

Rigid motions of \mathbb{R}^n

(revised January 2001)

Definition A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a rigid motion of \mathbb{R}^n if f preserves distances; that is, $|f(v) - f(w)| = |v - w|$ for all v, w in \mathbb{R}^n .

It is easy to see that if f is an element of $O(n)$ or f is a translation of \mathbb{R}^n (i.e. $f(x) = x + c$ for all $x \in \mathbb{R}^n$, where $c \in \mathbb{R}^n$ is a fixed vector), then f is a rigid motion of \mathbb{R}^n . We prove the following

Theorem The rigid motions of \mathbb{R}^n form a group under composition and every rigid motion $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can be written uniquely as $f = T \circ R$, where T is a translation of \mathbb{R}^n and R is an element of $O(n)$.

If $n = 2$ we can sharpen this result slightly.

Corollary Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a rigid motion. Then there exists a unique rotation R of \mathbb{R}^2 and a unique translation T of \mathbb{R}^2 such that either $f = T \circ R$ or $f = T \circ R \circ S$, where S denotes reflection in the x -axis given by $S(x, y) = (x, -y)$.

Proof of the Corollary We assume the Theorem and prove the Corollary first. It suffices to show that if f is any element of $O(2)$, then either f is a rotation of \mathbb{R}^2 or f can be written uniquely as $f = R \circ S$, where R is a rotation and S is reflection in the x -axis. Let $f(0, 0) = (a, b)$ and let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the translation by $c = (-a, -b)$. If $g = T \circ f$, then g is a rigid motion of \mathbb{R}^2 such that g fixes the origin. By the Lemma below it follows that g is an element of $O(2)$. Since $g(e_1)$ is a unit vector there exists a rotation R of \mathbb{R}^2 such that $h(e_1) = e_1$, where $h = R \circ g$. Since h is an element of $O(2)$ the vector $h(e_2)$ is a unit vector orthogonal to $h(e_1) = e_1$. Hence $h(e_2) = e_2$ or $-e_2$. In the first case h is the identity since it fixes both e_1 and e_2 . We conclude that $g = R^{-1}$ and $f = T^{-1} \circ g = T^{-1} \circ R^{-1}$. In the second case $h = S$ since both are linear transformations with the same values on the basis $\{e_1, e_2\}$ of \mathbb{R}^2 . Arguing as above we conclude that $f = T^{-1} \circ R^{-1} \circ S$. \square

Proof of the Theorem We need a preliminary

Lemma Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a rigid motion that fixes the origin. Then f is an element of $O(n)$; that is, f is a linear transformation such that $f(v) \cdot f(w) = v \cdot w$ (dot product) for all vectors v, w in \mathbb{R}^n .

Proof of the Lemma We show first that $|f(v)| = |v|$ for all vectors v in \mathbb{R}^n . If $\vec{0}$ denotes the origin, then by the definition of rigid motion $|f(v)| = |f(v) - f(\vec{0})| = |v - \vec{0}| = |v|$ for all vectors v in \mathbb{R}^n .

Next we show that $f(v) \cdot f(w) = v \cdot w$ for all vectors v, w in \mathbb{R}^n . From the observation above and the definition of rigid motion we see that $|v|^2 + |w|^2 - 2f(v) \cdot f(w) = |f(v)|^2 + |f(w)|^2 - 2f(v) \cdot f(w) = |f(v) - f(w)|^2 = |v - w|^2 = |v|^2 + |w|^2 - 2v \cdot w$. Comparing the first and last terms we reach the desired conclusion.

Next we show that a) $|f(v) + f(w) - f(v+w)|^2 = 0$ and b) $|f(cv) - cf(v)|^2 = 0$ for all vectors v, w in \mathbb{R}^n and all real numbers c , which will prove that f is linear. We prove only a) since the proof of b) is similar. By the discussion above f preserves dot products and lengths of vectors, and hence we obtain $|f(v) + f(w) - f(v+w)|^2 = |f(v)|^2 + |f(w)|^2 + |f(v+w)|^2 + 2f(v) \cdot f(w) - 2f(v) \cdot f(v+w) - 2f(w) \cdot f(v+w) = |v|^2 + |w|^2 + |v+w|^2 + 2v \cdot w - 2v \cdot (v+w) - 2w \cdot (v+w) = 2|v|^2 + 2|w|^2 + 4v \cdot w - 2|v|^2 - 2v \cdot w - 2w \cdot v - 2|w|^2 = 0$. \square

We now complete the proof of the Theorem. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a rigid motion, and let $f(\vec{0}) = c \in \mathbb{R}^n$. If T is translation by $-c$, then $g = T \circ f$ fixes the origin, and hence $g \in O(n)$ by the Lemma above. It follows that $f = T^{-1} \circ g$.

