Random Walks and Adic Systems

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- ramified—by a random-walk reinforcement scheme (such as the Euler and reverse Euler)

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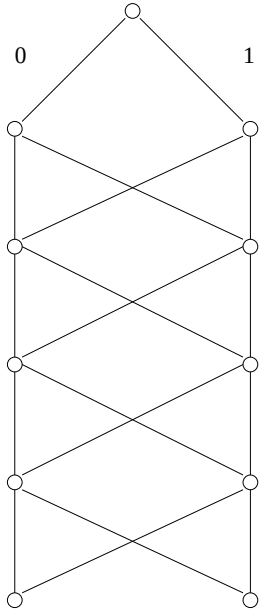

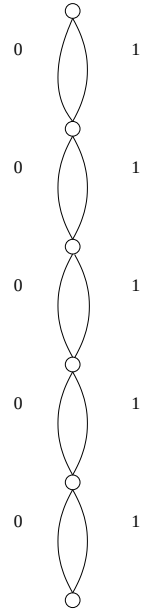

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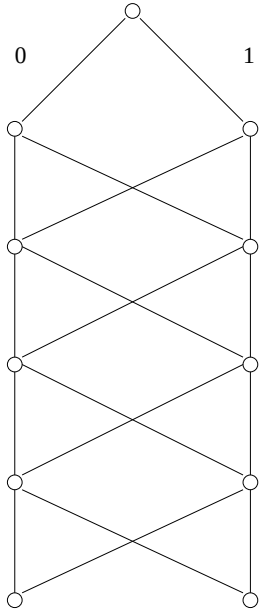

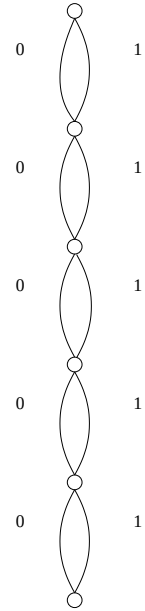

Graded graph, edges just from each level to the next.

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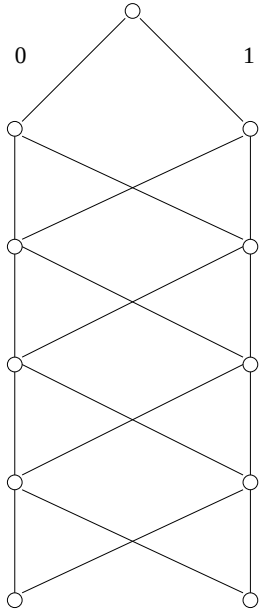

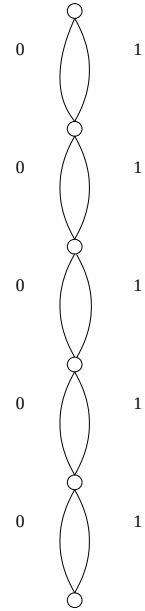

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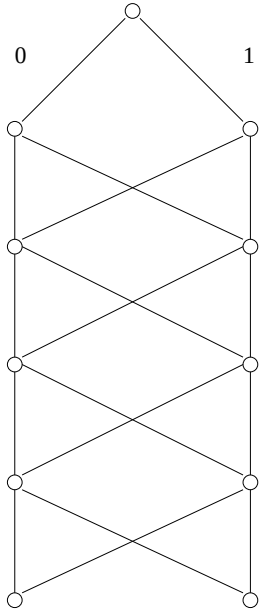

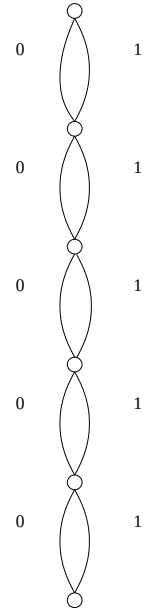

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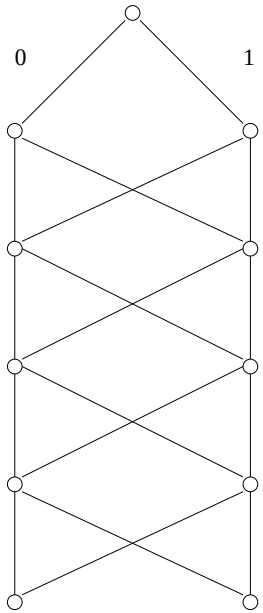

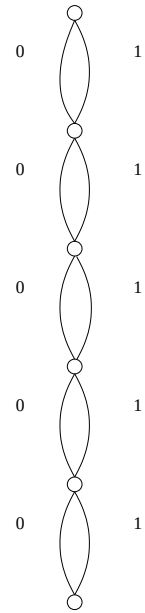

Graded graph, edges just from each level to the next.
Space of infinite paths is compact metric.
Label and order edges down to each vertex.
Two paths x and y are *comparable* if they agree from some level on downwards.
$Tx =$ the smallest $y > x$ (if there is one).

Adics and odometers

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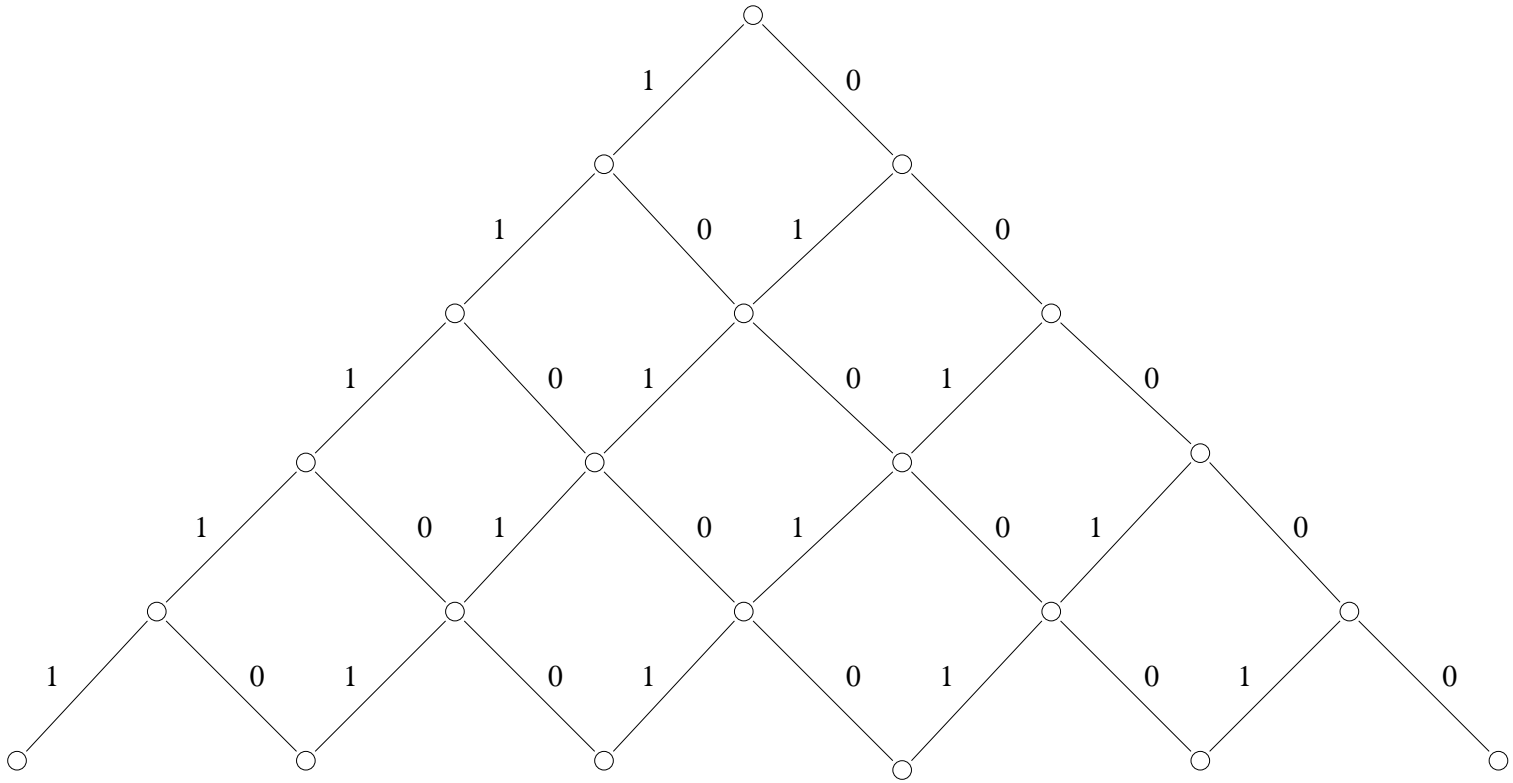
Adic is transverse to shift; translates in a self-similar tiling system, where the shift changes scale.

Pascal adic



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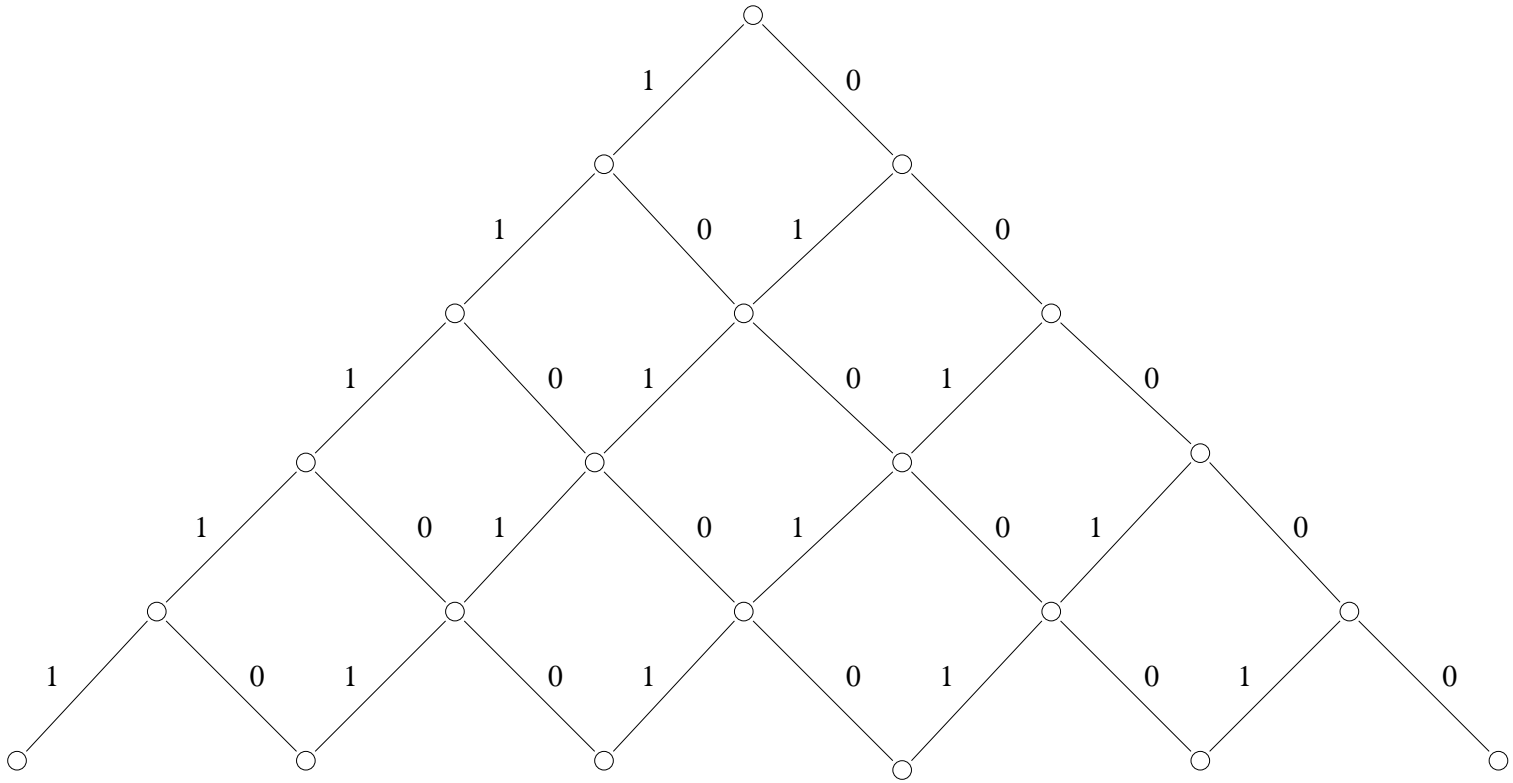

