AVERAGING ON COMPACT LIE GROUPS

1. HAAR MEASURE

Let G denote a connected Lie group of dimension n. Let $\{X_1,\ ...,\ X_n\}$ denote a collection of left invariant vector fields on G whose values at any point g of G form a basis of T$_g$G. The vector fields $\{X_1,\ ...,\ X_n\}$ determine an orientation for G, which shows that G is orientable.

Let $\{\theta_1,\ ...,\ \theta_n\}$ denote the dual basis of 1-forms on G. It is easy to check from the definitions that each θ_i is left invariant; that is L$_g^*\theta_i = \theta_i$ for all g $\in$ G. Let dG $= \theta_1\ \wedge\ ...\ \wedge \theta_n$, an n-form on G. Note that dG is left invariant since the $\{\theta_i\}$ are left invariant, and dG never vanishes on G since dG$(X_1,\ ...,\ X_n) = 1$ on G.

Lemma 1.1. *Let Ω be any left invariant n-form on G. Then $\Omega = \lambda\ dG$ for some real number λ. In particular, the vector space of left invariant n-forms is 1-dimensional.*

Proof. We know that $\Omega = f dG$ for some C$^\infty$ function f : G $\to \mathbb{R}$. Moreover, for every g $\in$ G we have $\Omega = L_g^*\Omega = (f \circ L_g)\ L_g^* dG = (f \circ L_g)\ dG$. Hence $f = f \circ L_g$ for every g $\in$ G, which proves that $f(g) = (f \circ L_g)(e) = f(e)$, where e denotes the identity of G. Hence f is a constant function. ☐

Now suppose that G is compact as well as connected. Then dG defines integration on G, namely $\int_G f dG$ is the integral of the n-form f dG over G for all C$^\infty$ functions f : G $\to \mathbb{R}$. Multiplying dG by a positive constant we obtain a left invariant n-form such that $\int_G dG = 1$.

The left invariant n-form dG such that $\int_G dG = 1$ is unique by the lemma above, and it defines the so called $Haar\ measure$ on G.

The Haar measure on G has nice properties.

Proposition 1.2. *Let G be a compact, connected Lie group, and let dG denote the Haar measure on G. Then*

1) dG is right invariant; that is $R_g^*\ dG = dG$ for all g $\in$ G.

2) If f : G $\to \mathbb{R}$ is any C$^\infty$ function and g $\in$ G is any element, then $\int_G f\ dG = \int_G (f \circ L_g)\ dG = \int_G (f \circ R_g)\ dG$.

Proof. 1) We shall need the following

Lemma 1.3. *Let G be a compact topological group, and let f : G $\to$ $(0, \infty)$ be a continuous homomorphism, where the group operation on $(0, \infty)$ is multiplication. Then $f(g) = 1$ for all g $\in$ G.*

Proof. The image f(G) is a compact subset of $(0, \infty)$ since f is continuous. In particular, there are positive constants a,b such that $a \leq f(g) \leq b$ for all g $\in$ G. If f(g) $\neq 1$ for some g $\in$ G, then $f(g^n) = f(g)^n \to 0$ or ∞ as n $\to \infty$, depending on whether $f(g) < 1$ or $f(g) > 1$. In either case we obtain a contradiction. ☐

Date: April 11, 2005.

We now complete the proof of 1). Since L_g and R_h commute for all g,h $\in$ G it follows that $R_h^* \, dG$ is a left invariant n-form on G for every h $\in$ G. By Lemma 1.1 we find that $R_h^* \, dG = f(h)dG$ for some nowhere zero function f : G $\rightarrow \mathbb{R}$. The function $f(h) = R_h^* \, dG(X_1, \ ..., \ X_n)$ is continuous on G, and clearly f(e) = 1. Hence f(G) $\subset (0, \infty)$ since f(G) is a connected subset of $\mathbb{R}$ that contains 1 but not 0.

By Lemma 1.3 it remains only to prove that f : G $\rightarrow (0, \infty)$ is a homomorphism. By definition we have $R_{gh}^* \, dG = f(gh) \, dG$ for all g,h $\in$ G. On the other hand, $R_{gh}^* \, dG = R_h^* \circ R_g^* \, dG = R_h^*(f(g) \, dG) = f(h) \, f(g) \, dG$. Hence f(gh) = f(g) f(h) for all g,h $\in$ G.

