&(b)\ [\mathcal{U}^0_z]=0\\&(c)\ \partial_t\psi^0-\ell(t,y)\cdot\mathcal{U}^0=\partial_t\psi^0-\ell(t,y)\cdot\mathcal{U}^0,\end{aligned} \tag{6.15}$$

$$\begin{aligned}&(a)\ [\mathcal{U}^1]=0\\&(b)\ [\mathcal{U}^1_z]=-[\partial_{x_d}U^0]\\&(c)\ \partial_t\psi^1-\triangle_y\psi^0+\ell(t,y)\cdot\mathcal{U}^1=-\triangle_y\psi^0,\end{aligned} \tag{6.16}$$

and for $j\geq 2$

$$\begin{aligned}&(a)\ [\mathcal{U}^j]=0\\&(b)\ [\mathcal{U}^j_z]=-[\partial_{x_d}U^{j-1}]\\&(c)\ \partial_t\psi^j-\triangle_y\psi^{j-1}+\ell(t,y)\cdot\mathcal{U}^j=0.\end{aligned} \tag{6.17}$$

6.2 Solution of the profile equations

The solution of the profile equations given below assumes strong transversality and the uniform Lopatinski condition, as well as the existence of a K-family of smooth inviscid symmetrizers. Recall from Theorem 5.15 and Corollary 5.3 that when $d\geq 2$, the first two conditions both follow from the low frequency standard Evans condition. Strong tranversality can be replaced by transversality as explained in Remark 6.1.

1. The interior equations $G^{-1}=0$ and $F^0=0$ and the boundary equations (6.15) are satisfied because of our assumptions about U^0, ψ^0 and $W(z,q)$.

2. Construction of $(\mathcal{U}^1, U^1, \psi^1)$. We construct the function $\mathcal{U}^1(t,y,x_d,z)$ as in (6.7) from the equations $G^0 = 0$, $F^1 = 0$, and the boundary equations (6.16). $\mathcal{U}^1$ will be a sum of three parts

$$\begin{aligned}&\mathcal{U}^1(t,y,x_d,z) = \mathcal{U}^1_a + \mathcal{U}^1_b + \mathcal{U}^1_c, \text{ where}\\ &\mathcal{U}^1_k(t,y,x_d,z) = U^1_k(t,y,x_d) + V^1_k(t,y,z),\ k = a,b,c,\end{aligned} \tag{6.18}$$

where we suppress $\pm$ subscripts.

First use the exponential decay of Q^0 to find an exponentially decaying solution $V^1_a(t,y,z)$ to

$$\begin{aligned}&\mathcal{L}_0(z,q(t,y),\partial_z)V^1_a = Q^0(t,y,z) \text{ on } \pm z \geq 0\\ &V^1_a \to 0 \text{ as } z \to \pm\infty,\end{aligned} \tag{6.19}$$

and define $U^1_a(t,y,x_d) \equiv 0$. This is the same type of problem as (3.25), which we solved by conjugating the corresponding first order system with $T_\pm(z,q,0)$.

Next, for $\mathcal{U}^1_a$ fixed as above, use (a,p)-transversality (recall Prop. 3.6) to solve for $\mathcal{U}^1_b(t,y,0,z)$ in

$$\begin{aligned}&\mathcal{L}_0(z,q(t,y),\partial_z)\mathcal{U}^1_b = 0 \text{ on } \pm z \geq 0\\ &[\mathcal{U}^1_a + \mathcal{U}^1_b] = 0\\ &[\partial_z\mathcal{U}^1_a + \partial_z\mathcal{U}^1_b] = -[\partial_{x_d}U^0]\\ &\ell(t,y)\cdot(\mathcal{U}^1_a + \mathcal{U}^1_b) = 0.\end{aligned} \tag{6.20}$$

Using (3.34) we see that $\mathcal{U}^1_b$ has limits as $z \to \pm\infty$. Define

$$\begin{aligned}&U^1_b(t,y,0) := \lim_{z\to\pm\infty} \mathcal{U}^1_b(t,y,0,z),\\ &V^1_b(t,y,z) := \mathcal{U}^1_b(t,y,0,z) - U^1_b(t,y,0),\end{aligned} \tag{6.21}$$

and let $U^1_b(t,y,x_d)$ be any smooth extension of $U^1_b(t,y,0)$ with compact support in x_d.

Finally, for an appropriate choice of $(U^1_c(t,y,0),\psi^1)$ we need $\mathcal{U}^1_c(t,y,0,x_d)$ to satisfy

$$\begin{aligned}&\mathcal{L}_0(z,q(t,y),\partial_z)\mathcal{U}^1_c = \mathcal{L}_{0,1}(z,q(t,y))d\psi^1\\ &[\mathcal{U}^1_c] = 0,\ [\partial_z\mathcal{U}^1_c] = 0,\ \partial_t\psi^1 + \ell(t,y)\cdot\mathcal{U}^1_c = 0\\ &\lim_{z\to\pm\infty} \mathcal{U}^1_c(t,y,0,z) = U^1_{c\pm}(t,y,0).\end{aligned} \tag{6.22}$$

According to the characterization of $T_q\mathcal{C}_\mathcal{B}$ given in Proposition 3.12, this is possible if and only if $(U^1_c(t,y,0),d\psi^1) \in T_{q(t,y)}\mathcal{C}_\mathcal{B}$. Thus, we first solve for $(U^1_c(t,y,x_d),\psi^1)$ satisfying the linearized inviscid problem

$$\begin{aligned}&L_0U^1_c = P^0 - L_0U^1_b \text{ on } \pm x_d \geq 0\\ &(U^1_c(t,y,0),d\psi^1(t,y)) \in T_{q(t,y)}\mathcal{C}_\mathcal{B}\end{aligned} \tag{6.23}$$

This problem requires an initial condition in order to be well-posed. The right side in the interior equation of (6.23) is initially defined just for $t \in [-T_0,T_0]$. With a C^∞ cutoff that is identically one in $t \geq -T_0/2$, we can modify the right side to be zero in $t \leq -T_0 + \delta$, say. Requiring (U^1_c, ψ^1) to be identically zero in $t \leq -T_0 + \delta$, we thereby obtain a problem for (U^1_c, ψ^1) that is forward well-posed since (U^0, ψ^0) satisfies the uniform Lopatinski condition and χ_p has full rank $N + k$ (see the proof of Theorem 1.6 in section 7). Thus, there exists a solution to (6.23) on $[-\frac{T_0}{2}, T_0]$. This allows us to obtain $\mathcal{U}^1_c(t, y, 0, z)$ satisfying (6.22) and to define

$$V^1_c(t, y, z) := \mathcal{U}^1_c(t, y, 0, z) - U^1_c(t, y, 0). \tag{6.24}$$

By construction the functions $(\mathcal{U}^1, U^1, \psi^1)$ satisfy the equations $G^0 = 0$, $F^1 = 0$, and the boundary conditions (6.16).