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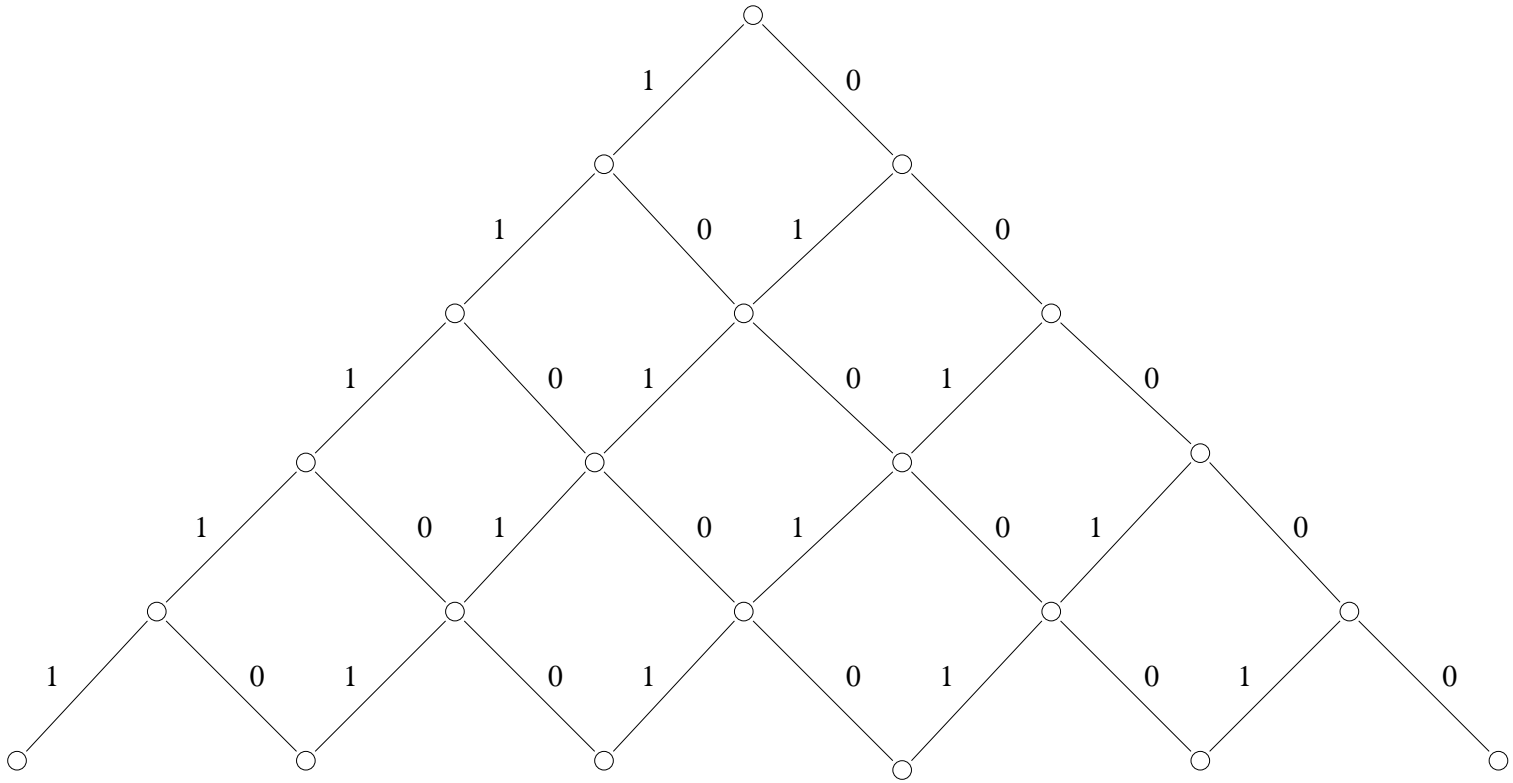

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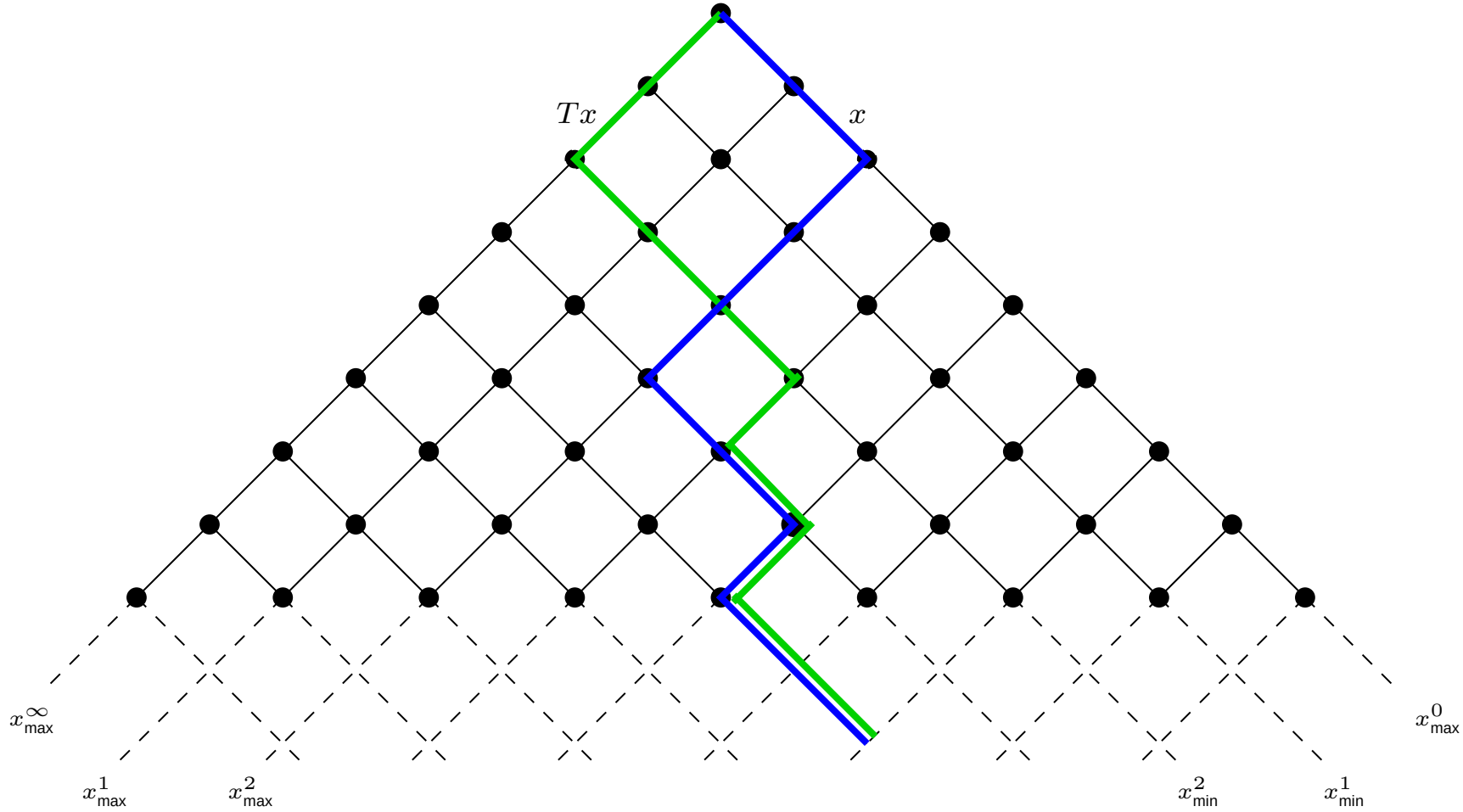
$$T(0^p 1^q 10*) = 1^q 0^p 01*, \quad p, q \geq 0$$

Action of the Pascal adic



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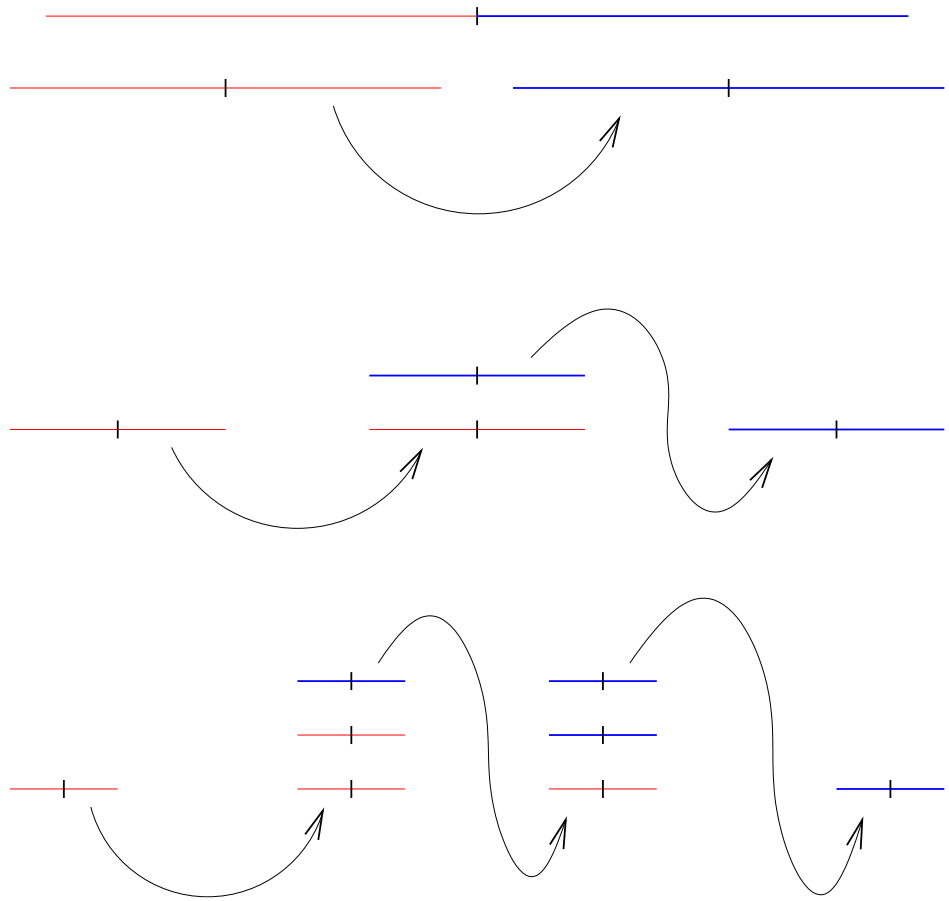
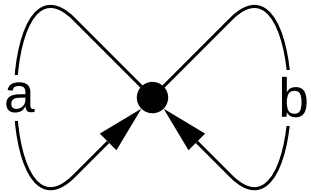
- It also leads to interesting examples, and the visual presentation can help with combinatorial analysis.

Pascal by Cutting and Stacking



C-adic



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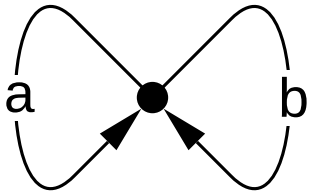

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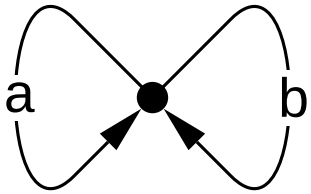

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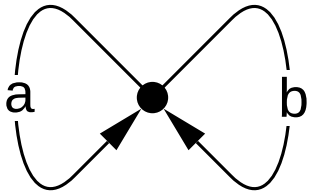

- Keeps track of symbol counts
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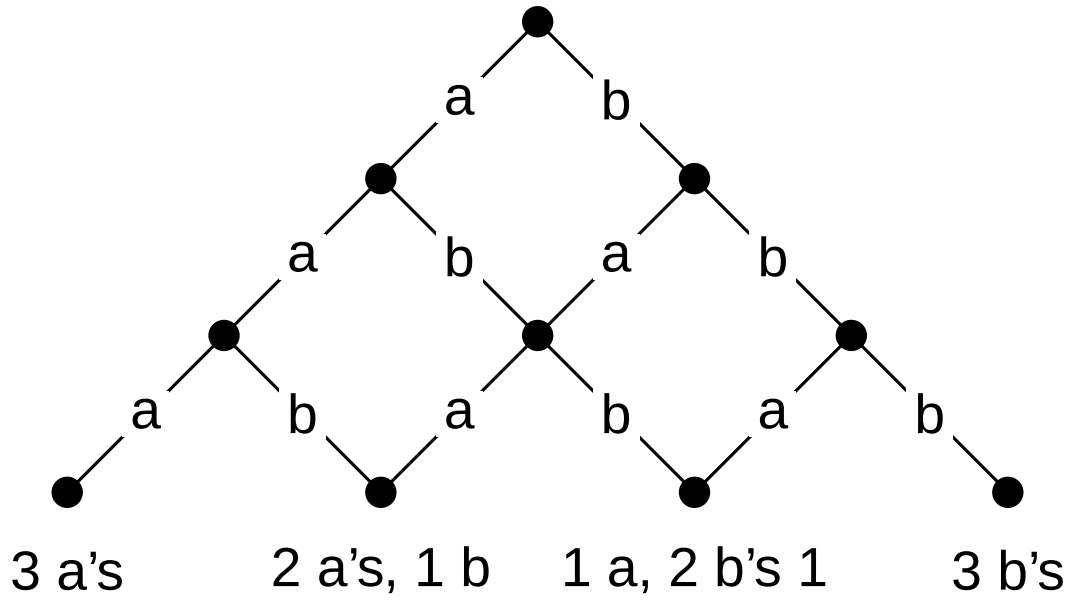
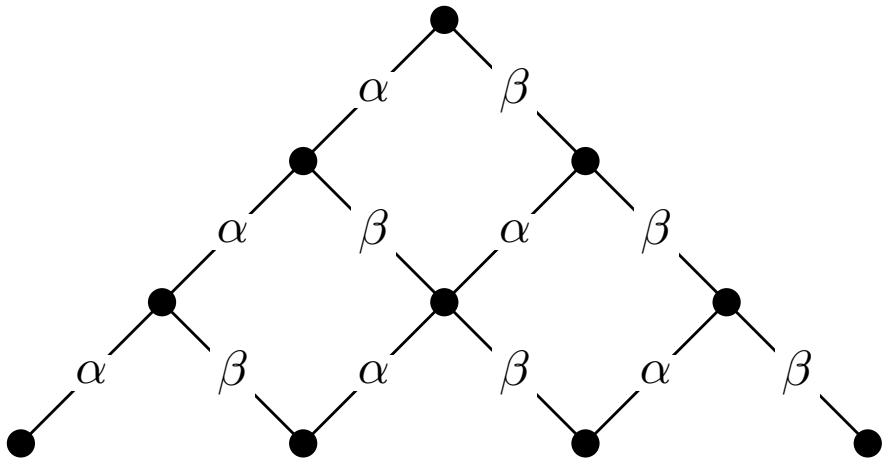
- Each path in the C-adic gives the history from time 0 of the random walk on the labeled edge graph