We prove 2). We recall the following differential forms version of the change of variables theorem.

Proposition. *Let f : M $\rightarrow$ M be a C^∞ diffeomorphism of a compact, orientable C^∞ manifold M of dimension n. Define $\epsilon(f) = 1$ if f is orientation preserving and $\epsilon(f) = -1$ if f is orientation reversing. Then $\int_M f^*\omega = \epsilon(f) \int_M \omega$ for every n-form ω on M.*

We shall apply this result to the diffeomorphisms L_g, R_g for each g $\in$ G. Note that each of these diffeomorphisms is orientation preserving since G is arc connected. If c(t) is a continuous curve in G with c(0) = e and c(1) = g, then the diffeomorphisms $L_{c(t)}, R_{c(t)}$ depend continuously on t and are orientation preserving at t $= 0$. Hence these diffeomorphisms are orientation preserving for all t.

Now let f : G $\rightarrow \mathbb{R}$ be a C$^\infty$ function and g $\in$ G any element. By the lemma above and the left invariance of dG we have $\int_G \ f \ dG = \int_G \ L_g^*(f \ dG) = \int_G \ (f \circ L_g) \ L_g^* \, dG = \int_G \ (f \circ L_g) \ dG$. A similar argument using the right invariance of dG shows that $\int_G \ f \ dG = \int_G \ (f \circ R_g) \ dG$ for all g $\in$ G. This completes the proof of Proposition 1.2 ☐

2. INVARIANT INNER PRODUCTS

Proposition 2.1. *Let U be a finite dimensional real vector space, and let G be a compact subgroup of GL(U). Then there exists an inner product $\langle , \rangle$ on U such that $\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle$ for all x,y $\in$ U and all $\varphi \in$ G.*

Proof. We first consider the case that G is connected. Let $\langle , \rangle_0$ be any inner product on U. We average the inner product $\langle , \rangle_0$ over G to obtain the desired inner product $\langle , \rangle$.

For v,w $\in$ U define $\langle v, w \rangle = \int_G \ f(g) \ dG$, where f(g) $= \langle g(v), g(w) \rangle_0$ and dG denotes the Haar measure on G. It is easy to check that $\langle , \rangle$ is a bilinear form on U, and $\langle v, v \rangle > 0$ for any nonzero tangent vector v since f(g) $= \langle g(v), g(v) \rangle_0$ is a continuous positive function on G.

In the case that G is connected it remains only to prove that $\langle h(v), h(w) \rangle = \langle v, w \rangle$ for all h $\in$ G, v,w $\in$ U. Let v,w,h be given and define f : G $\rightarrow \mathbb{R}$ as above by f(g) $= \langle g(v), g(w) \rangle_0$. Using 2) of Proposition 1.2 we compute $\langle h(v), h(w) \rangle = \int_G \langle g(h(v)), g(h(w)) \rangle_0 \, dG = \int_G \langle (gh)(v), (gh)(w) \rangle_0 \, dG = \int_G \ f(gh) \ dG = \int_G \ (f \circ R_h)(g) \ dG = \int_G \ f(g) \ dG = \langle v, w \rangle$.

Next we consider the case that G is an arbitrary compact subgroup of GL(U), not necessarily connected. We shall need the following standard result.

Lemma 2.2. *Let G be a Lie group, not necessarily compact, and let G$_0$ denote the connected component of G that contains the identity of G. Then G$_0$ is a closed, connected normal subgroup of G. In particular, G$_0$ is a Lie group.*

Proof. Clearly G$_0$ is connected, and G$_0$ is closed in G since connected components are always closed. It suffices to show that G$_0$ is a subgroup of G. Let μ : G x G $\rightarrow$ G denote the multiplication map. Then μ(G$_0$ x G$_0$) is a connected subset of G that contains the identity of G. The connected component of G that contains the identity e is the union of all connected subsets of G that contain e.

