

NOTES ON COUNTING

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It is important to be able to count exactly the number of elements in any finite set. We will see many applications of counting as we proceed (number of Enigma plugboard arrangements, computer passwords of a certain form, telephone area codes, automobile license plates, binary strings of a certain length needed to encode uniquely each possible English message of a certain length, etc.). In this section we list some of the basic techniques that are helpful for counting even in complicated situations.

Notation. If A is any set, we denote by $|A|$ the number of elements in A . (If A is infinite, we write $|A| = \infty$.) $\mathbb{N} = \{1, 2, \dots\}$ denotes the set of *natural numbers*.

Remark. Thus $|A| = n \in \mathbb{N}$ if and only if A can be put in one-to-one correspondence with $\{1, \dots, n\}$, i.e., “counted”. In the foundations of mathematics, following Georg Cantor, one defines the concept of cardinal number as follows. Two sets A and B are said to have the *same cardinal number* if they can be put in one-to-one correspondence, i.e., if there is a one-to-one onto function $f : A \rightarrow B$. Then even some infinite sets can be seen to have the same or different cardinal numbers. For example, the set $\mathbb{N} = \{1, 2, \dots\}$ of natural numbers, the set $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$ of integers, and the set \mathbb{Q} of rational numbers all have the same cardinal number—they are all *countable*—while the set \mathbb{R} has a different (in fact larger) cardinal number than \mathbb{N} and so is *uncountable*. A cardinal number is then an equivalence class under this concept of sameness. For example, 2 is the set of all sets that can be put in one-to-one correspondence with the set $\{\emptyset, \{\emptyset\}\}$.

1. DISJOINT SETS

If A and B are disjoint sets, i.e., $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$.

Example 1.1. If $A = \{2, 4, 8\}$ and $B = \{1, 5, 11, 12\}$, then $|A \cup B| = 3 + 4 = 7$.

Example 1.2. Here's a useful everyday example: If you have 8 shirts in the laundry and 2 in the drawer, then you have at least 10 shirts.

2. CARTESIAN PRODUCTS

Recall that the *Cartesian product* $A \times B$ of two sets A and B is the set of all ordered pairs with the first element from A and the second element from B :

$$(1) \quad A \times B = \{(a, b) : a \in A, b \in B\}.$$

By repeating this construction we can build up the Cartesian product of n sets for any $n \geq 2$:

$$(2) \quad A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) : a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}.$$

Proposition 2.1.

$$(3) \quad \begin{aligned} |A \times B| &= |A| |B| \\ |A_1 \times A_2 \times \cdots \times A_n| &= |A_1| |A_2| \cdots |A_n|. \end{aligned}$$

Example 2.1. How many “words” (arbitrary strings) are there on the 26 letters of the English alphabet consisting of a vowel followed by a consonant? Answer: $5 \cdot 21 = 105$.

Example 2.2. A (cheap) combination lock has three independently turning wheels, each of which can be set in any one of 8 different positions. How many possible combinations are there? Answer: $8 \cdot 8 \cdot 8 = 512$.

Example 2.3. How many license plate “numbers” can be made of three English letters followed by three digits? Answer: $26^3 \cdot 10^3 = 17,576,000$.

It is easy to see why Formula (2) is correct. Imagine n wheels on a spindle, like a combination lock, with n_i possible settings for wheel number $i, i = 1, \dots, n$. For each of the n_1 settings of the first wheel, there are n_2 settings of the second, thus $n_1 n_2$ ways to set the first pair of wheels. For each of these settings of the first two wheels, there are n_3 ways to set the third wheel, hence $n_1 n_2 n_3$ ways to set the third, and so on.

Exercise 2.1. Let L denote the number of license plate “numbers” that can be made of three English letters followed by four digits, and let N denote the number that can be made of four English letters followed by three digits. Find L/N —without first computing L or N .