Invariant measures for the Pascal adic



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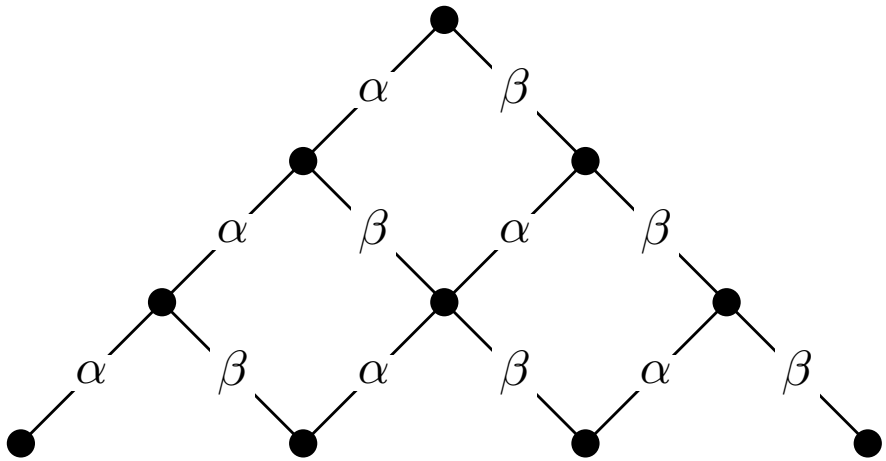

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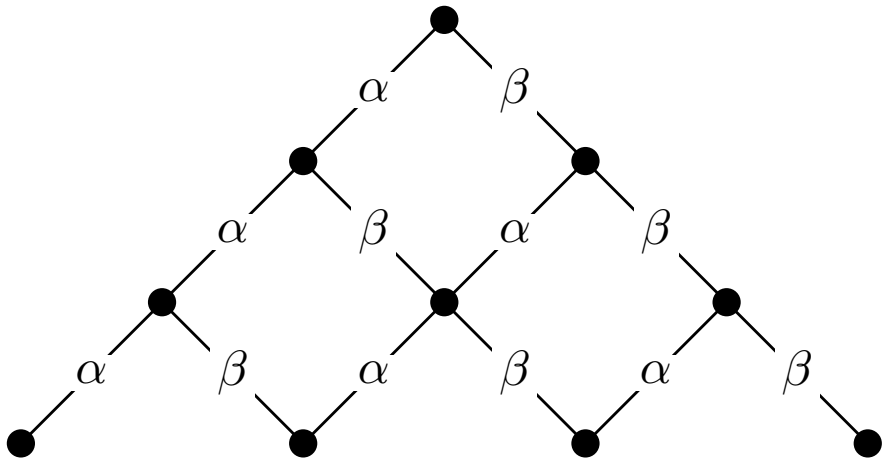

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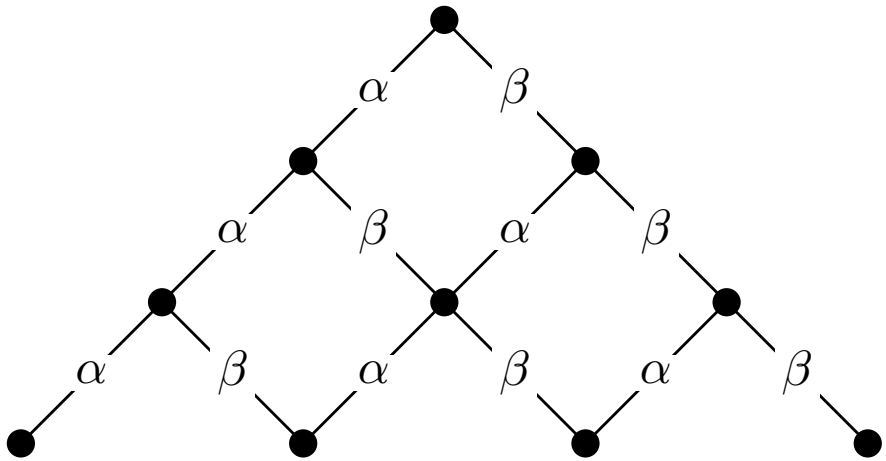

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- The diagram defines the “CCR" C^* algebra.

Pascal as a subshift

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$$b \quad b^5a \quad b^4ab^3ab^2aba^2 \quad b^3ab^2aba^2b^2aba^2ba^3 \quad b^2aba^2ba^3ba^4 \quad ba^5 \quad a$$

$$\ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots$$

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- Loosely Bernoulli (de la Rue and Janvresse; Frick for the Euler)

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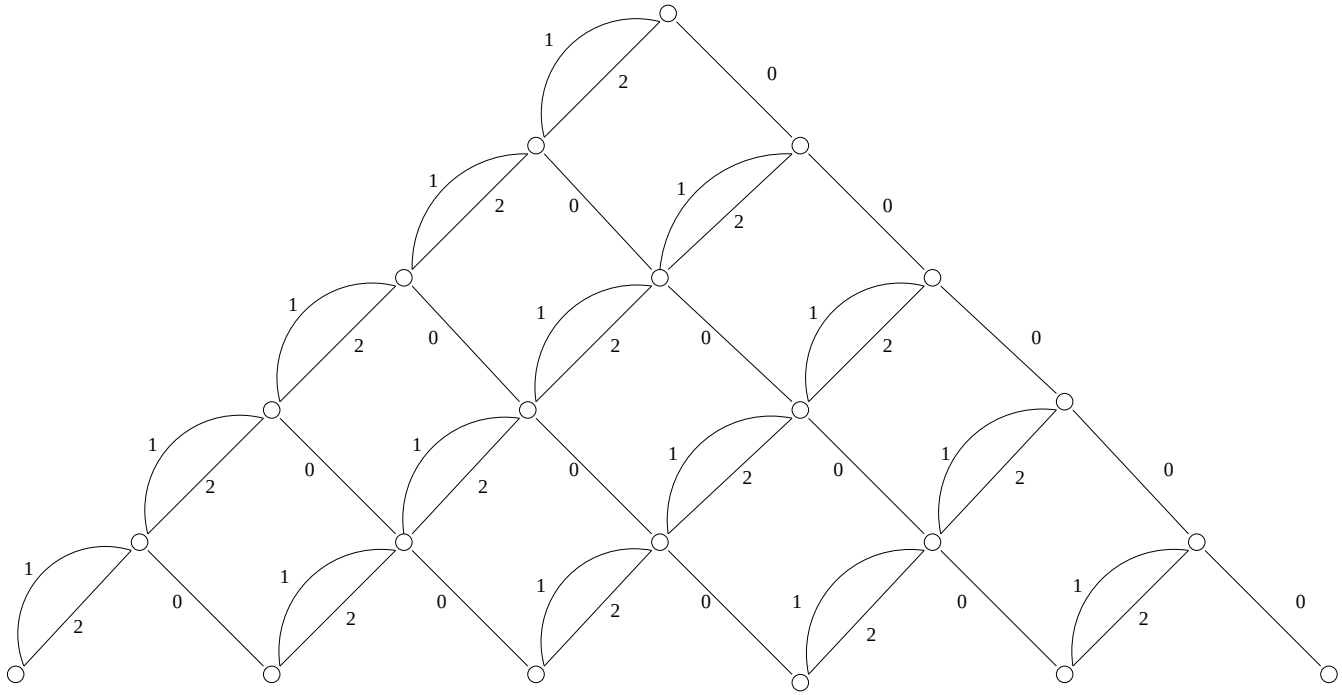

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- Is the system of infinite rank for each μ_α?
- Does it fail to have local rank 1?

Questions continued

- Local rank 1 would imply that there is $a > 0$ such that there are infinitely many m for which there is an m-block B such that

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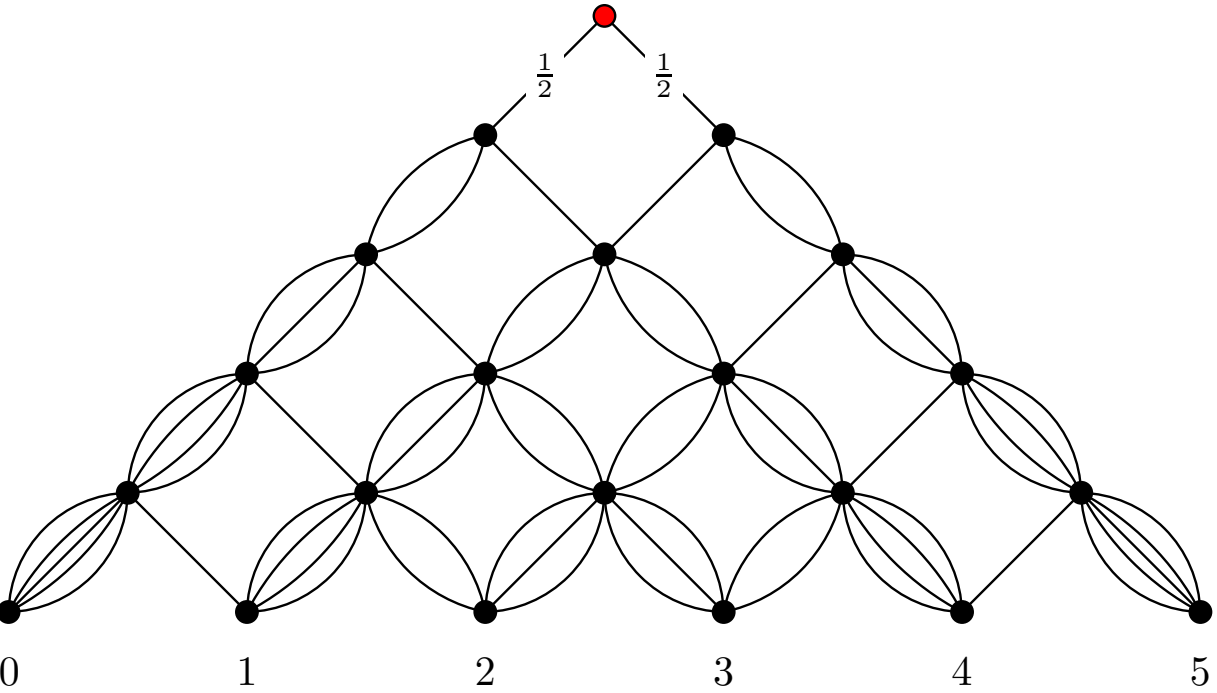
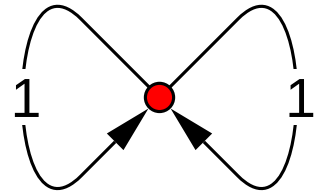
- Is the maximal spectral type singular?
- What are the joinings of μ_α and μ_β?
- Is the joint action of the Pascal and shift on $\{0,1\}^{\mathbb{Z}}$ *effective* (every nonidentity group element moves something)?

Positively reinforced random walk





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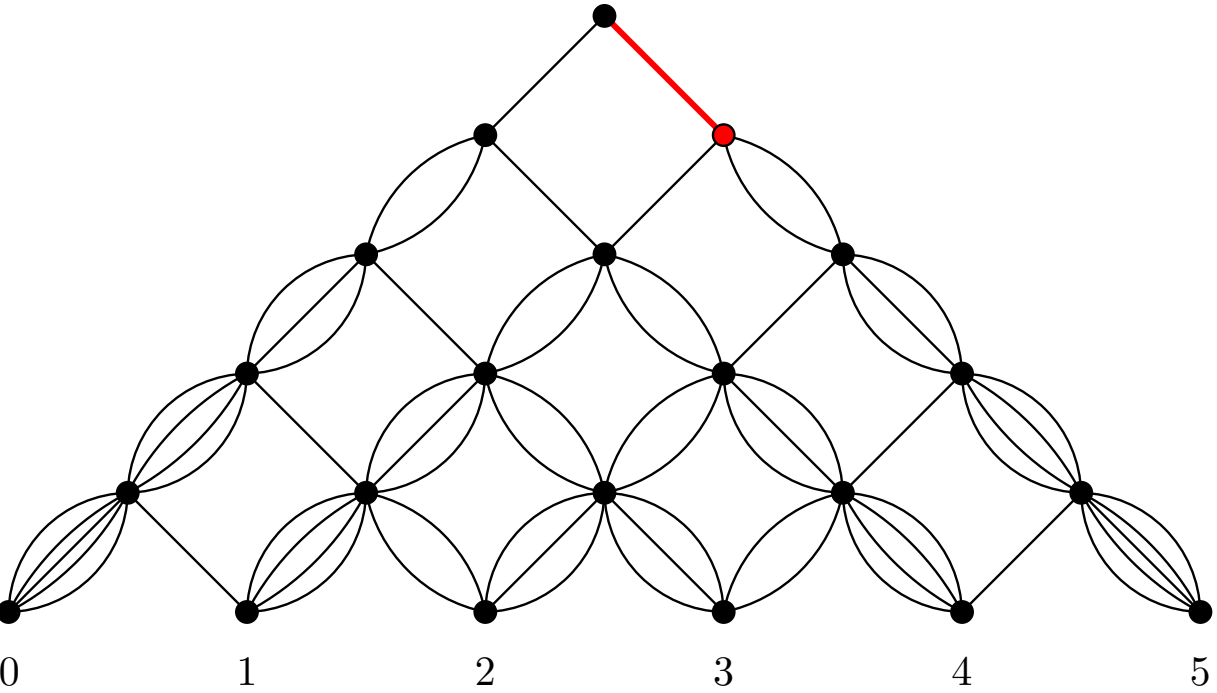