Hence μ(G$_0$ x G$_0$) $\subset$ G$_0$, which proves that G$_0$ is closed under multiplication. A similar argument using the inverse map λ : G $\rightarrow$ G shows that G$_0$ is closed under inverses. Hence G$_0$ is a subgroup of G. If g $\in$ G is any element, then gG$_0$g^{-1} is a connected subset of G that contains the identity. Hence gG$_0$g^{-1} $\subset$ G$_0$ for all g $\in$ G, which proves that G$_0$ is a normal subgroup of G. ☐

We now complete the proof of Proposition 2.1. Since G is compact it has only finitely many connected components (details omitted). Let $\{x_1, \ldots, x_N\}$ be elements such that G is the union of the cosets $x_iG_0 = G_0x_i$. By the work above we may choose an inner product $\langle,\rangle_0$ on U that is preserved by the connected subgroup G$_0$. Define an inner product on U by $\langle v,w\rangle = \sum_{i=1}^N \langle x_i(v), x_i(v)\rangle_0$ for all v,w $\in$ U. It is easy to check that the set of elements H in G that preserve the inner product $\langle,\rangle$ is a subgroup of G. Hence it suffices to show that H contains G$_0$ and $\{x_1, \ldots, x_N\}$, for then H = G.

We show first that H contains $\{x_1, \ldots, x_N\}$. Fix k with $1 \le k \le N$. Then there exists a permutation σ on N letters such that $G_0x_ix_k = Gx_{\sigma(i)}$ for $1 \le i \le N$. Hence for $1 \le i \le N$ there exists an element g$_{ik}$ $\in$ G$_0$ such that $x_ix_k = g_{ik}x_{\sigma(i)}$.

Let v,w $\in$ U and $1 \le k \le N$ be given. Then $\langle x_k(v), x_k(w)\rangle = \sum_{i=1}^N \langle x_ix_k(v), x_ix_k(w)\rangle_0 = \sum_{i=1}^N \langle g_{ik}x_{\sigma(i)}(v), g_{ik}x_{\sigma(i)}(w)\rangle_0 = \sum_{i=1}^N \langle x_{\sigma(i)}(v), x_{\sigma(i)}(w)\rangle_0 = \sum_{i=1}^N \langle x_i(v), x_i(w)\rangle_0 = \langle v,w\rangle$. Hence H contains $\{x_1, \ldots, x_N\}$.

We show that H contains G$_0$. Let g $\in$ G$_0$ be given. Since G$_0$ is normal in G there exist elements $\{g_1, \ldots, g_N\}$ $\in$ G$_0$ such that $x_ig = g_ix_i$ for $1 \le i \le N$. Let v,w $\in$ U be given. We compute $\langle g(v), g(w)\rangle = \sum_{i=1}^N \langle x_ig(v), x_ig(w)\rangle_0 = \sum_{i=1}^N \langle g_ix_i(v), g_ix_i(w)\rangle_0 = \sum_{i=1}^N \langle x_i(v), x_i(w)\rangle_0 = \langle v,w\rangle$. Hence H contains G$_0$. The proof of Proposition 2.1 is complete. ☐

As a corollary of the result above we obtain

Proposition 2.3. *Let n $\ge$ 2 be an integer, and let G be a compact subgroup of GL(n,$\mathbb{R}$). Then there exists an element g $\in$ GL(n,$\mathbb{R}$) such that gGg^{-1} $\subset$ O(n,$\mathbb{R}$).*

Remark A consequence of the Peter-Weyl theorem is that every compact Lie group is isomorphic to a compact subgroup of GL(n,$\mathbb{R}$) for a sufficiently large integer n. See for example Theorem 4.1 in the book *Representations of Compact Lie Groups* by Broecker-tom Dieck. The result above then implies that every compact Lie group is isomorphic to a compact subgroup of O(n,$\mathbb{R}$) for a sufficiently large integer n. Hence the compact Lie groups are precisely (up to isomorphism) the closed subgroups of O(n,$\mathbb{R}$) as n ranges over all positive integers. For n = 1 the circle S^1 of complex numbers of modulus 1 is the only connected 1-dimensional Lie group.