3. Contruction of $(\mathcal{U}^j, U^j, \Psi^j)$, $j \geq 2$. In the same way, for $j \geq 2$ we use the equations $G^{j-1} = 0$, $F^j = 0$, and the boundary conditions (6.17) to determine the functions $(\mathcal{U}^j, U^j, \psi^j)$.

Remark 6.1. It turns out that the profile construction can be carried out with only slight changes (e.g., a nonhomogeneous boundary condition in (6.23)) assuming just transversality, the uniform Lopatinski condition, and the existence of a K-family of smooth inviscid symmetrizers. In solving for $\mathcal{U}^1_b(t, y, 0, z)$ one uses the formulation of (a,p,s)-transversality given in Corollary 3.11. The inviscid problem (6.23) can be solved in $d \geq 1$ without explicit reliance on (a,p)-transversality using the approach described in [GMWZ6].

In the next Proposition we formulate a precise statement summarizing the construction of this section. The regularity assertions in the Proposition are justified as in [GMWZ4], Prop. 5.7, except that now we use the estimates of [Mo, Cou] in place of those of [Ma1]. Regularity is expressed in terms of the following spaces:

Definition 6.2. *1. Let H^s be the set of functions $U(t, y, x_d)$ on $[-T_0, T_0] \times \mathbb{R}^d$ such that the restrictions $U_\pm$ belong to $H^s([-T_0, T_0] \times \overline{\mathbb{R}}^d_\pm)$.*

2. Let $\tilde{H}^s$ be the set of functions $V(t, y, z)$ on $[-T_0, T_0] \times \mathbb{R}^{d-1} \times \mathbb{R}$ such that the restrictions $V_\pm$ belong to $C^\infty(\overline{\mathbb{R}}_\pm, H^s(t, y))$ and satisfy

$$|\partial^k_z V(t, y, z)|_{H^s(t,y)} \leq C_{k,s} e^{-\delta|z|} \text{ for all } k \tag{6.25}$$

for some $\delta > 0$.

Proposition 6.3 (Approximate solutions). *For given integers $m \geq 0$ and $M \geq 1$ let*

$$s_0 > m + \frac{7}{2} + 2M + \frac{d+1}{2}. \tag{6.26}$$

Suppose the given inviscid shock (U^0, ψ^0) has the properties assumed in Theorem 1.16, but now require only the standard low frequency Evans condition and the existence of a K-family of smooth inviscid symmetrizers. Assume $U^0 \in H^{s_0}$, $U^0_\pm(t, y, 0) \in H^{s_0}(t, y)$, and $\psi^0(t, y) \in H^{s_0+1}(t, y)$. Then one can construct (u^a, ψ^a) as in (6.5), (6.6)

$$\psi^a = \psi^0(t, y) + \epsilon\psi^1(t, y) + \cdots + \epsilon^M \psi^M(t, y), \tag{6.27}$$

$$(6.28)\qquad u^a = \left(\mathcal{U}^0(t,y,x_d,z) + \epsilon\mathcal{U}^1(t,y,x_d,z) + \cdots + \epsilon^M\mathcal{U}^M(t,y,x_d,z)\right)|_{z=\frac{x_d}{\epsilon}},$$

Let $\mathcal{E}$ denote the operator on the left side of (6.1)(a). The approximate solution (u^a,ψ^a) satisfies

$$(6.29)\qquad \begin{aligned} &\mathcal{E}(u^a,\psi^a) = \epsilon^M R^M(t,y,x_d) \textit{ on } [-\frac{T_0}{2},T_0]\times\overline{\mathbb{R}}^d_\pm \\ &[u^a]=0;\ \ [\partial_d u^a]=0 \textit{ on } x_d=0 \\ &\partial_t\psi^a - \epsilon\triangle_y\psi^a + \ell(t,y)\cdot u^a \\ &\qquad = \partial_t\psi^0 - \epsilon\triangle_y\psi^0 + \ell(t,y)\cdot\mathcal{U}^0(t,y,0,0) \textit{ on } x_d=0. \end{aligned}$$

We have

$$(6.30)\qquad \begin{aligned} &U^j(t,y,x_d)\in H^{s_0-2j},\ \psi^j(t,y)\in H^{s_0-2j+1}(t,y) \\ &V^j(t,y,z)\in\tilde{H}^{s_0-2j}, \end{aligned}$$

and $R^M(t,y,x_d)$ satisfies

$$(6.31)\qquad \begin{aligned} &(a)\ |(\partial_t,\partial_y,\epsilon\partial_{x_d})^\alpha R^M|_{L^2(t,y,x_d)} \le C_\alpha \textit{ for } |\alpha| \le m + \frac{d+1}{2} \\ &(b)\ |(\partial_t,\partial_y,\epsilon\partial_{x_d})^\alpha R^M|_{L^\infty(t,y,x_d)} \le C_\alpha \textit{ for } |\alpha|\le m. \end{aligned}$$

Remark 6.4. Observe that Proposition 6.3 can be applied to give high order approximate solutions even for slow MHD shocks.

7 Existence of nonclassical inviscid and viscous shocks

In this section we present the proofs of Theorems 1.6 and 1.16. The discussion will be brief since the proofs mainly involve combining results proved in previous sections with results proved elsewhere.