Exercise 2.2. How many *binary strings* (words on the alphabet $\{0, 1\}$) are there of length 7? How many are there of length less than or equal to 7 (counting the empty word, which has length 0)?

Exercise 2.3. A student shows up unprepared for a multiple-choice exam that has 10 questions with 4 possible answers for each one. The student takes random guesses on all the questions but gets a perfect score. How surprised should the student be?

Exercise 2.4. How many functions are there from a set with 4 elements to a set with 7 elements?

3. PERMUTATIONS

A *permutation* of a set $A = \{a_1, \dots, a_n\}$ is an ordering of that set.

Remark 3.1. Note that by definition the elements of any set are distinct. Thus there is no such set as $\{1, 1, 2\}$. Note also that the set $\{1, 2, 3\}$ is the *same* as the set $\{2, 3, 1\}$.

Example 3.1. If $A = \{a, b, c\}$, then abc and bac are two different permutations of A .

Example 3.2. Permutations of $\{1, \dots, n\}$ correspond to one-to-one onto functions $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$. To any such function π let correspond the ordering $\pi(1) \dots \pi(n)$. And given a permutation $i_1 \dots i_n$ of $\{1, \dots, n\}$, for each $j = 1, \dots, n$ define $\pi(j) = i_j$.

Proposition 3.1. *The number of permutations of any set with n elements is $n! = n(n-1) \cdots 2 \cdot 1$. ($1! = 1, 0! = 1$.)*

Proof. There are n choices for the first element in the ordering, and for each choice of the first element there are $n-1$ choices for the second, hence there are $n(n-1)$ ways to choose the first two elements. Then for each of the choices of the first two elements, there are $n-2$ ways to choose the third, and so on. \square

Example 3.3. The three (different) scramblers of an Enigma machine can be placed into their three slots in $3! = 3 \cdot 2 \cdot 1 = 6$ ways. The 6 arrangements are:

123 132 213 231 312 321.

An n -set is any set with n elements.

Definition 3.1. An r -permutation of an n -set ($n, r \in \mathbb{N}, 0 \leq r \leq n$) is an ordering of an r -element subset of the n -set.

Example 3.4. Let $A = \{1, 2, 3\}$ be a set with $n = 3$ elements, and let $r = 2$. Then the 2-permutations of A are

$$12 \quad 21 \quad 13 \quad 31 \quad 23 \quad 31.$$

Proposition 3.2. The number of r -permutations of an n -set is

$$(4) \quad P(n, r) = n(n-1) \cdots (n-r+1).$$

Proof. The proof is the same as for Proposition 3.1. □

Example 3.5. The senior class wants to choose a president, vice president, and secretary from among its 4000 members. How many ways are there to do this (assuming no person can serve simultaneously in two positions)? Answer: $4000 \cdot 3999 \cdot 3998$ ways.

Exercise 3.1. A weekly bridge group decides it should get organized by having each member serve either as scheduler, food provider, treasurer, or scorekeeper, and that in order to even out the workloads they should have a new arrangement every month. How long can this four-member group keep this up without repeating an arrangement?

Exercise 3.2. How many 4-letter words (arbitrary strings) are there on the 26 letters of the English alphabet? How many are there which use 4 different letters?

Exercise 3.3. A baseball team has 17 players. How many ways are there to choose a starting lineup of 9 players plus determine the order in which they will bat? How many ways are there to do this and in addition assign to each player one of the 9 defensive positions (catcher, pitcher, first base, etc.)?

4. SUBSETS (COMBINATIONS)

By an r -subset of a set A we mean any set $B \subset A$ such that $|B| = r$, i.e., B has r elements. Notice that the elements of any r -subset are *unordered*, as they are for any set.