Proof. Let $\mathfrak{P}_n$ denote the collection of all positive definite inner products on $\mathbb{R}^n$. The group GL(n,$\mathbb{R}$) acts transitively on $\mathfrak{P}_n$ by (gB)(x,y) = B(g^{-1}x, g^{-1}y) for x,y $\in$ $\mathbb{R}^n$, g $\in$ GL(n,$\mathbb{R}$) and B $\in$ $\mathfrak{P}_n$. By Proposition 2.1 there exists an element B of $\mathfrak{P}_n$ such that hB = B for all h $\in$ G. Since GL(n,$\mathbb{R}$) acts transitively on $\mathfrak{P}_n$ we may choose g $\in$ GL(n,$\mathbb{R}$) such that B = $g^{-1}\langle,\rangle$, where $\langle,\rangle$ denotes the standard dot product on $\mathbb{R}^n$. If h $\in$ G, then $ghg^{-1}(\langle,\rangle) = gh(B) = g(B) = \langle,\rangle$. Hence the elements of gGg^{-1} preserve the dot product $\langle,\rangle$ on $\mathbb{R}^n$. ☐

3. INVARIANT RIEMANNIAN STRUCTURES

Proposition 3.1. *Let M be a C$^\infty$ manifold and let G be a compact Lie group that is a subgroup of Diff(M), the group of diffeomorphisms of M. Then there exists a Riemannian structure $\langle,\rangle$ on M such that the elements of G are isometries of $\{M, \langle,\rangle\}$.*

Proof. The proof is a small variation of the proof of Proposition 2.1, and we omit most of the details. As in (2.1) we first consider the case that G is connected. Let $\langle,\rangle_0$ be any Riemannian structure on

M. For m $\in$ M and v,w $\in$ T$_m$M define $\langle v, w\rangle = \int_G\ f(g)\ dG$, where f(g) $= \langle g_*(v), g_*(w)\rangle_0$ and dG denotes the Haar measure on G. The proof of Proposition 2.1 shows that the elements of G preserve the inner product $\langle v, w\rangle$.

Next, let G be an arbitrary compact subgroup of Diff(M), not necessarily connected, and let G$_0$ denote the connected component of G that contains the identity. Let $\{x_1,\ ...,\ x_N\}$ be elements such that G is the union of the cosets $x_iG_0 = G_0x_i$. By the work above we may choose a Riemannian structure $\langle,\rangle_0$ on M that is preserved by the connected subgroup G$_0$. Define a Riemannian structure $\langle,\rangle$ on M by $\langle v, w\rangle = \sum_{i=1}^N \langle (x_i)_*(v), (x_i)_*(v)\rangle_0$ for all v,w $\in$ T$_m$M and all m $\in$ M. The proof of Proposition 2.1 now shows that G preserves the inner product $\langle,\rangle$ on all tangent spaces of M. ☐

4. G-INVARIANT METRICS ON COSET SPACES G/K

Let X be a set on which a Lie group G acts transitively, and let x$_0$ be a point of X. If K = G$_{x_0}$ $= \{g \in G : g(x_0) = x_0\}$, then there is a bijection φ of the coset space G/K onto X given by $\varphi(g) = g(x_0)$. It is known that the coset space G/K has the structure of a C$^\infty$ manifold of dimension dim G $-$ dim K.

In this section we show that if Ad(K) has compact closure in GL($\mathfrak{G}$), then G/K admits a Riemannian structure $\langle,\rangle$ such that G is a transitive group of isometries of $\{G/K, \langle,\rangle\}$. The action of G on G/K is given by g(hK) = ghK for all g,h $\in$ G.

Here is the first step in proving the result stated above. It follows from the definition of K below that K is a closed subgroup of G, but we don't assume that K is compact.