Proof of Theorem 1.6. We will show how the proofs given in [Mo, Cou, Ma2] can be adapted to our case. We discuss the case $d\ge 2$. The case $d=1$ can be treated using results of [LY], for example.

Part 1. Let χ be a local defining function for the shock manifold $\mathcal{C}$, and suppose $q\in\mathcal{C}$ is a point where the uniform Lopatinski condition holds. The existence of a smooth K-family of inviscid symmetrizers implies inviscid continuity, so from the uniform Lopatinski condition at q we deduce that there exists a $c>0$ such that

$$(7.1)\qquad \begin{aligned} &|\chi_p'(q)u + \chi_s'(q)(i\hat{\tau}+\hat{\gamma})\psi + \chi_h'(q)i\hat{\eta}\psi| \ge c|u,\psi| \\ &\qquad\text{for all } u=(u_+,u_-)\in\mathbb{E}_-(H_0(q,\hat{\zeta})),\ \psi\in\mathbb{C},\ \hat{\zeta}\in\overline{S}^d_+. \end{aligned}$$

This implies

$$(7.2)\qquad b(q,\hat{\zeta}) := \chi_s'(q)(i\hat{\tau}+\hat{\gamma}) + \chi_h'(q)i\hat{\eta} \neq 0 \text{ for all } \hat{\zeta}\in\overline{S}^d_+,$$

so we may define $\Pi(q,\hat\zeta):\mathbb{C}^{N+k}\to\mathbb{C}^{N+k}$, the smooth orthogonal projector onto $b(q,\hat\zeta)^\perp$. Setting

$$M(q)u := \chi'_p(q)u \tag{7.3}$$

and applying $\Pi(q,\hat\zeta)$ to the boundary condition in (4.1) we obtain the projected boundary condition (whose analogue in [Cou] is equation (17a)):

$$\Pi(q,\hat\zeta)M(q)u = 0. \tag{7.4}$$

Moreover, we deduce from (7.1) the corresponding uniform Lopatinski condition for the problem (4.1)(a) with the new boundary condition (7.4):

$$|\Pi(q,\hat\zeta)M(q)u|\geq c|u| \text{ for all } u\in\mathbb{E}_-(H_0(q,\hat\zeta)). \tag{7.5}$$

Recall from Proposition 1.18 that

$$M(q)=\chi'_p(q) \text{ has full rank } N+k \tag{7.6}$$

as a consequence of the uniform Lopatinski condition. The analogue of (7.6) in the conservative case was stated as "Assumption 4" in [Cou], and was used there to construct a suitable adjoint boundary condition as a step in proving existence of solutions to the linearized shock problem by a duality argument. We are now in a position to complete the proof of the first part of Theorem 1.6 by repeating arguments used in [Mo, Cou] to establish Theorem 5.2 of that paper.

Part 2. First we must define and construct *shock front initial data compatible to order* $s-1$. Such data is constructed in Proposition 2.2 of [Ma2] in the conservative Lax case. To carry out a similar construction here we need to use local defining functions, and in order to patch together locally defined initial data, we must check that compatibility is independent of the choice of local defining function.

One obtains local corner compatibility conditions as in [Ma2] by supposing (u_+,u_-,ψ) is a smooth solution to the nonlinear problem

$$\begin{aligned}
&(a)\ \sum_{j=0}^{d-1}A_j(u_\pm)\partial_j u_\pm+\mathcal{A}_d(u_\pm,d\psi)\partial_d u_\pm=0 \text{ in } \pm x\geq 0\\
&(b)\ \chi(u_+,u_-,\partial_t\psi,\partial_y\psi)=0 \text{ on } x=0,\\
&(c)\ u_\pm(0,y,x_d)=u^0_\pm(y,x_d),\ \partial_t\psi(0,y)=\sigma(y),\ \psi(0,y)=\psi^0(y),
\end{aligned} \tag{7.7}$$

computing relations between $\partial_t^{k+1}\psi(0,y)$ and $\partial_t^k u_\pm(0,y,0)$ by differentiating (7.7)(b), and using (7.7)(a) to express time derivatives of u in terms of space derivatives of u (and derivatives of ψ involving ∂_t^j, $j\leq k$). The relations at $t=0$, $x_d=0$ have the form

$$\partial_t^{k+1}\psi\,\chi'_s+\chi'_{p+}\ \left(-A_0(u^0_+)^{-1}\mathcal{A}_d\right)^k\partial^k_{x_d}u^0_+ +\chi'_{p-}\ \left(-A_0(u^0_-)^{-1}\mathcal{A}_d\right)^k\partial^k_{x_d}u^0_- = I_k, \tag{7.8}$$

where derivatives of χ are evaluated at $(u^0(y,0),\sigma(y),\partial_y\psi^0(y))$, and I_k is an explicitly computable function involving no t (resp. x_d) derivatives of ψ (resp. $u_\pm$) of order greater than k (resp. $k-1$).

Definition 7.1. *A function* $(u^0_+(y,x_d), u^0_-(y,x_d), \sigma(y), \psi^0(y))$ *satisfying* (1.10) *is said to determine compatible shock front initial data to order* $s-1$ *when there exists a function* $\psi(t,y)$ *such that*

$$\psi(0,y) = \psi^0(y),\ \partial_t\psi(0,y) = \sigma(y) \tag{7.9}$$

and the relations (7.8) *hold for* $1 \leq k \leq s-1$.

It is not immediately clear that the notion of compatibility is well-defined. To see that it is, note that if χ_1 and χ_2 are two local defining functions for $\mathcal{C}$ near q, we must have

$$\chi_2(q) = B(q)\chi_1(q) \tag{7.10}$$

for some invertible $(N+k)\times(N+k)$ matrix $B(q)$. Using (7.10) and a straightforward induction on k, one shows that if $\psi(t,y)$ and $u^0_\pm(y,x_d)$ satisfy the relations (7.8) for $1 \leq k \leq s-1$ with $\chi = \chi_1$, then these same functions satisfy the relations with $\chi = \chi_2$.