Suppose now that f is a rigid motion such that $f = T \circ R = T^* \circ R^*$, where T and T^* are translations by c and c^* respectively, and R and R^* are elements of $O(n)$. Since R and R^* fix the origin we conclude that $c^* = T^*(\vec{0}) = (T^* \circ R^*)(\vec{0}) = (T \circ R)(\vec{0}) = T(\vec{0}) = c$. Hence $T = T^*$ and $R^* = (T^*)^{-1} \circ f = T^{-1} \circ f = R$. This completes the proof of the second part of the Theorem.

It remains only to prove that the set of rigid motions of \mathbb{R}^n forms a group under composition. Clearly the identity is a rigid motion, and it is easy to see that the set of rigid motions is closed under composition of functions. To conclude, we need to show that every rigid motion has an inverse that is also a rigid motion. Given a rigid motion f there exist a translation T and an element R of $O(n)$, both invertible, such that $f = T \circ R$. Hence $f^{-1} = R^{-1} \circ T^{-1}$ exists and is a rigid motion since it is a composition of rigid motions. \square

Remark

The lemma above can be strengthened in a way that is sometimes useful. Specifically we have the following

Proposition

Let U be an open subset of \mathbb{R}^n and let $f : U \rightarrow \mathbb{R}^n$ be a distance preserving function; that is $|f(x) - f(y)| = |x - y|$ for all vectors x, y in U . Then f is the restriction of a rigid motion of \mathbb{R}^n .

Proof

We first reduce to the case that U contains the origin. If x^* is any point of U , then $U^* = U - x^*$ is an open set that contains the origin, and $f^* = f \circ T_{x^*} : U^* \rightarrow \mathbb{R}^n$ is a distance preserving function, where $T_{x^*} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the translation by x^* . If we can prove that f^* is the restriction of a rigid motion, then the same is true for f .

By the previous paragraph we may assume that the open set U contains the origin $\vec{0}$. Let $x^* = f(\vec{0})$. If $f^* = T_{-x^*} \circ f$, where $T_{-x^*} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ denotes the translation by $-x^*$, then $f^* : U \rightarrow \mathbb{R}^n$ is a distance preserving function that fixes the origin $\vec{0}$ in U . If we can prove that f^* is the restriction of a rigid motion, then the same is true for f .

It suffices to prove that if $f : U \rightarrow \mathbb{R}^n$ is a distance preserving function that fixes the origin $\vec{0}$ in U , then f is the restriction of an element of $O(n)$. The proof of the lemma above shows that

- 1) $f(v) \cdot f(w) = v \cdot w$ for all vectors v, w in U .
- 2) $f(v + w) = f(v) + f(w)$ if v, w and $v+w$ are vectors in U .
- 3) $f(cv) = cf(v)$ if v and cv are vectors in U with c in \mathbb{R} .

Since U is open we may choose $\varepsilon > 0$ such that U contains the ε - ball centered at the origin. Now define $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows. Let $g(\vec{0}) = \vec{0}$. If v is a nonzero vector, then define $g(v) = (1/\lambda) f(\lambda v)$, where λ is any positive number such that $|\lambda v| < \varepsilon$.

To prove the Proposition it suffices to show

- a) g is well defined
- b) g is an element of $O(n)$
- c) $g(v) = f(v)$ whenever v lies in U .

To prove a) we let λ and μ be any positive numbers such that $|\lambda v| < \varepsilon$ and $|\mu v| < \varepsilon$. By 3) above we have $(\mu / \lambda) f(\lambda v) = f((\mu / \lambda) (\lambda v)) = f(\mu v)$. Hence $(1 / \lambda) f(\lambda v) = (1 / \mu) f(\mu v)$, which proves a).

To prove b) we show first that $g(v) \cdot g(w) = v \cdot w$ for all nonzero vectors v, w of \mathbb{R}^n . Given nonzero vectors v, w choose a positive number λ such that $|\lambda v| < \varepsilon$ and $|\lambda w| < \varepsilon$. Then from the definition of g and 1) above we obtain $g(v) \cdot g(w) = (1 / \lambda^2) f(\lambda v) \cdot f(\lambda w) = (1 / \lambda^2) (\lambda v) \cdot (\lambda w) = v \cdot w$. Setting $v = w$ we conclude that $|g(v)| = |v|$ for all vectors v in \mathbb{R}^n . It now follows from the proof of the lemma above that g is a linear map of \mathbb{R}^n , and we conclude that g is an element of $O(n)$.

We prove c). Let v be any nonzero vector of U and let λ be a positive number such that $|\lambda v| < \varepsilon$. By 3) we have $\lambda f(v) = f(\lambda v)$, and hence $f(v) = (1 / \lambda) f(\lambda v) = g(v)$. \square