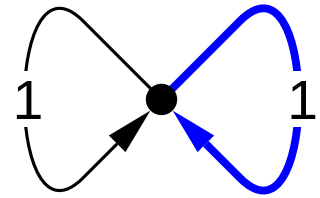

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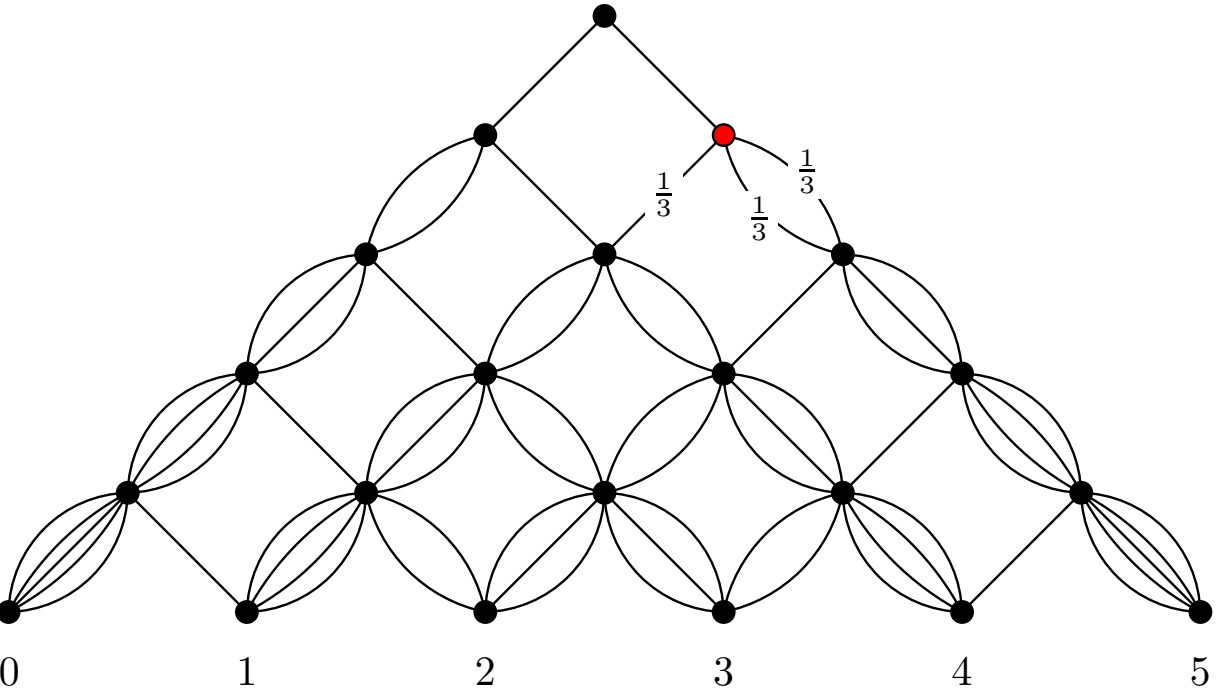

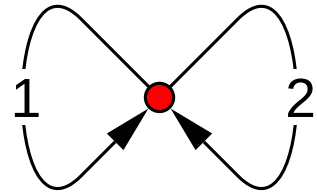

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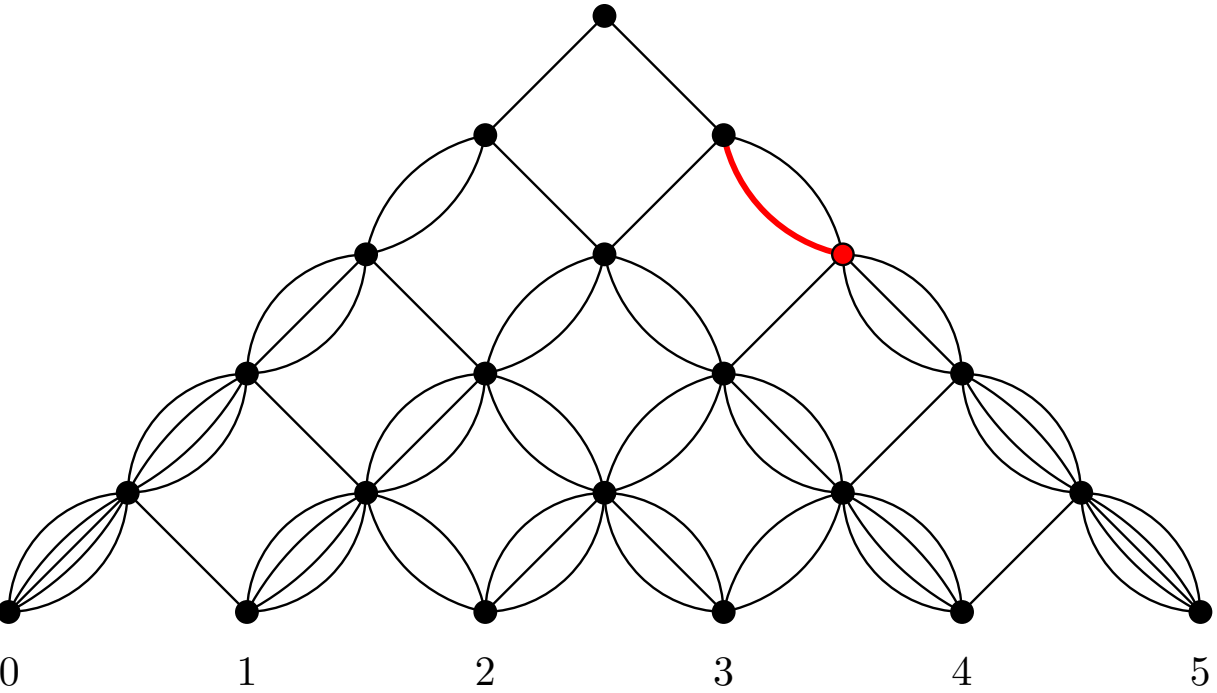

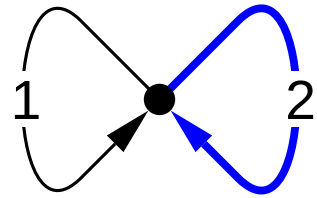

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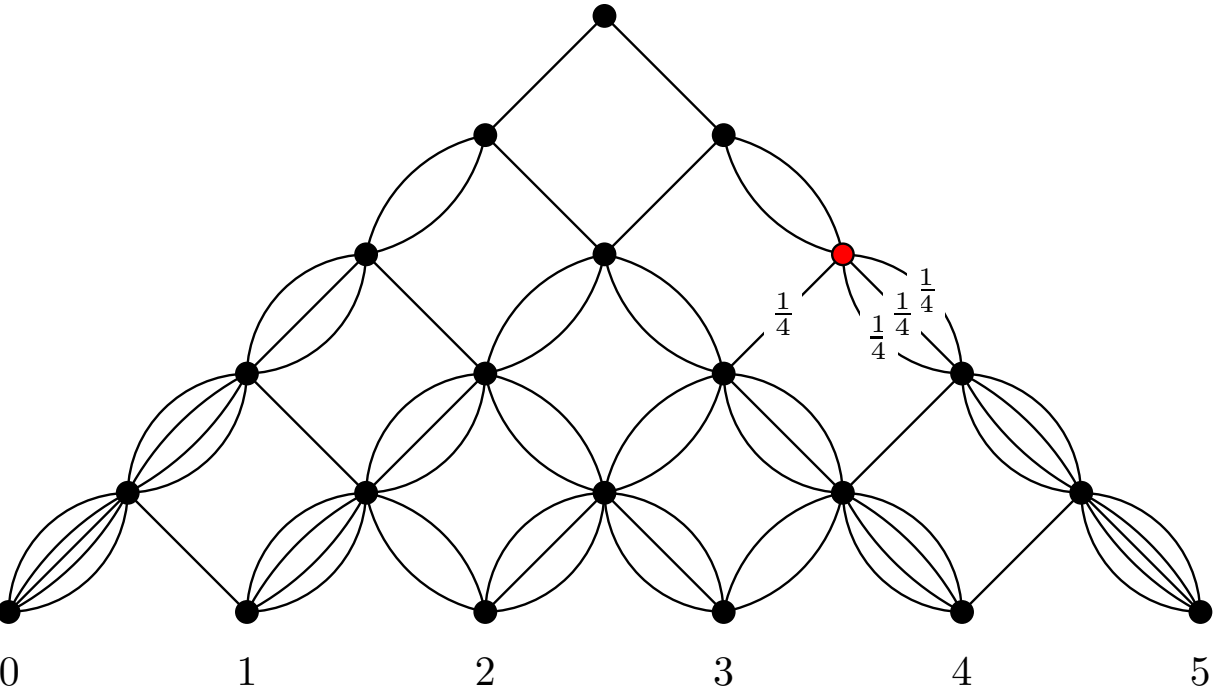

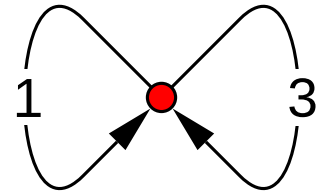

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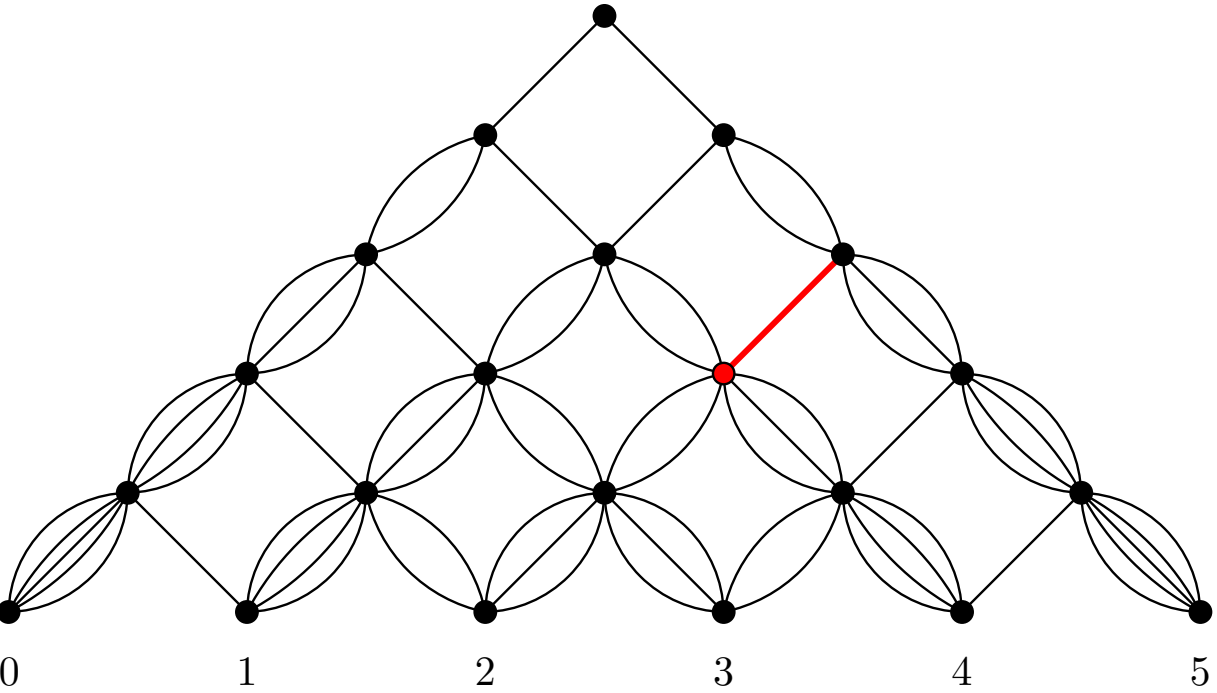

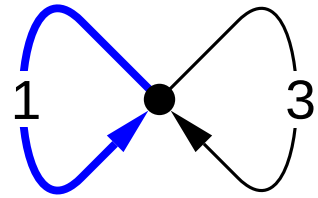

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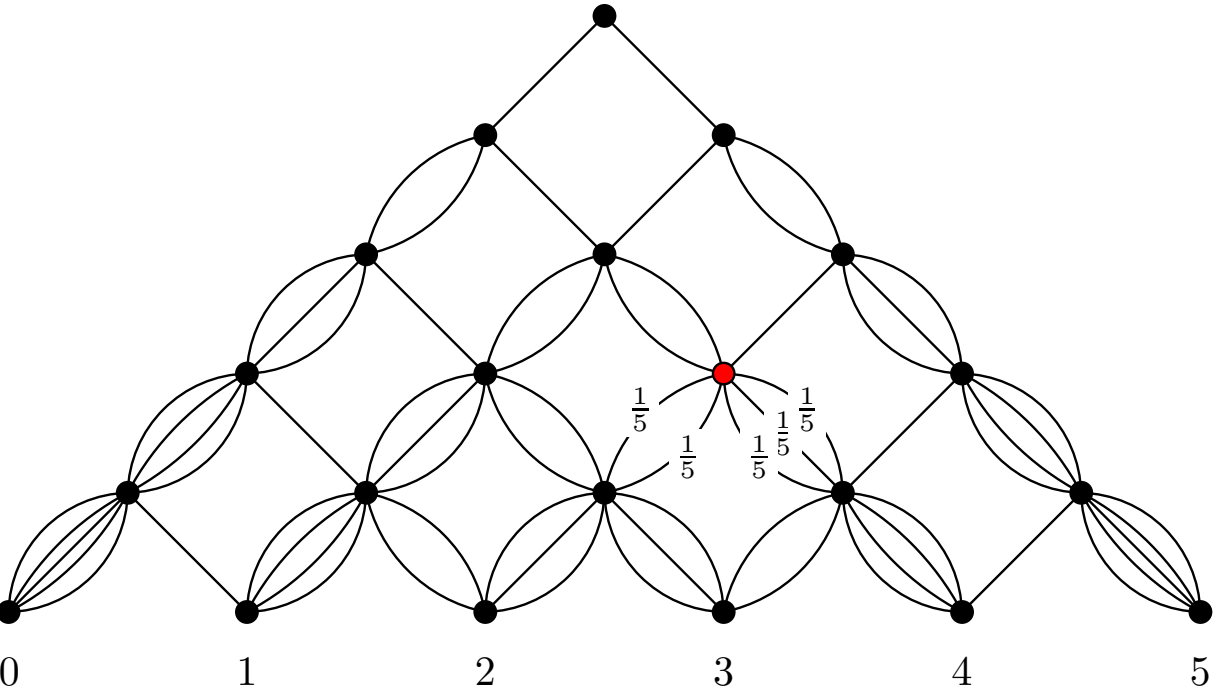