Majda's construction of compatible data relies on the observation that the uniform Lopatinski condition implies *one dimensional stability* (Lemma 2.1 of [Ma2]). In our context this is the statement that for $\hat\zeta' = (\hat\tau', \hat\gamma', \hat\eta') := (0,1,0)$, the map

$$\hat\Gamma_\chi(q,\hat\zeta') : \mathbb{E}_-(H_0(q,\hat\zeta')) \times \mathbb{C} \to \mathbb{R}^{N+k} \tag{7.11}$$

is invertible, which follows immediately from the uniform Lopatinski condition (7.1). This allows us to carry out Majda's construction of compatible initial data for nonclassical shocks. The rest of the proof of part (2) of Theorem 1.6 can now be completed by following [Mo, Cou].

□

Proof of Theorem 1.16. The theorem is proved by constructing exact solutions of the nonlinear transmission problem (6.1) which are close to an approximate solution (u^a, ψ^a) as constructed in Proposition 6.3. The key step is to obtain good estimates for solutions (v,ϕ) to the transmission problem obtained by linearizing (6.1) with respect to both u and ψ at (u^a,ψ^a):

$$\begin{aligned} &\mathcal{E}'_u(u^a,\psi^a)v + \mathcal{E}'_\psi(u^a,\psi^a)\phi = f \text{ on } [-T_0,T_0]\times\overline{\mathbb{R}}^d_\pm \\ &[v]=0,\ [\partial_d v]=0,\ \partial_t\phi - \varepsilon\triangle_y\phi + \ell(t,y)\cdot v = 0 \text{ on } x_d=0, \\ &v=0,\ \phi=0,\ f=0 \text{ in } t<-T_0/3. \end{aligned} \tag{7.12}$$

The desired estimate is stated (in the conservative, Lax, constant multiplicity case) in Theorem 7.2 of [GMWZ3]. We can use the same argument to prove the identical estimate in our context provided the linearized problem (7.12) satisfies the modified uniform Evans condition. The modified uniform Evans condition then allows us to construct a Kreiss-type symmetrizer for (7.12) from the K-family of smooth viscous symmetrizers by taking K large enough.

By Theorem 5.15(a) of this paper, the low frequency standard Evans condition implies the low frequency modified Evans condition. The fact that nonvanishing of $D_s(q,\hat\zeta,\rho)$ for $\rho>0$ implies nonvanishing of $D_m(q,\hat\zeta,\rho)$ for $\rho>0$ can easily be proved just as in Proposition

2.16 of [GMWZ3]. Observe that since $(u^a, d\psi^a)$ differs from $(W(\frac{x_d}{\varepsilon}, q(t,y)), d\psi^0)$ by an error which is small in L^∞ for ε and $|x_d|$ small (recall (6.4)-(6.6)), we need to deduce nonvanishing of $D_m(q, \hat{\zeta}, \rho)$ for q near $\mathcal{C}_{\mathcal{B}}$ but not in $\mathcal{C}_{\mathcal{B}}$; but this follows from viscous continuity.

With the linear estimate of Theorem 7.2 of [GMWZ3] in hand, the proof is completed by the same fixed point argument used to prove Theorem 7.7 of [GMWZ3].

□

8 Appendix A: Extension to real viscosity

In this appendix we describe the changes needed to treat real viscosities. Consider the $N \times N$ viscous system on $\mathbb{R}^{d+1}$ given by (1.12).

1. Structural assumptions. The assumptions made here in order to define and treat the case of real, or partially parabolic, viscosities are all satisfied, for example, by the compressible Navier-Stokes and viscous MHD equations. A detailed discussion of the assumptions is given in [GMWZ4].

We again make Assumptions 1.1 and 1.2, but now add the following block form requirement:

Assumption 8.1. *Possibly after a change of variables u and multiplying the system on the left by an invertible constant coefficient matrix, there is $N' \in \{1, \dots, N\}$ and there are coordinates $u = (u^1, u^2) \in \mathbb{R}^{N-N'} \times \mathbb{R}^{N'}$ such that*

$$A_0(u) = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix}, \quad B_{j,k} = \begin{pmatrix} 0 & 0 \\ 0 & B_{j,k}^{22} \end{pmatrix}. \tag{8.1}$$

Assumption 1.8 should now be replaced by

Assumption 8.2. *(H2')(Partial parabolicity.) The $B_{j,k}^{22}$ are C^∞ functions on $\mathcal{U}^*$ valued in $\mathbb{R}^{N' \times N'}$. There is $c > 0$ such that for all $u \in \mathcal{U}^*$ and $\xi \in \mathbb{R}^d$ the eigenvalues of $\overline{B}^{22}(u, \xi) = \sum_{j,k=1}^d \overline{B}_{j,k}^{22}(u)$ satisfy $\Re\mu \geq c|\xi|^2$.*

(H3')(Strict dissipativity.) There is $c > 0$ such that for all $u \in \mathcal{U}$ and $\xi \in \mathbb{R}^d$ the eigenvalues μ of $i\overline{A}(u, \xi) + \overline{B}(u, \xi)$ satisfy

$$\Re\mu \geq \frac{c|\xi|^2}{1 + |\xi|^2}. \tag{8.2}$$

Assumption 8.3. *For the low and medium frequency analysis it is enough to assume:*

$(H4)_{\ell,m}$ For all $u \in \mathcal{U}^*$ and all $\xi \in \mathbb{R}^d \setminus 0$, $\overline{A}^{11}(u, \xi) := \sum_{j=1}^d \overline{A}_j^{11}(u)\xi_j$ has only real eigenvalues.

For the high frequency analysis we must strengthen this to:

$(H4)_h$ For all $u \in \mathcal{U}^*$ and $\xi \in \mathbb{R}^d \setminus 0$ the eigenvalues of $\overline{A}^{11}(u, \xi)$ are real and semisimple with constant multiplicities.

The profile equation in the case of a real viscosity $\mathcal{B}(u)$ can still be written as (1.16), and profiles $W(z, q)$ are defined and associated to points q of a shock manifold $\mathcal{C}_{\mathcal{B}}$ as before.