Proposition 4.1. *Let X be a C$^\infty$ manifold on which a Lie group G acts transitively. Let x$_0$ be a point of X, and let K = G$_{x_0}$ $= \{g \in G : g(x_0) = x_0\}$. Suppose T$_{x_0}$X admits an inner product $\langle,\rangle_0$ such that K leaves invariant $\langle,\rangle_0$; that is, $\langle (\varphi)_*(v), (\varphi)_*(w)\rangle_0 = \langle v, w\rangle_0$ for all v,w $\in$ T$_{x_0}$X. Then $\langle,\rangle_0$ extends uniquely to a Riemannian structure $\langle,\rangle$ on X such that the elements of G are isometries of $\{X, \langle,\rangle\}$.*

Proof. Let x be a point of X, and let g be an element of G (typically not unique) such that g(x$_0$) $=$ x. Given vectors v,w of T$_x$X, there are unique elements v$_0$, w$_0$ in T$_{x_0}$X such that $(g)_*(v_0) =$ v and $(g)_*(w_0) =$ w. We define $\langle v, w\rangle = \langle v_0, w_0\rangle_0$. This is the only possible definition of the Riemannian structure $\langle,\rangle$ on X if the elements of G are to be isometries. Hence $\langle,\rangle$ is unique if it exists.

We must show first that the Riemannian structure $\langle,\rangle$ above is well defined. Suppose g$'$ is another element of G such that $g'(x_0) = x$, and let v$'_0$ and w$'_0$ be the unique elements of T$_{x_0}$X such that $(g')_*(v'_0) = v$ and $(g')_*(w'_0) = w$. We must show that $\langle v_0, w_0\rangle_0 = \langle v'_0, w'_0\rangle_0$.

If $k = g^{-1}g'$, then $k \in K = G_{x_0}$. We compute $v = (g')_*(v'_0) = (g)_*(k)_*(v'_0)$. On the other hand we also have $v = (g)_*(v_0)$. It follows that $(k)_*(v'_0) = v_0$ since $(g)_* : T_{x_0}X \rightarrow T_xX$ is an isomorphism. Similarly, $(k)_*(w'_0) = w_0$. Finally, $\langle v_0, w_0\rangle = \langle (k)_*(v'_0), (k)_*(w'_0)\rangle = \langle v'_0, w'_0\rangle$ since by hypothesis K leaves invariant $\langle,\rangle_0$.

It remains to show that the elements of G preserve the inner product $\langle,\rangle$ defined above. We have shown that $\langle g_*(v_0), g_*(v_0)\rangle = \langle v_0, w_0\rangle_0$ for all g $\in$ G and all v$_0$, w$_0$ $\in$ T$_{x_0}$X. Let g,h $\in$ G and v,w $\in$ T$_{g(x_0)}$X be given. Choose v$_0$, w$_0$ $\in$ T$_{x_0}$X such that $g_*(v_0) = v$ and $g_*(w_0) = w$. Then $(hg)_*(v_0) = h_*(v)$ and $(hg)_*(w_0) = h_*(w)$. Hence $\langle h_*(v), h_*(w) = \langle v_0, w_0\rangle_0 = \langle g_*(v_0), g_*(w_0)\rangle = \langle v, w\rangle$. The proof is complete. ☐

Before proving the main result stated at the beginning of this section we apply the result above in a case where the point x$_0$ and the inner product $\langle,\rangle_0$ can be given explicitly. Let n $\geq$ 2 be an integer, and let $\mathfrak{P}_n$ be the set of positive definite, real, symmetric n x n matrices. Then $\mathfrak{P}_n$ is an open, convex subset of the vector space Sym(n,$\mathbb{R}$) of n x n real, symmetric n x n matrices. Hence $\mathfrak{P}_n$ is a C$^\infty$ manifold of dimension $\frac{n(n+1}{2}$.

Remark There is a one-one correspondence between $\mathfrak{P}_n$ and the set of positive definite inner products on $\mathbb{R}^n$. Given an element B of $\mathfrak{P}_n$ we define an inner product $\langle , \rangle$ by $\langle v, w\rangle = \langle Bv, w\rangle_0$, where $\langle , \rangle_0$ denotes the standard dot product on $\mathbb{R}^n$ and v,w are elements of $\mathbb{R}^n$ regarded as column vectors.