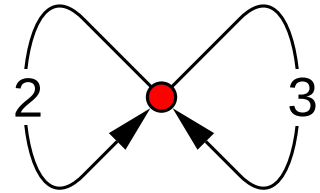

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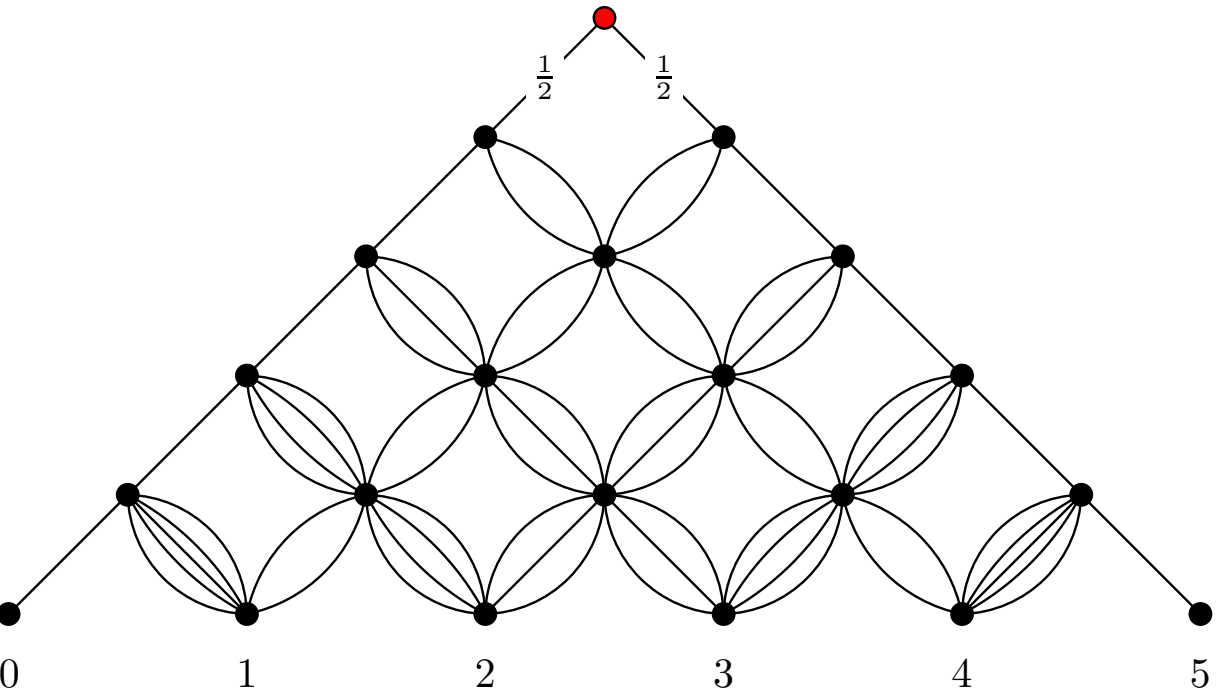
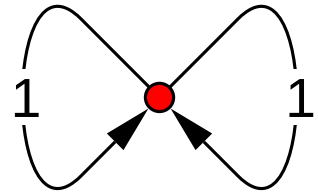
Negatively reinforced random walk, R-adic





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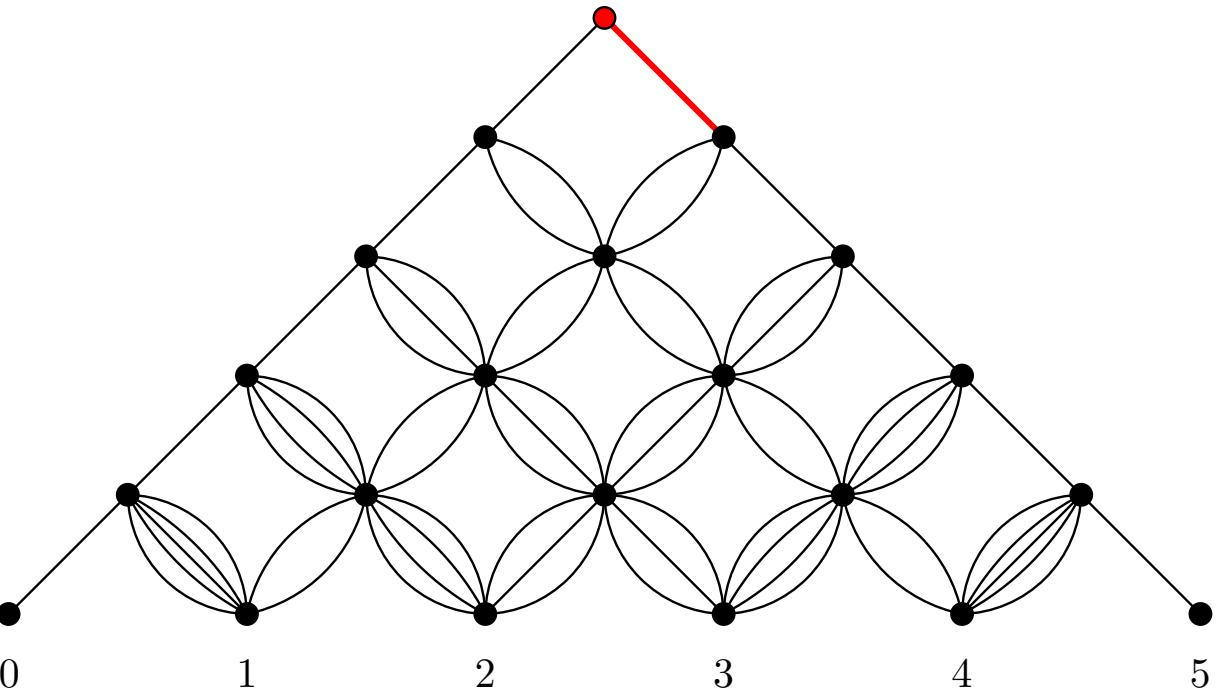

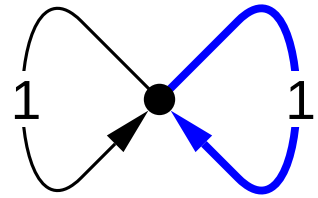

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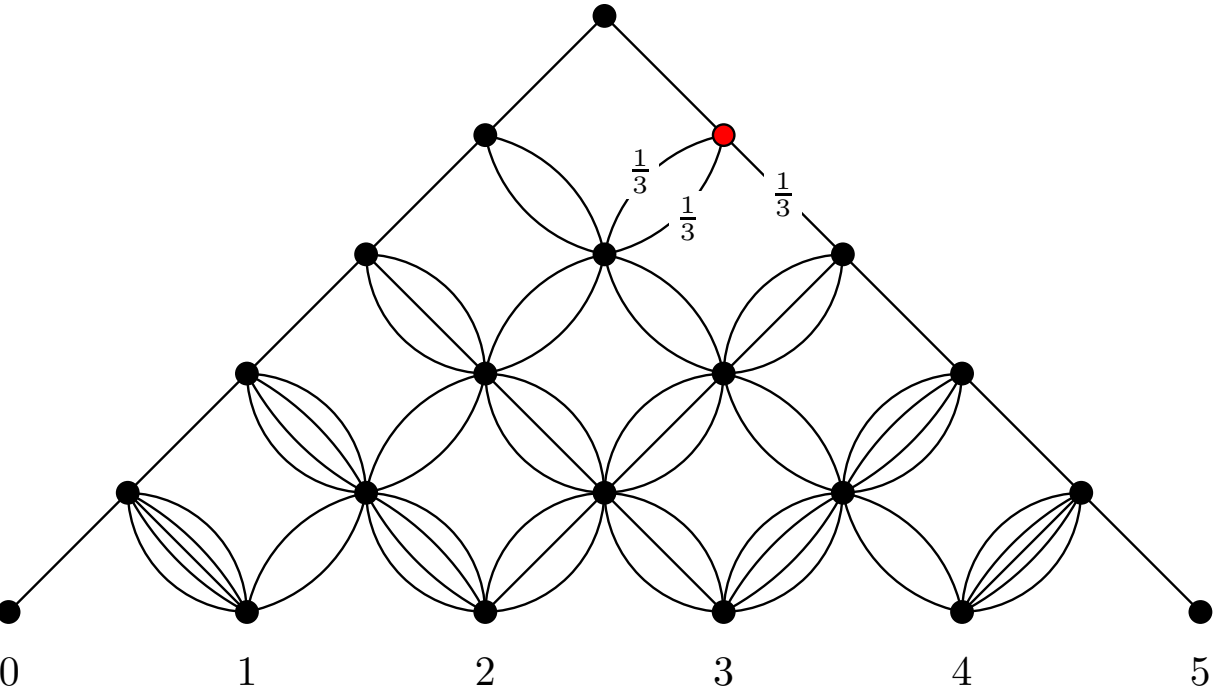

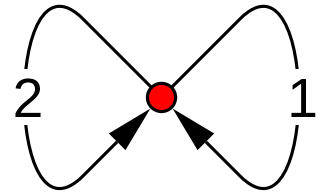

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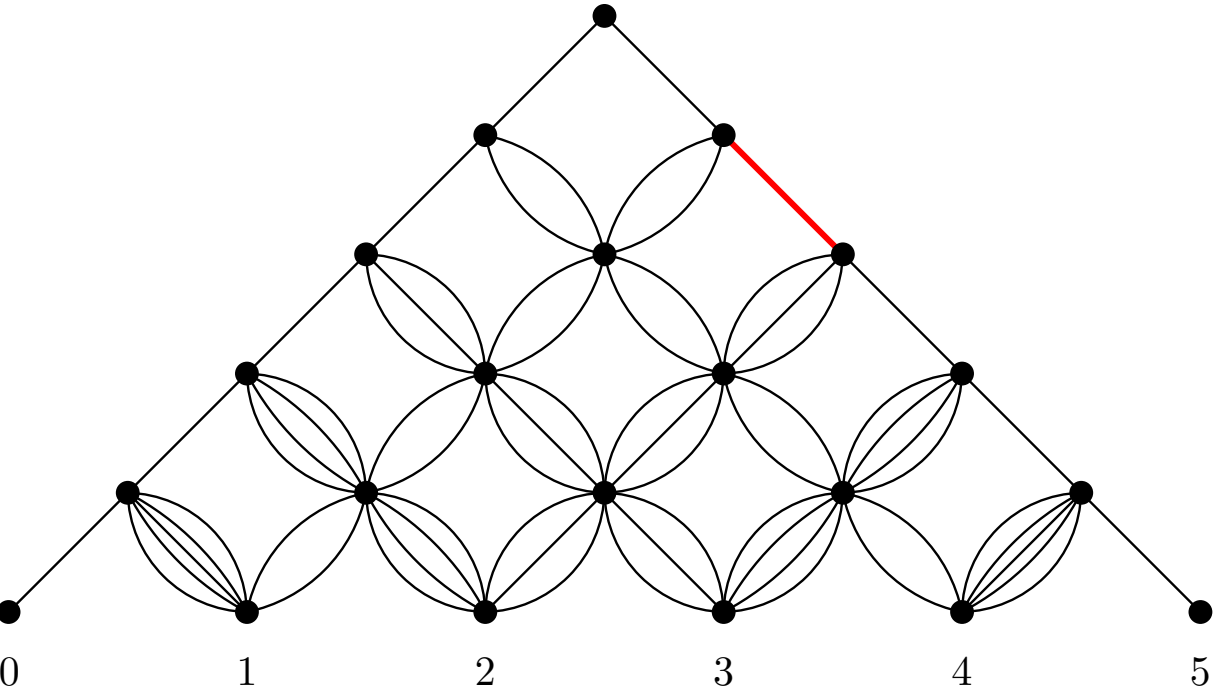

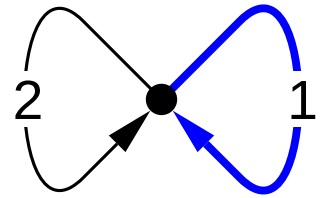

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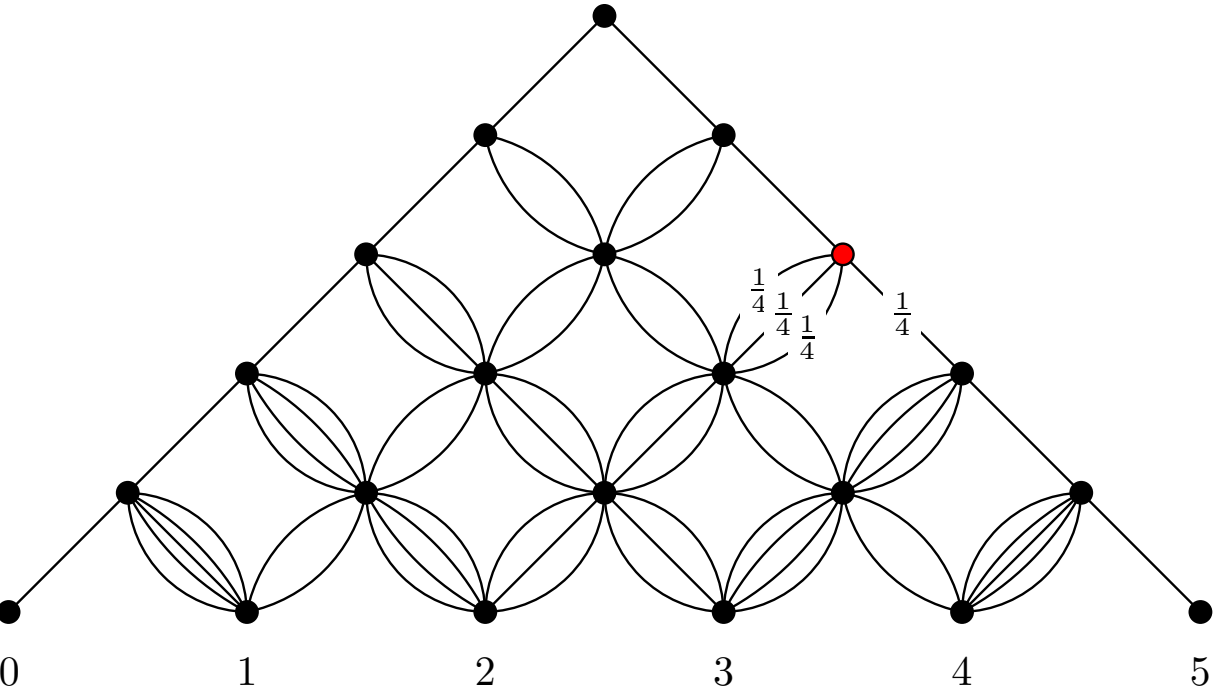