We discuss the local construction of $\mathcal{C}_\mathcal{B}$ for real viscosities $\mathcal{B}(u)$ below. One difference is that now we restrict the undercompressive index k to satisfy

$$0 \leq k \leq N' - 1. \tag{8.3}$$

In addition we now add the following assumption:

Assumption 8.4. *Suppose we are given a shock manifold $\mathcal{C}_\mathcal{B}$. For the low and medium frequency analysis it is enough to suppose:*

$(H5)_{\ell,m}$ *For any planar shock $q = (p_+, p_-, s, h) \in \mathcal{C}_\mathcal{B}$ with normal direction $\nu = \nu(s,h) = (-s,-h,1)$,*

$$\det \left(\sum_{j=0}^{d} A_j^{11}(W(z,q))\nu_j \right) \neq 0 \textit{ for all } z \in \mathbb{R} \cup \{\pm\infty\}. \tag{8.4}$$

For the high frequency analysis we must strengthen this to

$(H5)_h$ *For any $q \in \mathcal{C}_\mathcal{B}$ and $z \in \mathbb{R} \cup \{\pm\infty\}$ the polynomial in ξ*

$$\det \left(\sum_{j=0}^{d} A_j^{11}(W(z,q))\xi_j \right) \tag{8.5}$$

is hyperbolic in the direction $\nu(s,h)$.

Remark 8.5. 1. With $\mathcal{A}_d(u,s,h) = \sum_{j=0}^{d} A_j(u)\nu_j$ as before, let $\overline{\mathcal{A}}_d = (A_0)^{-1}\mathcal{A}_d$. By Assumptions 8.3 and 8.4 the eigenvalues of $\overline{\mathcal{A}}_d^{11}(W(z,q),s,h)$ are real and nonzero. Let N_+^1 be the number of positive eigenvalues of $\overline{\mathcal{A}}_d^{11}(W(z,q),s,h)$. By connectedness this number is independent of $q \in \mathcal{C}_\mathcal{B}$, $z \in \mathbb{R} \cup \{\pm\infty\}$.

2. If instead of a shock manifold we are given just a single transversal profile $\underline{w}(z)$ associated to a planar shock $\underline{q}$, then we make Assumption 8.4 with $W(z,q)$ replaced by $\underline{w}(z)$. As explained below this will allow us to construct a shock manifold near $\underline{q}$.

2. Construction of $\phi_\pm(z, p_\pm, s, h, a_\pm)$. With $w = (w^1, w^2)$ and $w^3 := \partial_z w^2$ the profile transmission problem equivalent to (1.16) can be written

$$\begin{aligned} &\partial_z w^1 = -(\mathcal{A}_d^{11})^{-1}\mathcal{A}_d^{12} w^3 \\ &\partial_z w^2 = w^3 \\ &\partial_z(\mathcal{B}_{d,d}^{22} w^2) = \left(\mathcal{A}_d^{22} - \mathcal{A}_d^{21}(\mathcal{A}_d^{11})^{-1}\mathcal{A}_d^{12}\right) w^3 \\ &[w] = 0,\ [w^3] = 0 \text{ on } z = 0, \end{aligned} \tag{8.6}$$

where the matrices are evaluated at $(w_\pm(z), s, h)$. For $q = (p_+, p_-, s, h)$ one again looks for profiles $w = W(z,q)$, with endstates $p_\pm$ and satisfying (8.6), near a given profile $\underline{w} = W(z, \underline{q})$ where $\underline{q} = (\underline{p}_+, \underline{p}_-, 0, 0)$. In place of (2.7) we consider

$$\begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix}' = \begin{pmatrix} -(\mathcal{A}_d^{11})^{-1}\mathcal{A}_d^{12} w^3 \\ w^3 \\ G_d w^3 \end{pmatrix}, \tag{8.7}$$

where

$$G_d := (\mathcal{B}^{22}_{d,d})^{-1}\left(\mathcal{A}^{22}_d - \mathcal{A}^{21}_d(\mathcal{A}^{11}_d)^{-1}\mathcal{A}^{12}_d\right) \tag{8.8}$$

and the matrices are now evaluated at $(p_\pm, s, h)$. The interior problems (8.6) in $\pm z \geq 0$ are solved by considering them as perturbations, quadratic in $(w_\pm - p_\pm, w^3_\pm)$, of (8.7). G_d is clearly nonsingular and, in fact, strict dissipativity implies $G_d(p_\pm, s, h)$ has no purely imaginary eigenvalues ([GMWZ4], Lemma 3.39).

Definition 8.6. *1. Let r_- (resp. ℓ_+) denote the number of eigenvalues μ of $G_d(p_+, s, h)$ (resp. $G_d(p_-, s, h)$) with $\Re\mu < 0$ (resp. $\Re\mu > 0$).*

2. Parallel to (2.8), (2.9) *we define invariant subspaces $\mathbb{E}_\mp(G_d(p_\pm, s, h)) \subset \mathbb{R}^{N'}$ of dimensions r_- (resp. ℓ_+) with corresponding projections $\Pi_\mp(p_\pm, s, h) : \mathbb{R}^{N'} \to \mathbb{E}_\mp(p_\pm, s, h)$, and fix isomorphisms linear in $a_\pm \in \mathbb{E}_\mp(G_d(\underline{p}_\pm, 0, 0))$:*

$$\alpha_\pm(p_\pm, s, h; a_\pm) : \mathbb{E}_\mp(G_d(\underline{p}_\pm, 0, 0)) \to \mathbb{E}_\mp(G_d(p_\pm, s, h)) \tag{8.9}$$

Functions $\phi_\pm(z, p_\pm, s, h, a_\pm)$ as in (2.26) satisfying the obvious analogue of Proposition 2.2 are now constructed just as before. The function $\tilde{\Psi}(p, s, h, a)$ corresponding to (2.34) is given by

$$\tilde{\Psi}(p, s, h, a) := \begin{pmatrix} \phi_+(0,\cdot) - \phi_-(0,\cdot) \\ \phi^2_{+,z}(0,\cdot) - \phi^2_{-,z}(0,\cdot) \\ s + \phi^2_+(0,\cdot)\cdot \underline{w}^2_z(0) - \underline{w}^2(0)\cdot\underline{w}^2_z(0) \end{pmatrix} (p, s, h, a) \in \mathbb{R}^{N+N'+1}. \tag{8.10}$$