Proposition 4.2. *There is a Riemannian structure $\langle , \rangle$ on $\mathfrak{P}_n$ such that GL(n,$\mathbb{R}$) is a transitive group of isometries of $\{\mathfrak{P}_n, \langle , \rangle\}$.*

Proof. We define a left action of G = GL(n,$\mathbb{R}$) on $\mathfrak{P}_n$ by g(A) = gAg^t for all A $\in \mathfrak{P}_n$ and all g $\in$ G. To see that G acts transitively on $\mathfrak{P}_n$ let A $\in \mathfrak{P}_n$ be given. A standard construction yields an element B of $\mathfrak{P}_n \subset GL(n, \mathbb{R})$ such that B^2 = A. (One may reduce to the case that A is diagonal, and then the choice of B is obvious.) Now $B(I) = BIB^t = B^2 = A$, where I denotes the identity matrix. If x$_0$ = I, then $K = G_{x_0} = O(n, \mathbb{R})$, the orthogonal group.

For X $\in$ M(n,$\mathbb{R}$) let X$_I$ $\in T_I M(n, \mathbb{R})$ denote the initial velocity of the curve $t \rightarrow I + tX$. On T$_I \mathfrak{P}_n$ we define a positive definite inner product $\langle , \rangle_0$ by $\langle A_I, B_I\rangle_0$ = trace AB, for all A,B $\in \mathfrak{P}_n$. Since the elements of G act by linear transformations on the vector space Sym(n,$\mathbb{R}$) of n x n symmetric matrices it follows that each g in G equals its own differential map g$_*$: $T_A Sym(n, \mathbb{R}) \rightarrow T_{g(A)} Sym(n, \mathbb{R}) \simeq Sym(n, \mathbb{R})$. The action of K = O(n,$\mathbb{R}$) on Sym(n,$\mathbb{R}$) is by conjugation since $g^t = g^{-1}$ for all g $\in$ K. Since trace $(gAg^{-1})(gBg^{-1})$ = trace AB for all A,B $\in \mathfrak{P}_n$ and all g $\in$ K it follows that K leaves invariant the inner product $\langle , \rangle_0$ on T$_I \mathfrak{P}_n$. The assertion of (4.2) now follows from (4.1). ☐

We now reach the two main results of this section. The first result is clearly a corollary of the second, but the proof we present of the first result is simpler than the proof of the second.

Proposition 4.3. *Let G be a connected Lie group and let K be a compact subgroup of G. Then the coset manifold G/K admits a Riemannian structure $\langle , \rangle$ such that G is a transitive group of isometries of $\{G/K, \langle , \rangle\}$.*

Proof. G acts transitively on G/K since g(eK) = gK for all g $\in$ G. If x$_0$ = eK, then G$_{x_0} = \{g \in G : g(x_0) = x_0\}$ = K. By Proposition 4.1 it suffices to show that there exists a K-invariant inner product $\langle , \rangle_0$ on T$_{x_0}$G/K. Since K is compact, by Proposition 3.1 there exists a Riemannian structure $\langle , \rangle$ on G/K such that K is a group of isometries of $\{G/K, \langle , \rangle\}$. In particular, since K fixes x$_0$ if follows that K leaves invariant $\langle , \rangle_0 = \langle , \rangle_{eK}$. ☐

Proposition 4.4. *Let G be a connected Lie group, and let K be a closed subgroup of G such that Ad(K) has compact closure in GL($\mathfrak{G}$). Then the coset manifold G/K admits a Riemannian structure $\langle , \rangle$ such that G is a transitive group of isometries of $\{G/K, \langle , \rangle\}$.*

Proof. Again, G acts transitively on G/K. Now let $\mathfrak{G}, \mathfrak{K}$ denote the Lie algebras of G,K respectively. If H is the closure in GL($\mathfrak{G}$) of the subgroup Ad(K), then H is also a subgroup of GL($\mathfrak{G}$). By Proposition 2.1 there exists an inner product $\langle , \rangle$ on $\mathfrak{G}$ that is left invariant by the elements of H and in particular by the elements of Ad(K).

Regard $\mathfrak{K}$ as a subalgebra of $\mathfrak{G}$ by setting $\mathfrak{K} = \{X \in \mathfrak{G} : X(e) \in T_e K \subset T_e G\}$. It then follows from the left invariance of X that X is tangent to K at every point of K, and hence X restricts to a left invariant vector field of K.