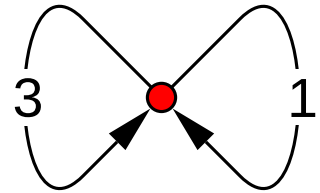

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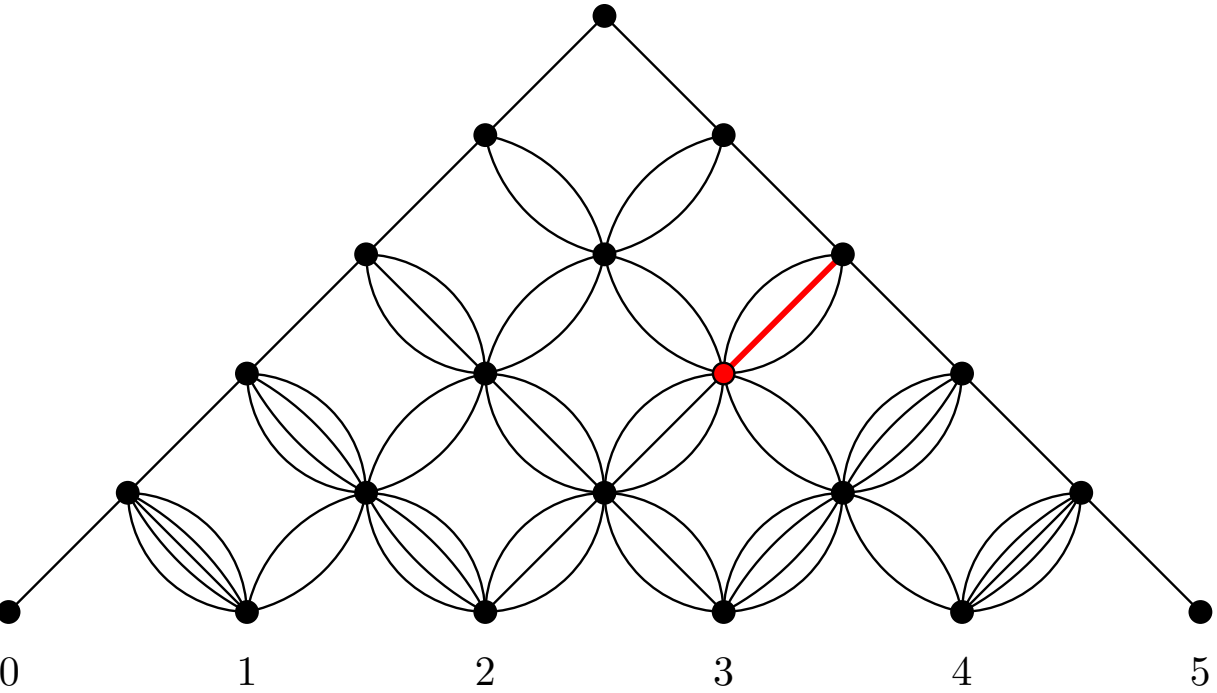

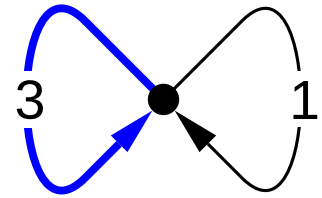

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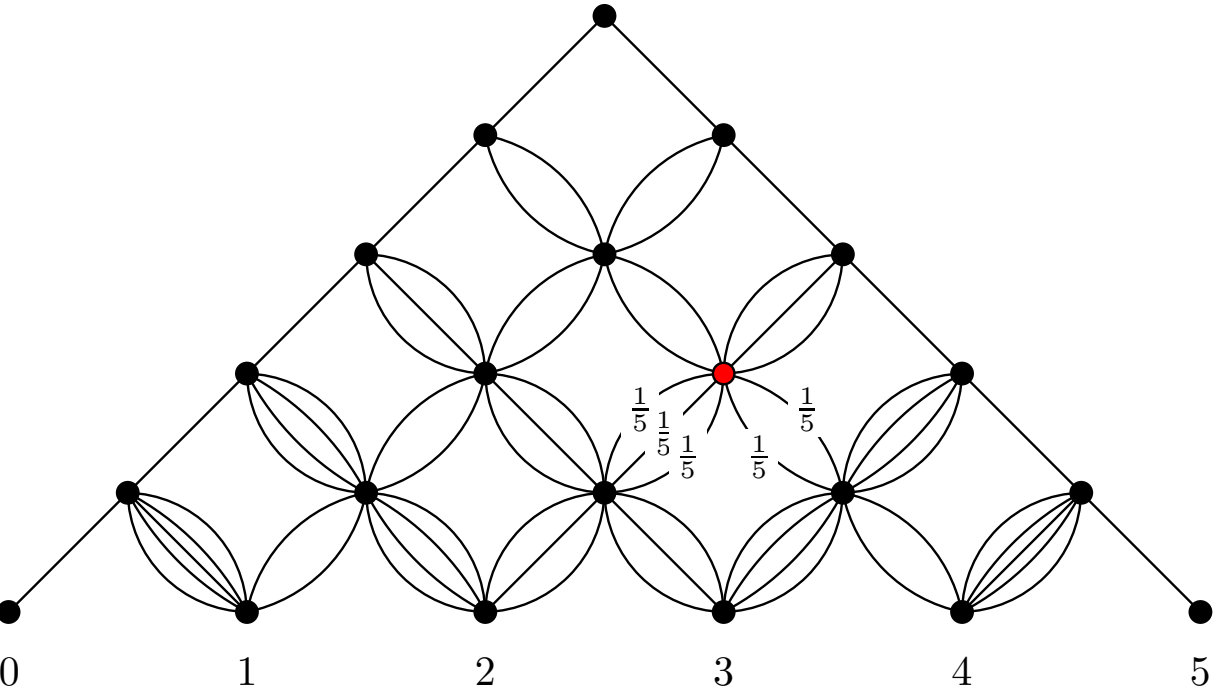

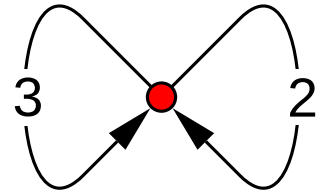

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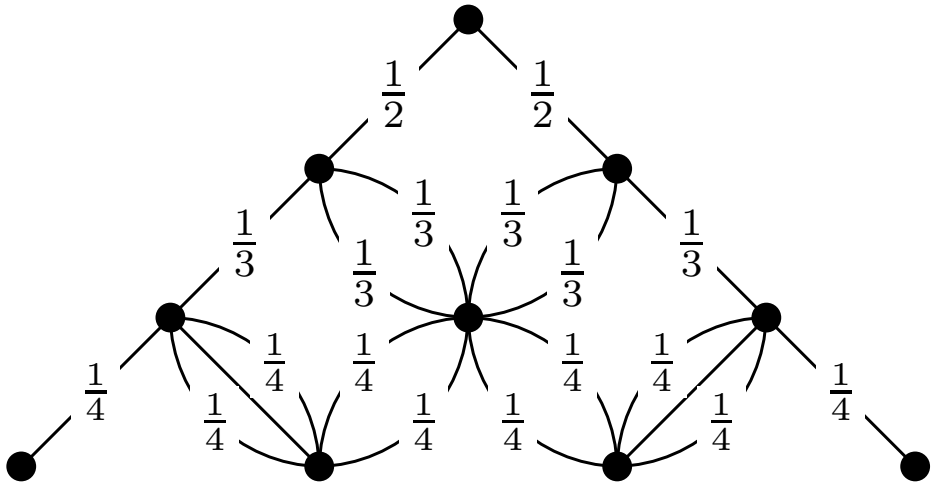
The Walk Measures

The Euler Walk Measure



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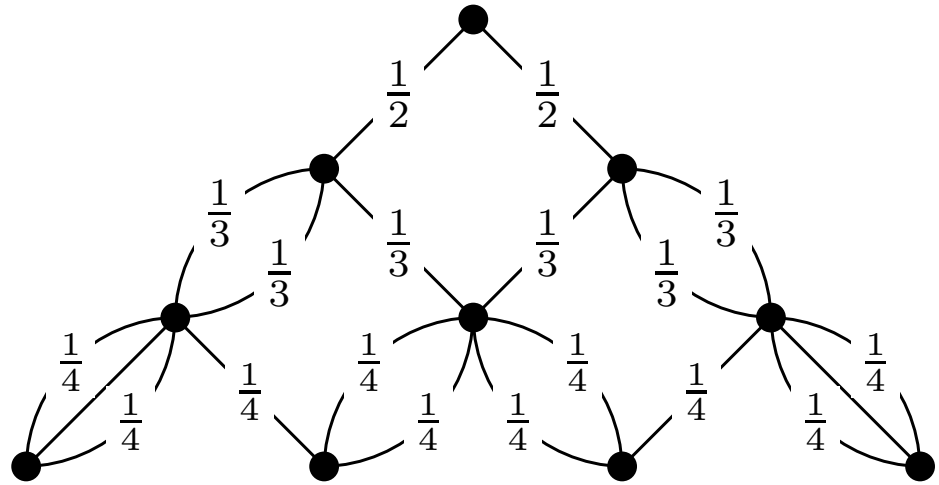
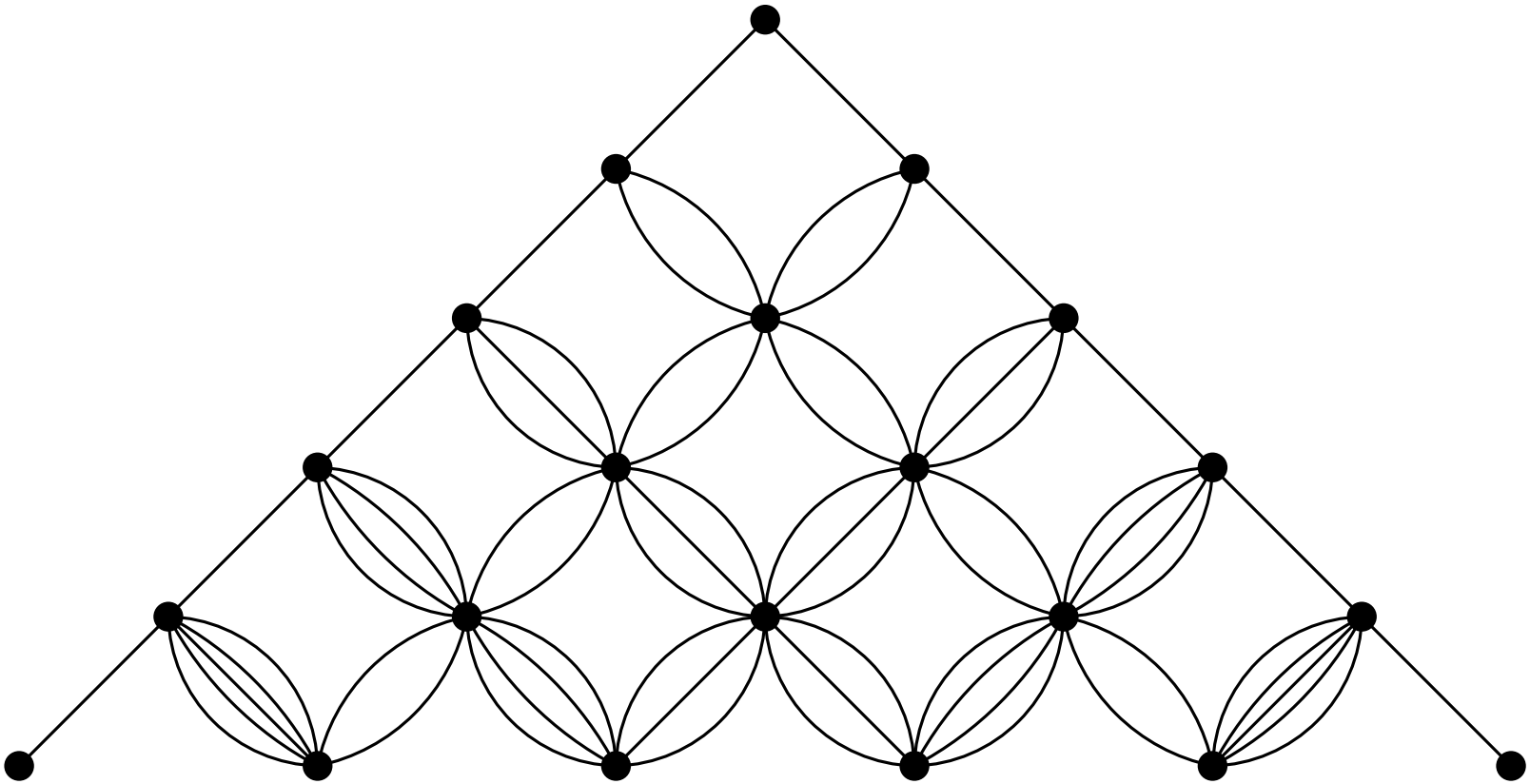