3. Linearization and HP form. The rescaled transmission problem corresponding to (1.45) is now

$$\begin{aligned} &\sum_{j=0}^{d-1} A_j(u)\partial_j u + \mathcal{A}_d(u, d\psi)\partial_z u - \sum_{j,k=1}^{d} D_j(B_{j,k}(u)D_k u) = 0 \text{ on } \pm z \geq 0 \\ &[u] = 0,\ [\partial_z u^2] = 0 \text{ on } z = 0, \end{aligned} \tag{8.11}$$

where $D_j := \partial_j - (\partial_j\psi)\partial_z$ for $j = 1, \ldots, d-1$ and $D_d = \partial_z$. Assuming we have an exact solution of (8.11) given by a profile $W(z, q)$ and front $\psi = st + hy$, consider the partially and fully linearized (Fourier-Laplace transformed) transmission problems

$$\begin{aligned} &\mathcal{L}(z, q, \zeta, \partial_z)u = f \text{ on } \pm z \geq 0 \\ &[u] = 0, [u^2_z] = 0 \text{ on } z = 0, \end{aligned} \tag{8.12}$$

and

$$\begin{aligned} &\mathcal{L}(z, q, \zeta, \partial_z)u - \psi\mathcal{L}_1(z, q, \zeta) = f \\ &[u] = 0,\ [u^2_z] = 0,\ c_0(\zeta)\psi + \underline{w}^2_z(0)\cdot u^2_+ = 0 \text{ on } z = 0, \end{aligned} \tag{8.13}$$

where $\mathcal{L}(z, q, \zeta, \partial_z)$ and $\mathcal{L}_1(z, q, \zeta)$ are written out explicitly in [GMWZ4], equations (3.14) and (2.65) (in the latter case replace $\partial_z f_j(W)$ by $A_j(W)\partial_z W$, $j = 0, \ldots, d-1$).

With $U = (u, u_z^2)$ we rewrite (8.12) as a first-order $(N + N') \times (N + N')$ transmission problem

$$\begin{aligned} &\partial_z U - G(z,q,\zeta)U = F \\ &[U] = 0 \text{ on } z = 0, \end{aligned} \tag{8.14}$$

where the components of G are written out in [GMWZ4], equation (3.36). As in (3.20), (4.38) for $|\zeta|$ small we conjugate (8.14) to HP form

$$\begin{aligned} &\partial_z \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} = \begin{pmatrix} H_\pm(q,\zeta) & 0 \\ 0 & P_\pm(q,\zeta) \end{pmatrix} \begin{pmatrix} u_{H\pm} \\ u_{P\pm} \end{pmatrix} + \tilde{F} \\ &\tilde{\Gamma}_H(q,\zeta)u_H + \tilde{\Gamma}_P(q,\zeta)u_P = 0 \text{ on } z = 0, \end{aligned} \tag{8.15}$$

where, as before,

$$H_\pm(q,\zeta) = -\mathcal{A}_d(p_\pm,s,h)^{-1}\left(A_0(p_\pm)(i\tau+\gamma) + \sum_{j=1}^{d-1} A_j(p_\pm)i\eta_j\right) + O(\rho^2), \tag{8.16}$$

but now the lower right $N' \times N'$ block is

$$P_\pm(q,\zeta) = G_d(p_\pm,s,h) + O(\rho) \tag{8.17}$$

for G_d as in (8.8).

Proposition 8.7 (Relations between indices). *Consider the indices R_-, L_+ (as in Assumption 1.2), r_-, ℓ_+ (Defn. 8.6), N_+^1 (Remark 8.5), and the undercompressive index k. We have*

$$\begin{aligned} &(a)\ N' + N_+^1 = r_- + (N - R_-) \\ &(b)\ N - N_+^1 = \ell_+ + (N - L_+) \\ &(c)\ r_- + \ell_+ - N' = R_- + L_+ - N = 1 - k. \end{aligned} \tag{8.18}$$

Proof. Part (c) follows by adding (a) and (b). To prove (a), let

$$G_+(q,\zeta) = \lim_{z\to+\infty} G(z,q,\zeta). \tag{8.19}$$

Strict dissipativity (H3') implies that for $\zeta \neq 0$, $G_+(q,\zeta)$ has no eigenvalues on the imaginary axis. For ρ small, we can count the eigenvalues μ with $\Re\mu < 0$ using (8.15) (recall (3.17), (3.18)), and the number is clearly $r_- + (N - R_-)$. For ρ large one can show as in [GMWZ4], Lemma 3.38 that this number is $N' + N_+^1$. This implies (a), and (b) is proved similarly. □

4. Transversality and the manifold $\mathcal{C}_\mathcal{B}$. With $\tilde{\Psi}$ as defined in (8.10), we have now in place of (2.37)

$$\begin{aligned} &\text{rank}\nabla_a \tilde{\Psi}(\underline{p},0,0,\underline{a}) = r_- + \ell_+ = N' + 1 - k \\ &\text{rank}\nabla_{a,p} \tilde{\Psi}(\underline{p},0,0,\underline{a}) = N + N' + 1, \end{aligned} \tag{8.20}$$

as the conditions that define a-transverality and (a,p)-transversality respectively. To define (a,p,s)-transversality we just replace $\nabla_{a,p}$ by $\nabla_{a,p,s}$ in (8.20). The notions of *transversality* and *strong transversality* may now be defined just as before (Defn. 2.14). Moreover, the proof of Prop. 2.8 can now be repeated to yield an $N+d-k$ dimensional shock manifold $\mathcal{C}_{\mathcal{B}}$ near $\underline{q}$ when transversality holds. In place of Prop. 2.11 we now have

$$\begin{aligned} &\dim \mathcal{S}_+ = N + r_-,\ \ \dim \mathcal{S}_- = N + \ell_+,\ \ \dim \mathcal{S} = 2N + (N'+1-k) \\ &\dim \mathcal{S}^0_+ = r_-,\ \ \dim \mathcal{S}^0_- = \ell_+. \end{aligned} \tag{8.21}$$

Sections 3-6, and Appendix B now extend in a straightforward way to the case of real viscosity. In many cases only a change of index is needed (e.g., N' in place of N), or u_z should be replaced by u_z^2 in a boundary condition. We'll briefly indicate some of these changes below.