Proposition 4.1. Let $n, r \in \mathbb{N}$ with $0 \leq r \leq n$. The number of r -subsets of a set with n elements is

$$(5) \quad C(n, r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. We sneak up on the result—counting the r -subsets of a set A with n elements—by counting instead, in a different way than before, the number $P(n, r)$ of r -permutations of A . We know already that

$$P(n, r) = n(n-1) \cdots (n-r+1).$$

But each r -permutation of A is formed by first selecting an r -subset of A (there are $C(n, r)$ ways to make this choice) and then ordering that particular r -element subset (there are $r!$ ways to do this). Therefore

$$(6) \quad P(n, r) = C(n, r) \cdot r!,$$

and hence

$$(7) \quad C(n, r) = \frac{n!}{r!(n-r)!}.$$

□

Example 4.1. The number of 2-subsets of a 4-set is $4!/(2!2!) = 6$. If the 4-set is $A = \{1, 2, 3, 4\}$, then these 2-subsets are

$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}.$$

Note that each of them can be ordered in $r! = 2$ ways, leading to $6 \cdot 2 = P(4, 2) = 4 \cdot 3$ 2-permutations of A .

$C(n, r) = \binom{n}{r}$ is sometimes read “ n choose r ”, since it gives the number of ways to choose an unordered set of r objects from a pool of n . These numbers are also known as *binomial coefficients*, because of the following Proposition.

Proposition 4.2. For any $n \in \mathbb{N}$ and any $a, b \in \mathbb{R}$,

$$(8) \quad (a+b)^n = \sum_{r=0}^n C(n, r) a^r b^{n-r}.$$

The binomial coefficients have many fascinating properties, the most basic of which is the *Pascal Identity*:

Proposition 4.3. Let $n, k \in \mathbb{N}$ with $k \leq n$. Then

$$(9) \quad C(n+1, k) = C(n, k-1) + C(n, k).$$

Proof. This equation can be checked easily by calculation, using the definition of $C(n, k)$ as a quotient of factorials. But the following *combinatorial proof* can give more insight into the idea behind the identity.

Let A be a set with $n + 1$ elements, fix a particular $a \in A$, and let $B = A \setminus \{a\}$, so that B consists of all elements of A except a , and $|B| = n$.

We want to count the k -element subsets of A . These fall into two disjoint classes, the ones that contain a and the ones that don't contain a . The ones that contain a are obtained by choosing a $(k - 1)$ -element subset of B —there are $C(n, k - 1)$ ways to do this—and then adding a to it. The ones that do not contain a are obtained by choosing a k -element subset of B —and there are $C(n, k)$ ways to do this. \square

This relation leads to the familiar and fascinating *Pascal Triangle*, in which each entry is the sum of the two immediately above it and the n 'th row lists the coefficients in the expansion of $(a + b)^n$:

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 1 & 1 \\
 & & & & 1 & 2 & 1 \\
 & & 1 & 3 & 3 & 1 \\
 1 & 4 & 6 & 4 & 1 \\
 & & & & & \vdots
 \end{array}$$

Exercise 4.1. From a basketball team consisting of 12 players, how many ways are there to select a starting lineup consisting of five players? How many ways are there to select a starting lineup of five, if each player is assigned a different position—center, power forward, small forward, shooting guard, or point guard?

Exercise 4.2. How many binary strings of length seven have exactly three 0's?

Exercise 4.3. An intramural basketball team consists of four women and six men. The games are four on four. How many ways are there to pick a starting lineup that includes at least one man and at least one woman?

Exercise 4.4. An Enigma machine plugboard joins up 6 different pairs of the 26 letters. (For example a is joined to x, k to r, m to t, o to v, q to y, and s to z.) How many different plugboard settings are there?

Exercise 4.5. How many license plates that have three letters followed by three digits use no letter or digit twice?

Exercise 4.6. From a Congressional committee consisting of 13 Republicans and 12 Democrats it is necessary to choose a subcommittee of 4 Republicans and 3 Democrats. How many possibilities are there?