- Adic invariant
- Gives each edge connecting level n to level $n+1$ weight $\frac{1}{n+2}$.
- Gives each cylinder of length n measure $\frac{1}{(n+1)!}$

The Euler adic



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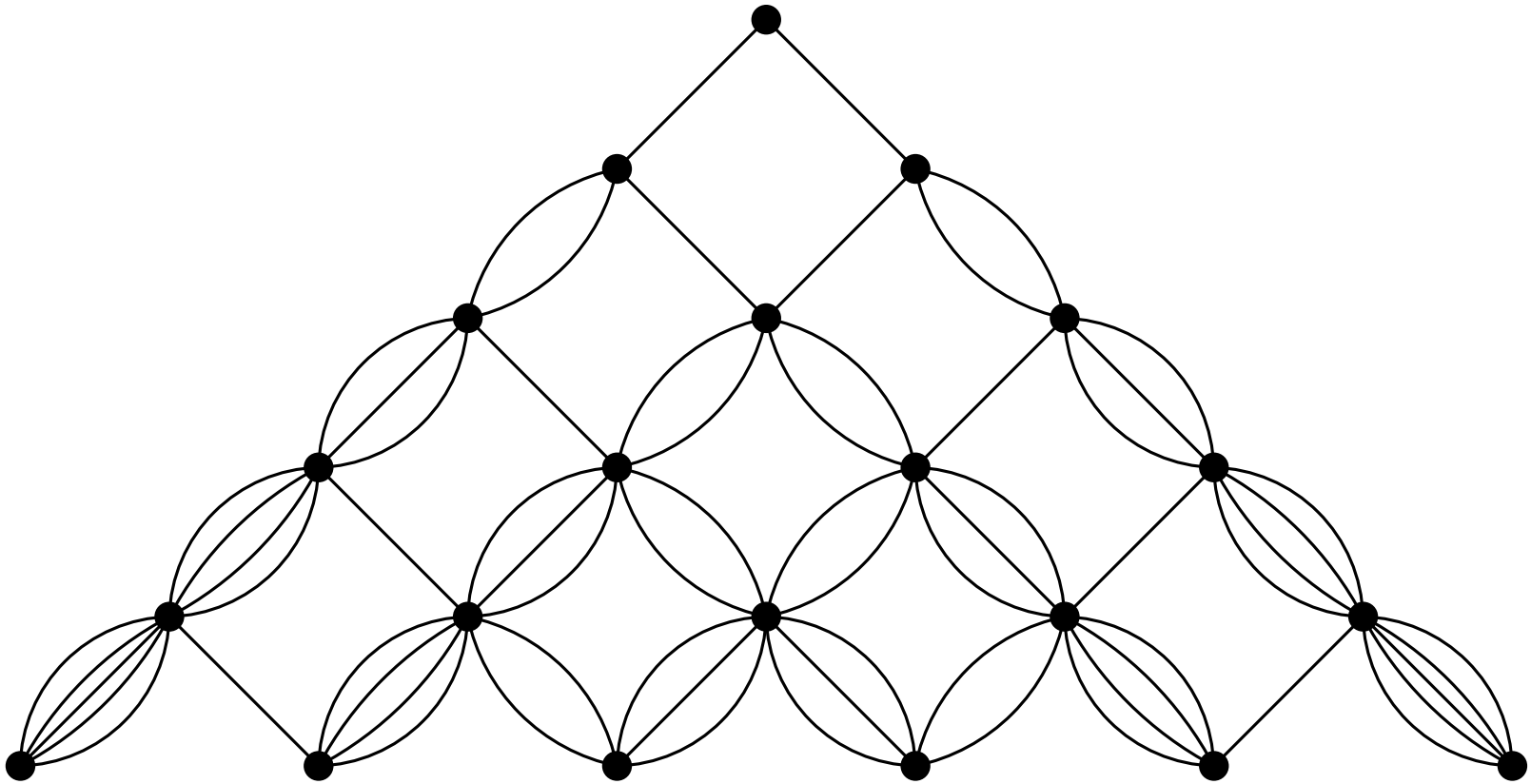
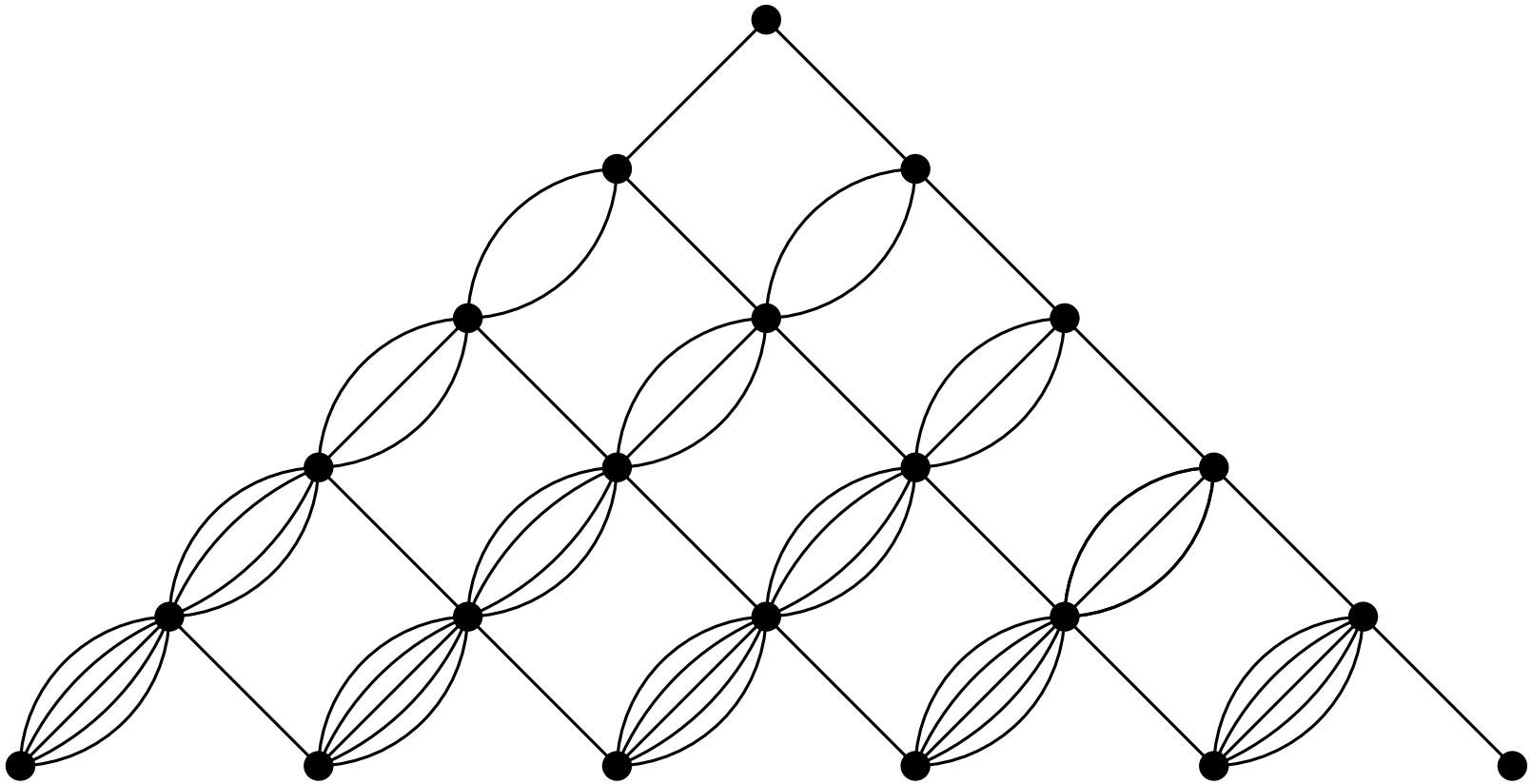


The Stirling adic



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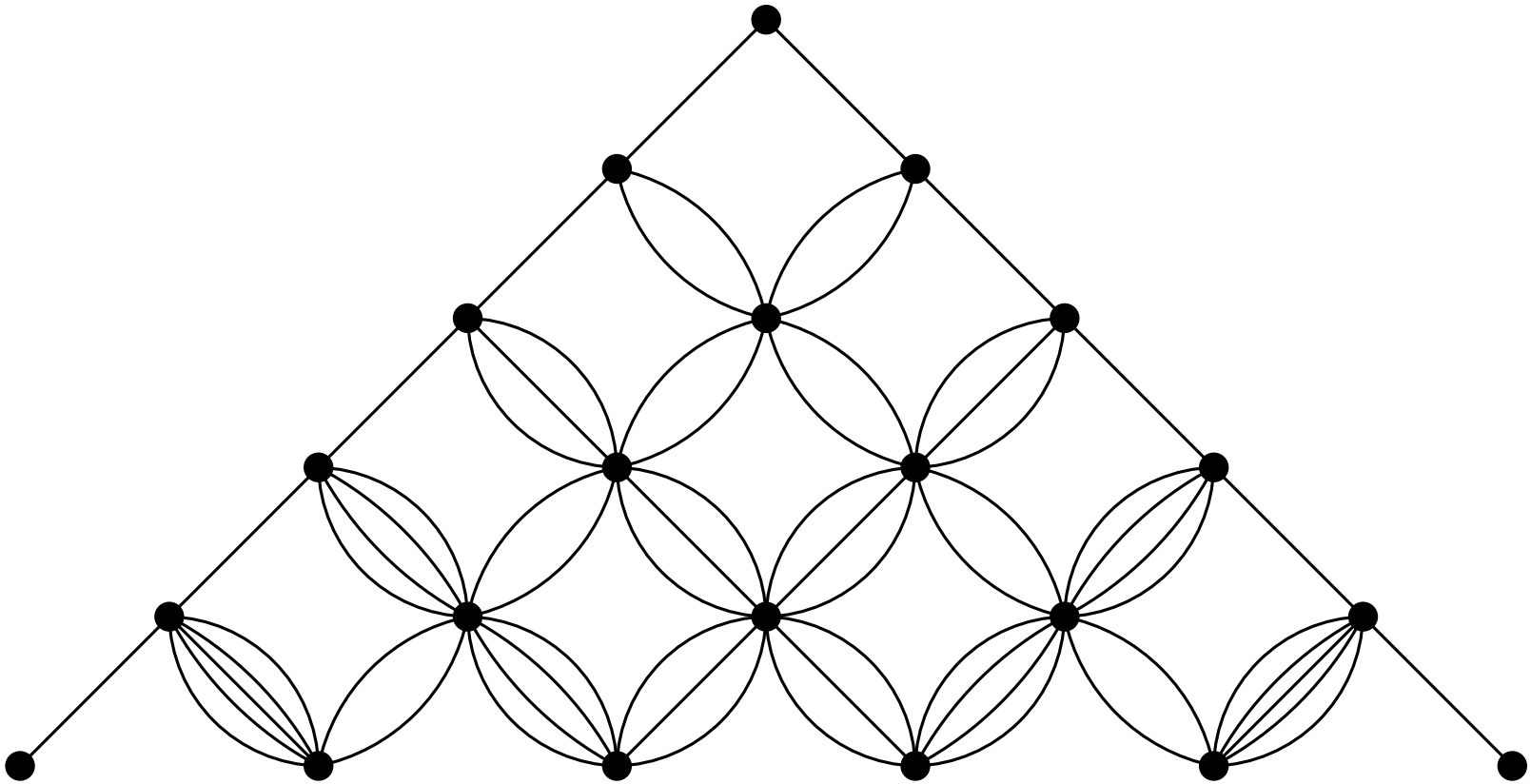
- Proved by Ergodic Theorem or Reverse Martingale Theorem.

Counting paths in the Euler adic



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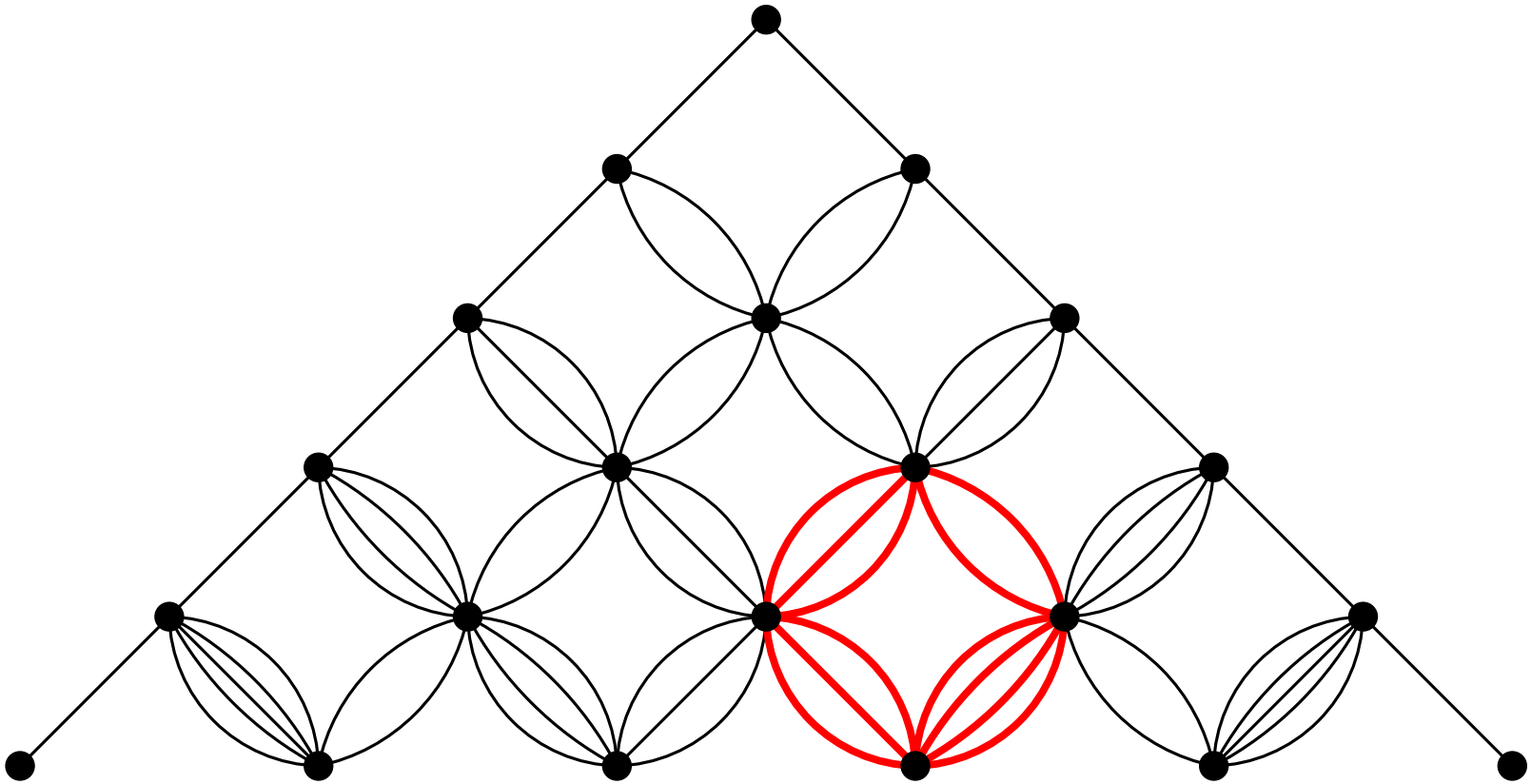