5. Reduced transmission conditions. In place of (3.40) we have

$$\mathbb{C}^{N+N'+1} = \mathbb{F}_{H,\mathcal{R}}(q) \oplus \mathbb{F}_P(q), \tag{8.22}$$

where

$$\dim \mathbb{F}_P(q) = N' + 1 - k,\ \ \dim \mathbb{F}_{H,\mathcal{R}}(q) = N + k. \tag{8.23}$$

We still have

$$\Gamma_{0,red}(q)(u_H, \dot{s}, \dot{h}) := \pi_{H,\mathcal{R}}(q)\left(\Gamma_{0,H}(q)u_H + \Gamma_{\mathcal{R}}(\dot{s}, \dot{h})\right), \tag{8.24}$$

but, of course, $\mathcal{R}_z$ is replaced by $\mathcal{R}_z^2$ in the definition (3.37) of $\Gamma_{\mathcal{R}}$.

6. Stability determinants. Using Proposition 8.7 we see that the spaces in (4.41) now have dimensions

$$\begin{aligned} &\dim \mathbb{E}_-(H(q,\zeta)) = (N - R_-) + (N - L_+) = N + k - 1 \text{ as before}, \\ &\dim \mathbb{E}_-(P(q,\zeta)) = r_- + \ell_+ = N' + 1 - k, \end{aligned} \tag{8.25}$$

so $\mathbb{D}_s(q, \hat{\zeta}, \rho)$ (4.50) is an $(N+N') \times (N+N')$ determinant. Similarly, D_s (4.34) and $\tilde{D}_s$ (4.45) are of size $2(N+N') \times 2(N+N')$. The modified Evans functions $\tilde{D}$ (4.58) and D_m (4.72) are of size $(2(N+N')+1) \times (2(N+N')+1)$.

With $\tilde{\Psi}$ as in (8.10), the proof of the nonconservative Zumbrun-Serre works without any substantial change. Now, for example, the block $\Psi^1_{\tilde{a}}$ in (5.27) is of size $(N'-k) \times (N'-k)$.

The spaces $\hat{\mathcal{C}}$ and $\mathbb{E}_+(P)$ that enter into the block decomposition of D_m (see (5.75),(5.76)) are now respectively of dimensions $N+1-k$ (as before) and $N'-1+k$. Thus, $\beta(q,\zeta)$ (5.78) is given by a $2N' \times 2N'$ determinant.

The space $\mathcal{C}^*$ that enters into the definition of $\tilde{D}_s$ (see (5.101), (5.102)) has dimension $N+1-k$ as before.

Theorem 5.15 summarizing the low frequency results can be repeated verbatim in the case of real viscosity.

7. Viscous $\mathcal{C}_\mathcal{B}$-shocks. The construction of approximate viscous shocks in section 6 can be repeated with no significant changes. Theorem 1.16, giving the existence of a family of viscous $\mathcal{C}_\mathcal{B}$-shocks converging to a given inviscid $\mathcal{C}_\mathcal{B}$-shock as $\varepsilon \to 0$, remains true as stated when we substitute the new structural Assumptions 8.1, 8.2, 8.3, and 8.4. The estimates and iteration scheme of [GMWZ4] (in particular, the high frequency analysis, which is the main difference between the fully and partially parabolic cases) can be repeated without change to complete the proof of Theorem 1.16 for real viscosities. We recall that the high frequency analysis of [GMWZ4] requires both hypotheses $(H4)_h$ and $(H5)_h$.

9 Appendix B: Uniqueness of $\mathcal{C}_\mathcal{B}$

Some choices were made in the definition of $\mathcal{C}_\mathcal{B}$, and it is not immediately clear that $\mathcal{C}_\mathcal{B}$ is independent of these choices. We show this now using properties of the functions $\Phi(z,p,s,h,a)$ arising from translation invariance of the profile equation.

For example, recall the choice of $\underline{z}$, which determines $\underline{a}$, in (2.25). By the argument of Prop. 2.2, part (b), $-\underline{z} \geq 0$ should be large enough so that for all $z_0 \geq -\underline{z}$

$$\begin{aligned} &|\partial_z \underline{w}|_{L^1(|z|\geq z_0)} \leq R,\ |\partial_z \underline{w}|_{L^\infty(|z|\geq z_0)} \leq R \\ &|\Pi_\mp(\underline{p}_\pm, 0, 0)\partial_z \underline{w}(\pm z_0)| \leq r, \end{aligned} \tag{9.1}$$

where R and r are the constants in (2.10).

Proposition 9.1. *The manifold $\mathcal{C}_\mathcal{B}$ defined in Proposition 2.8 is, in a sufficiently small neighborhood of $(\underline{p},0,0)$, independent of the choice of $\underline{z}$ as above.*

There is also some freedom in the choice of the third component of $\tilde{\Psi}(p,s,h,a)$ in (2.34); for example, the s in the third component could be replaced by any smooth function $g(s,h)$ such that $g(0,0)=0$.

Proposition 9.2. *The manifold $\mathcal{C}_\mathcal{B}$ defined in Proposition 2.8 is, in a sufficiently small neighborhood of $(\underline{p},0,0)$, independent of the choice of $g(s,h)$ as above replacing s in the third component of $\tilde{\Psi}(p,s,h,a)$. More generally, the same manifold $\mathcal{C}_\mathcal{B}$ near $(\underline{p},0,0)$ is obtained for any choice of the third component of $\tilde{\Psi}$, so long as the rank conditions assumed in Prop. 2.8 hold and $\tilde{\Psi}(\underline{p},0,0,\underline{a})=0$.*

We present the proof of Proposition 9.2. The proof of Proposition 9.1 is similar but easier.