Let $\mathfrak{P}$ denote the orthogonal complement of $\mathfrak{K}$ in $\mathfrak{G}$. If $\pi : G \rightarrow G/K$ is the projection map, then by Proposition 4.1 of the handout on the manifold structure of G/H it follows that $\pi_* : \mathfrak{P}(e) \rightarrow$

$T_{eK}G/K$ is a linear isomorphism, where $\mathfrak{P}(e) = \{X(e) : X \in \mathfrak{P}\}$. Let $\langle , \rangle^*$ be the unique inner product on $T_{eK}G/K$ such that $\pi_* : \{\mathfrak{P}(e), \langle , \rangle\} \to \{T_{eK}G/K, \langle , \rangle^*\}$ is a linear isometry. It suffices to show that K preserves the inner product $\langle , \rangle^*$ on $T_{eK}G/K$. The proof will then be completed by Proposition 4.1.

We need two preliminary results.

Lemma 4.5. *Ad(K) leaves invariant the subspaces $\mathfrak{K}$ and $\mathfrak{P}$ of $\mathfrak{G}$.*

Lemma 4.6. *Let elements $\varphi \in$ K and X $\in \mathfrak{P}$ be given. Then $\varphi_* \; \pi_* \; X(e) = \; \pi_* \; (Ad(\varphi)X)(e)$.*

Assuming for the moment that these lemmas have been proved, we complete the proof of Proposition 4.4. Let elements $\xi, \eta \in T_{eK}G/K$ be given, and choose elements X,Y $\in \mathfrak{P}$ uniquely such that $\pi_* X(e) = \xi$ and $\pi_* Y(e) = \eta$. Recall that the elements of Ad(K) preserve the inner product $\langle , \rangle$ on $\mathfrak{G}$ and hence on $\mathfrak{P}$. For any element $\varphi \in$ K we have $\langle \varphi_*(\xi), \varphi_*(\eta)\rangle^* = \langle \varphi_* \; \pi_* \; X(e), \varphi_* \; \pi_* \; Y(e)\rangle^* = \langle \pi_* \; (Ad(\varphi)X)(e), \pi_* \; (Ad(\varphi)Y)(e)\rangle^* = \langle Ad(\varphi)X, Ad(\varphi)Y\rangle = \langle X, Y\rangle = \langle \pi_* \; X(e), \pi_* \; Y(e)\rangle^* = \langle \xi, \eta\rangle^*$. Hence K preserves the inner product $\langle , \rangle^*$ on $T_{eK}G/K$. ☐

We now prove the Lemmas 4.5 and 4.6.

Proof of Lemma 4.5 It suffices to prove that Ad(K) leaves $\mathfrak{K}$ invariant. Since Ad(K) leaves $\langle , \rangle$ invariant it then follows that Ad(K) leaves invariant the orthogonal complement $\mathfrak{P}$ of $\mathfrak{K}$ in $\mathfrak{G}$.

Let X $\in \mathfrak{K}$ and $\varphi \in$ K be given. Then X(e) $= \alpha'(0)$, where $\alpha(t) = exp(tX)$ is a curve in K since $\mathfrak{K}$ is the Lie algebra of K. Hence $(Ad(\varphi)X)(e) = (c_\varphi)_* X(e) = \beta'(0) \in T_eK$ since $\beta(t) = (c_\varphi \circ \alpha)(t)$ is also a curve in K. It follows that Ad(φ)X $\in \mathfrak{K}$.

Proof of Lemma 4.6 Let elements $\varphi \in$ K and X $\in \mathfrak{P}$ be given. Then $\pi_* \; X(e) = \alpha'(0)$, where $\alpha(t) = exp(tX)K = \pi(exp(tX)$. Hence $\varphi_* \; \pi_* \; X(e) = \beta'(0)$, where $\beta(t) = (\varphi \circ \alpha)(t) = \varphi(exp(tX)K = \varphi(exp(tX)\varphi^{-1}K = exp(tAd(\varphi)X)K = \pi(exp(tAd(\varphi)X))$. However, the final equality above shows that $\beta'(0) = \pi_* \; (Ad(\varphi)X)(e)$, which completes the proof of Lemma 4.6.