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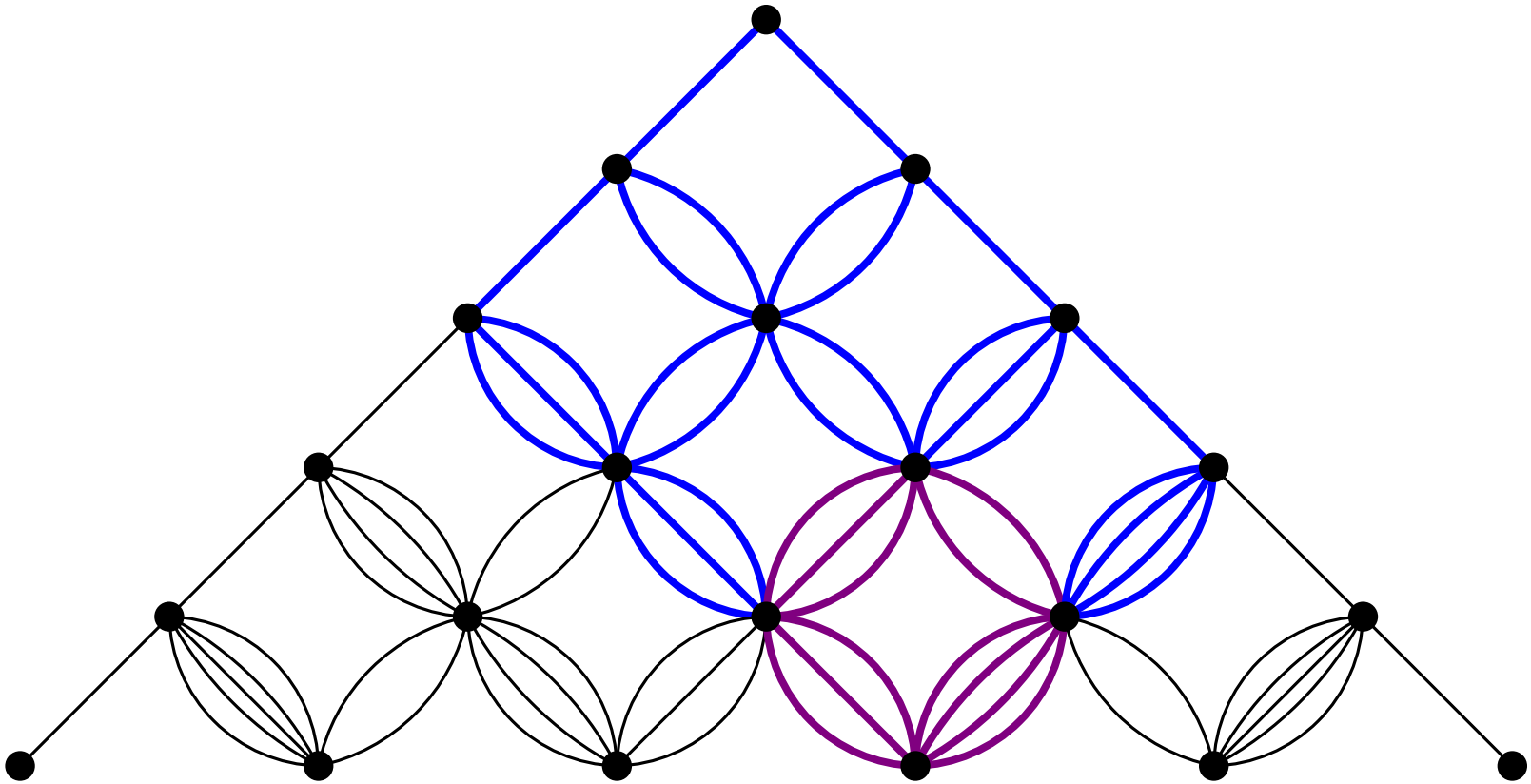
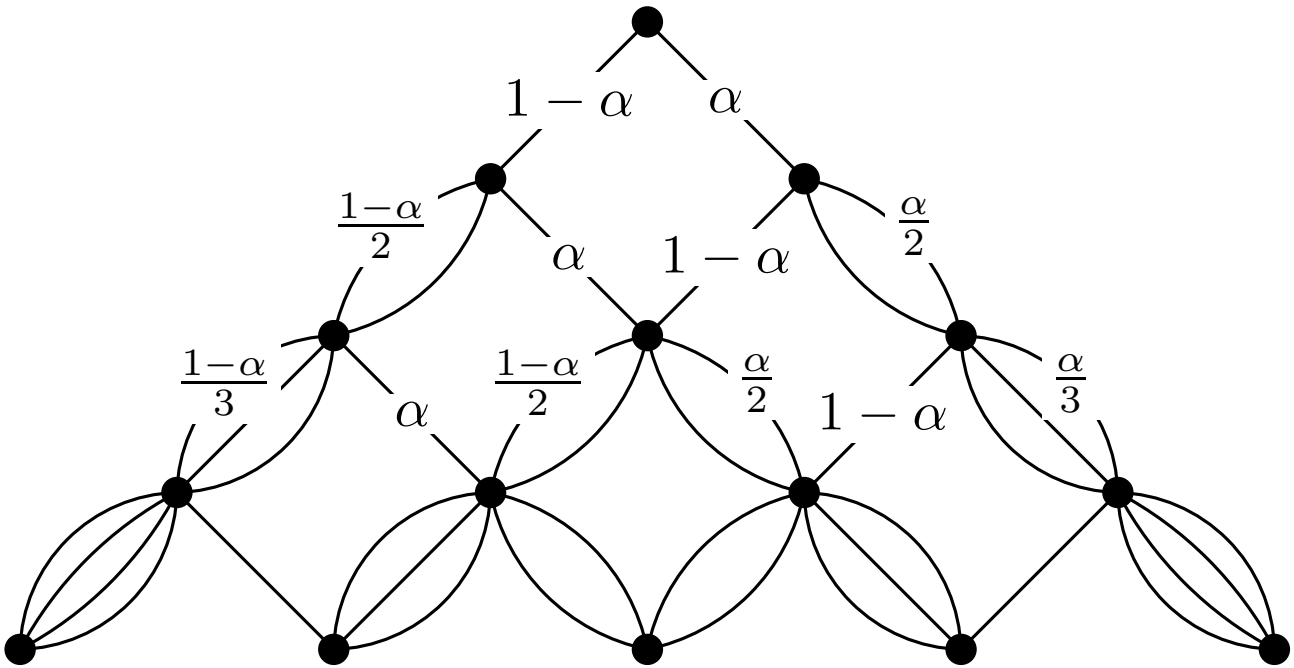


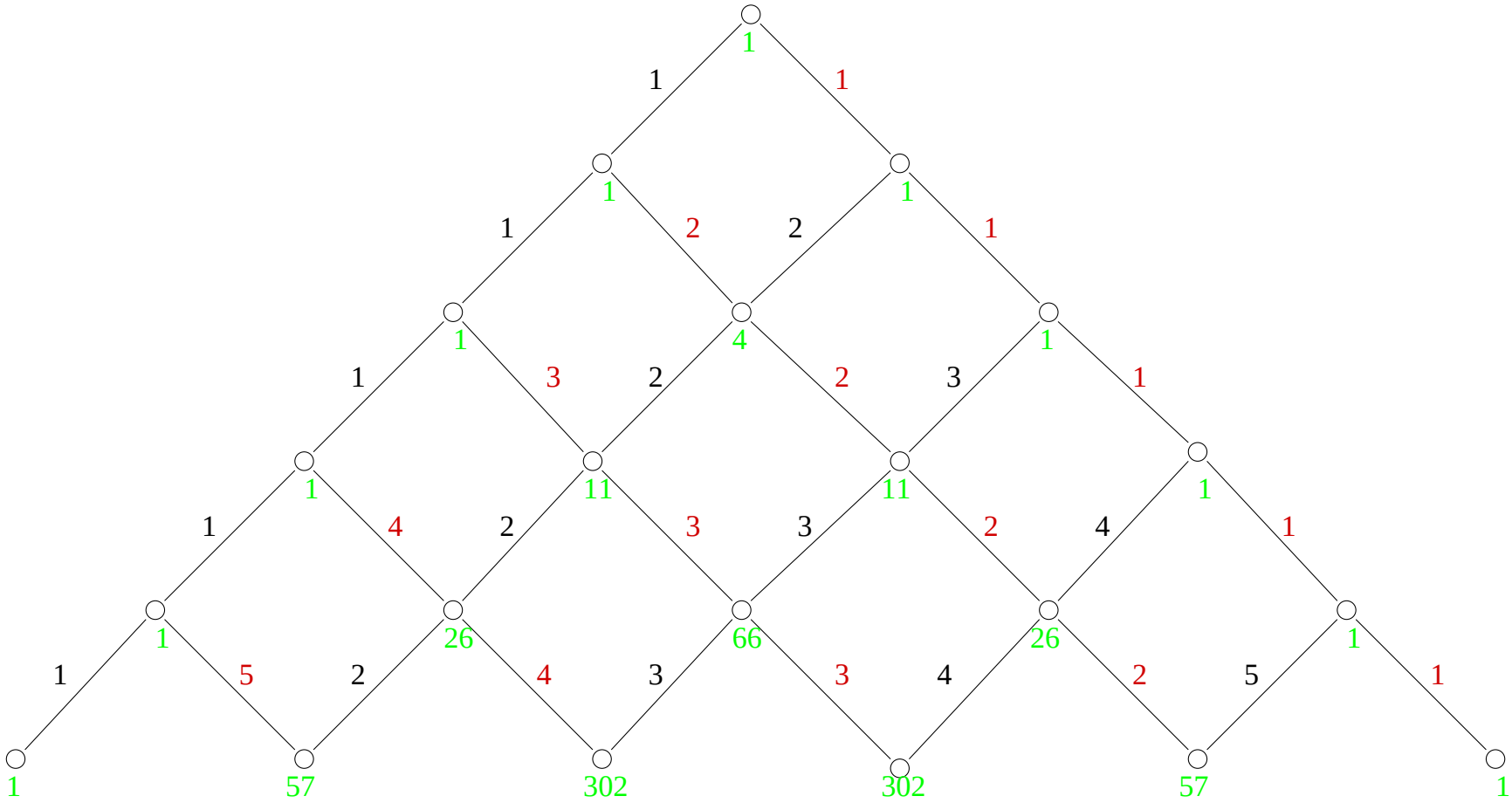
Ergodic Measures on the Reverse Euler

- Let μ be an ergodic measure on X_R and C a cylinder in X_R, for μ-a.e. $x \in X_R$,

$$\mu[C] = \lim_{n\to\infty} \frac{\dim(C,(n,k))}{n!} = \frac{(\alpha)^{k_0}(1-\alpha)^{n_0-k_0}}{k_0!(n_0-k_0)!}.$$



Dimensions of vertices in the Euler graph



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- Also by Varchenko and Petersen using a formula for path counts in the Eulerian graph, monotonicity of ratios, a two-dimensional induction argment, and a one-to-one correspondence between sets of “good" paths

A formula generalizing the one for Eulerian numbers.

Theorem 1. *For $p \geq 0$, $q \geq 1$, and $i, j \geq 0$, let $B_{p,q}(j+i, i)$ denote the number of paths in the Euler graph from the vertex $(p+q, q)$ to the vertex $(p+j+q+i, q+i)$. Then for all p, q, i, j we have*

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Then we got a much shorter argument, satisfying boundary conditions for a recurrence equation by checking equality of two degree i polynomials in p at $i+1$ points.

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and explicitly calculate the density in particular cases, including d-dimensional ones.

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This in turn has as a corollary an identity relating Stirling numbers of the first and second kinds:

An identity involving Stirling numbers

For $1 \leq k \leq n$, $0 \leq r \leq k$,

$$\binom{r+n-k-1}{r} s_1(n, r+n-k) =$$

$$\sum_{m=0}^{k} \binom{m+n-k}{m+1} \sum_{i=0}^{r} \binom{i+n-k+m-1}{i} \frac{(-1)^{m+r-i}}{n^{r-i+1}} (r-i+1)! \times$$

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Thanks to Xavier Méla and Sarah Bailey Frick for many of the pictures.