Exercise 4.7. Construct a combinatorial argument to prove that if $k, r, n \in \mathbb{N}$ with $k \leq r \leq n$, then $C(n, r)C(r, k) = C(n, k)C(n-k, r-k)$.

5. A FEW MORE COUNTING PRINCIPLES

5.1. Inclusion-Exclusion.

$$(10) \quad |A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

This is because when we add $|A_1|$ and $|A_2|$, the elements that are in both sets, i.e. in $A_1 \cap A_2$, are counted in twice.

$$(11) \quad \begin{aligned} |A_1 \cup A_2 \cup A_3| &= |A_1| + |A_2| + |A_3| \\ &- (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &+ |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

This is because in forming $|A_1| + |A_2| + |A_3|$ the elements that are in more than one of the sets A_i are counted in either twice, if they are in exactly two of these sets, or three times if they are in all three sets. Subtracting off $(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|)$ accomplishes the correction for elements that are in exactly two of the sets, since it subtracts off the extra 1 for each of them. But this term subtracts off 3 for each element that is in all 3 sets, so the cardinality of $A_1 \cap A_2 \cap A_3$ has to be added back in.

A similar formula applies to the cardinality of the union of n sets for any $n \in \mathbb{N}$.

Example 5.1. Let's determine how many words (arbitrary strings) of length 6 on the 26 English letters either start with a or end with ed . The number starting with a is the number of arbitrary 5-letter words which can be appended to the initial a , namely 26^5 . The number that end with ed is 26^4 . And the number that start with a and end with ed is 26^3 . So the answer is $26^5 + 26^4 - 26^3$.

Exercise 5.1. How many functions are there from $\{1, 2, \dots, n\}$ to $\{0, 1\}$ which assign 0 to either 1 or n ?

5.2. Counting the complement. Sometimes it is easier to count the number of elements that are *not* in a set A and then subtract from the total number to find out how many are in A .

Example 5.2. How many binary strings of length 10 contain at least one 1? Well, there are 2^{10} binary strings of length 10 and only one (all 0's) that does not contain any 1's, so the answer is $2^{10} - 1$.

Exercise 5.2. How many strings of 3 decimal digits do not contain the same digit 3 times?

Exercise 5.3. How many strings of 8 decimal digits contain a repeated digit?

5.3. The Pigeonhole Principle. This principle is so obvious that it hardly seems worth talking about:

Pigeonhole Principle. *If $n + 1$ objects are placed into n boxes (or “pigeonholes”—see the array next to the elevator on the third floor of Phillips Hall), then at least one of the boxes has to contain at least two of the objects.*

Example 5.3. How many strings of 12 decimal digits contain a repeated digit? Answer: All 10^{12} of them. Think of each of the 12 places in the string as an “object” which is placed into one of the “boxes” 0,1,2,3,4,5,6,7,8,9 according to the entry in the string at that place. Since $12 > 10$, at least two of the 12 objects have to end up in the same box.

Exercise 5.4. Early one morning you are groping in the dark in a drawer full of unpaired blue and black socks. How many do you have to pull out in order to be certain that you have two that match?

Exercise 5.5. Six people are stranded on an island. Each pair of the six consists either of two allies or two enemies. Show that there must exist either a clique of three mutual allies or a set of three mutual enemies.

Exercise 5.6. For each real number x , denote by $\lfloor x \rfloor$ the greatest integer less than or equal to x , by $\{x\} = x - \lfloor x \rfloor$ the fractional part of x , and by $d(x, \mathbb{Z})$ the distance from x to the nearest integer.

Let $N \in \mathbb{N}$, and let α be any irrational number. Show that there is $q \in \mathbb{N}$ with $q \leq N$ such that $d(q\alpha, \mathbb{Z}) < 1/N$.

Hint: Consider the “objects” $0, (\alpha), (2\alpha), \dots, (N\alpha)$ and the “boxes” $\{x : i/N \leq x < (i + 1)/N\}, i = 0, 1, \dots, N - 1$.