Proof of Proposition 9.2. **1.** At first we take $g(s,h)=s$ as in the definition (2.34) of $\tilde{\Psi}$.

The proof of Prop. 2.8 provides us with functions

$$p_+(p_\beta,s,h), p_-(p_\beta,s,h), a(p_\beta,s,h) \tag{9.2}$$

such that $\mathcal{C}_{\mathcal{B}}$ consists of points of the form

$$(p_+(p_\beta, s, h), p_-(p_\beta, s, h), s, h), \tag{9.3}$$

where (p_β, s, h) varies in a neighborhood $\mathcal{N}_1$ determined by the implicit function theorem. Define

$$\mathcal{C}_{\tilde{\Psi}} = \{(p_+(p_\beta, s, h), p_-(p_\beta, s, h), s, h, a(p_\beta, s, h) : (p_\beta, s, h) \in \mathcal{N}_1\}, \tag{9.4}$$

which is precisely the zero set near $(\underline{p}, 0, 0, \underline{a})$ for the equation

$$\tilde{\Psi}(p, s, h, a) = 0. \tag{9.5}$$

Similarly, for Ψ as in (2.31) and using part 3 of Remark 2.15, we can use the implicit function theorem to define the zero set near $(\underline{p}, 0, 0, \underline{a})$ for the equation

$$\Psi(p, s, h, a) = 0, \tag{9.6}$$

which we write

$$\mathcal{C}_\Psi = \{(p_+(p_\gamma, s, h, a_1), p_-(p_\gamma, s, h, a_1), s, h, a_1, a'(p_\gamma, s, h, a_1) : (p_\gamma, s, h, a_1) \in \mathcal{N}_2\}. \tag{9.7}$$

Here we set $a = (a_1, a')$ and take a_i in Remark 2.15 to be a_1.

Let π be the projection

$$\pi(p_+, p_-, s, h, a) = (p_+, p_-, s, h). \tag{9.8}$$

It suffices to show

$$\pi\mathcal{C}_{\tilde{\Psi}} = \pi\mathcal{C}_\Psi \text{ near } (\underline{p}, 0, 0). \tag{9.9}$$

2. We clearly have

$$\mathcal{C}_{\tilde{\Psi}} \subset \mathcal{C}_\Psi \text{ near } (\underline{p}, 0, 0, \underline{a}), \text{ and thus } \pi\mathcal{C}_{\tilde{\Psi}} \subset \pi\mathcal{C}_\Psi \text{ near } (\underline{p}, 0, 0), \tag{9.10}$$

simply because Ψ gives the first two components of $\tilde{\Psi}$.

3. A particular point

$$P = (p_+(p_\gamma, s, h, a_1), p_-(p_\gamma, s, h, a_1), s, h, a_1, a'(p_\gamma, s, h, a_1) \tag{9.11}$$

of $\mathcal{C}_\Psi$ has the property that the function of z given by (recall (2.26))

$$f_P(z) := \phi(z, p_+(p_\gamma, s, h, a_1), p_-(p_\gamma, s, h, a_1), s, h, a_1, a'(p_\gamma, s, h, a_1)) \tag{9.12}$$

satisfies at $z = 0$ the first two parts of the boundary condition (2.33), but not necessarily the third. In particular it defines a smooth solution of (1.16) on $\mathbb{R}$.

For $|g(s, h)| = |s|$ small enough, the function

$$f_P(z + r) = \phi(z + r, p_+(p_\gamma, s, h, a_1), p_-(p_\gamma, s, h, a_1), s, h, a_1, a'(p_\gamma, s, h, a_1)) \tag{9.13}$$

will satisfy the third boundary condition at $z = 0$ for some translate r (here we suppress the dependence of r on the other parameters). Thus, by translation invariance of (1.16), $f_P(z + r)$ is a smooth solution of (1.16) on $\mathbb{R}$ and satisfies the third boundary condition at $z = 0$. In other words, it satisfies (2.1)(a), (2.4), and all three boundary conditions at $z = 0$.

Now, for $-\underline{z}$ initially chosen large enough, the argument of (2.2) yields an a^* near $\underline{a}$ such that

$$f_P(z + r) = \phi(z, p_+(p_\gamma, s, h, a_1), p_-(p_\gamma, s, h, a_1), s, h, a^*). \tag{9.14}$$

But then

$$P^* = (p_+(p_\gamma, s, h, a_1), p_-(p_\gamma, s, h, a_1), s, h, a^*) \tag{9.15}$$

is a point near $(\underline{p}, 0, 0, \underline{a})$ satisfying $\tilde{\Psi} = 0$.

Hence $P^* \in \mathcal{C}_{\tilde{\Psi}}$, and this implies that for P as in (9.11)

$$\pi P \in \pi \mathcal{C}_{\tilde{\Psi}}. \tag{9.16}$$

Thus, $\pi\mathcal{C}_\Psi \subset \pi\mathcal{C}_{\tilde{\Psi}}$ near $(\underline{p}, 0, 0)$, so we obtain (9.9).

4. The same argument applies if the third boundary condition (and hence $\tilde{\Psi}$) is redefined by replacing s with any smooth function $g(s, h)$ satisfying $g(0, 0) = 0$. Indeed, the argument works for any choice of the third component of $\tilde{\Psi}$, so long as the rank conditions (2.37) hold and $\tilde{\Psi}(\underline{p}, 0, 0, \underline{a}) = 0$.

□

The following description of $\mathcal{C}_\mathcal{B}$ in terms of Ψ provided by the above proof is worth emphasizing:

Corollary 9.3. *Assume $\underline{w}$ is transversal (Defn. 2.14) and let $\mathcal{C}_\mathcal{B}$ be the manifold defined in Prop. 2.8. For $\mathcal{C}_\Psi$ as in* (9.7) *and the projection π as in* (2.14) *we have*

$$\mathcal{C}_\mathcal{B} = \pi\mathcal{C}_\Psi \text{ near } (\underline{p}, 0, 0). \tag{9.17}$$

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