Quaternionic Structures

Definition A finite dimensional real vector space U admits a quaternionic structure if there exist linear transformations I, J and K such that

(#) $I^2 = J^2 = K^2 = -\mathrm{Id}$

$IJ = -JI = K$; $JK = -KJ = I$; $KI = -IK = J$.

If U admits a quaternionic structure, then U becomes a free $\mathbb{H}$–module by defining $(a + bi + cj + dk)(u) = au + b(Iu) + c(Ju) + d(Ku)$ for $u \in U$ and $a + bi + cj + dk \in \mathbb{H}$.

Remark U is a free R-module where the ring R = $\mathbb{H}$ is actually a division algebra. The noncommutativity of $\mathbb{H}$ makes it harder to do linear algebra than in a vector space U where the ring R is a field.

Example $U = \mathbb{H}^n = \{(h_1, \dots, h_n) : h_i \in \mathbb{H} \text{ for } 1 \le i \le n\}$. Since $\mathbb{H}$ can be regarded as a 4-dimensional vector space over $\mathbb{R}$ it follows that $\mathbb{H}^n$ is a 4n-dimensional vector space over $\mathbb{R}$. The transformations I, J and K are defined by $I(h_1, \dots, h_n) = (i\,h_1, \dots, i\,h_n)$, $J(h_1, \dots, h_n) = (j\,h_1, \dots, j\,h_n)$ and $K(h_1, \dots, h_n) = (k\,h_1, \dots, k\,h_n)$ for all $(h_1, \dots, h_n) \in \mathbb{H}^n$. It is obvious that I, J and K satisfy the conditions (#) above.

Definition Let U be a real vector space with a quaternionic structure defined by {I,J,K} as above. Let $\mathrm{End}_{\mathbb{H}}(U) = \{T \in \mathrm{End}(U) : T(xu) = x(Tu) \text{ for all } x \in \mathbb{H}.\}$. The set $\mathrm{End}_{\mathbb{H}}(U)$ is a real subspace of End(U), and $\mathrm{End}_{\mathbb{H}}(U)$ becomes an $\mathbb{H}$–module if one defines $(xT)(u) = x(Tu)$ for all $x \in \mathbb{H}$, $u \in U$ and $T \in \mathrm{End}_{\mathbb{H}}(U)$.

Let $M(n,\mathbb{H})$ denote the set of n x n matrices A with entries (A_{ij}) in $\mathbb{H}$. $M(n,\mathbb{H})$ is also a vector space over $\mathbb{R}$, and it becomes a free $\mathbb{H}$–module if one defines $(xA)_{ij} = x\,A_{ij}$.

Since U is finite dimensional over $\mathbb{R}$ it has finite rank as a free $\mathbb{H}$– module. Let $\mathfrak{B} = \{u_1, \dots, u_n\}$ be an $\mathbb{H}$– basis for U; that is, every element $u \in U$ can be written uniquely as $u = \sum_{k=1}^{n} h_i\, u_i$ for elements $\{h_1, \dots, h_n\}$ in $\mathbb{H}$. For each element $T \in \mathrm{End}_{\mathbb{H}}(U)$ we associate the .matrix A(T) in $M(n,\mathbb{H})$ such that $T(u_i) = \sum_{k=1}^{n} A(T)_{ki}\, u_k$ for $1 \le i \le n$. It is easy to check that the map $A : \mathrm{End}_{\mathbb{H}}(U) \to M(n,\mathbb{H})$ is an $\mathbb{R}$–linear vector space isomorphism and also an $\mathbb{H}$–module isomorphism.

We wish to define matrix multiplication $\cdot$ in $M(n,\mathbb{H})$ so that $A(S \circ T) = A(S)\cdot A(T)$ for all S,T in $\mathrm{End}_{\mathbb{H}}(U)$; that is, $A : \mathrm{End}_{\mathbb{H}}(U) \to M(n,\mathbb{H})$ is an $\mathbb{R}$–algebra isomorphism. The definition that makes this possible looks peculiar because of the noncommutativity of $\mathbb{H}$.

Definition If $A,B \in M(n,\mathbb{H})$, then let $A \cdot B$ denote the unique element of $M(n,\mathbb{H})$ such that $(A \cdot B)_{ij} = \sum_{k=1}^{n} B_{kj} A_{ik}$.

Remark This definition of matrix multiplication for $M(n,\mathbb{R})$ and $M(n,\mathbb{C})$ is equivalent to the usual one since $\mathbb{R}$ and $\mathbb{C}$ are commutative.

Proposition $A(S \circ T) = A(S) \cdot A(T)$ for all $S,T \in \mathrm{End}_{\mathbb{H}}(U)$.

Proof By definition $(S \circ T)(e_i) = \sum_{j=1}^{n} A(S \circ T)_{ji}\, e_j$. On the other hand $(S \circ T)(e_i) = S(\sum_{k=1}^{n} A(T)_{ki}\, e_k) = \sum_{k=1}^{n} A(T)_{ki}\, S(e_k) = \sum_{k=1}^{n} A(T)_{ki} \{ \sum_{j=1}^{n} A(S)_{jk} e_j \} = \sum_{j=1}^{n} \{ \sum_{k=1}^{n} A(T)_{ki} A(S)_{jk} \}\, e_j = \sum_{j=1}^{n} \{A(S) \cdot A(T)\}_{ji}\, e_j$. It follows that $A(S \circ T)_{ji} = \{A(S) \cdot A(T)\}_{ji}$ for all i,j.□

The transpose operation in $M(n,\mathbb{H})$

Definition For $A \in M(n,\mathbb{H})$ we let A^* be that element of $M(n,\mathbb{H})$ such that $A^*_{ij} = \overline{A_{ji}}$ for all i,j.

Proposition $(A \cdot B)^* = B^* \cdot A^*$ for all $A,B \in M(n,\mathbb{H})$.

Proof $(A \cdot B)^*_{ij} = \overline{(A \cdot B)_{ji}} = \overline{(\sum_{k=1}^{n} B_{ki}\, A_{jk})} = \sum_{k=1}^{n} \overline{(B_{ki}\, A_{jk})} = \sum_{k=1}^{n} \overline{A_{jk}}\ \overline{B_{ki}} = \sum_{k=1}^{n} A^*_{kj}\, B^*_{ik} = (B^* \cdot A^*)_{ij}$.□

The Type of an irreducible Clifford representation

Let U be a finite dimensional real vector space. For a positive integer n let $C\ell(n)$ denote the Clifford algebra of dimension 2^n determined by $\{\mathbb{R}^n, \langle\ ,\ \rangle\}$, where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

A map $\rho : C\ell(n) \to \mathrm{End}(U)$ is an algebra homomorphism if ρ is an $\mathbb{R}$–linear map and in addition $\rho(xy) = \rho(x) \circ \rho(y)$ for all x,y in $C\ell(n)$. We say that U is a $C\ell(n)$–module. If $\rho(C\ell(n))$ leaves no proper subspace of U invariant, then we say that U is an irreducible $C\ell(n)$–module. An algebra homomorphism $\rho : C\ell(n) \to \mathrm{End}(U)$ is also called a representation of $C\ell(n)$ on U, and $\rho : C\ell(n) \to \mathrm{End}(U)$ is an irreducible representation if U is an irreducible $C\ell(n)$–module.

Two representations $\rho_1 : C\ell(n) \to \mathrm{End}(U_1)$ and $\rho_2 : C\ell(n) \to \mathrm{End}(U_2)$ are said to be <u>equivalent</u> if there exists an isomorphism $T : U_1 \to U_2$ such that $\rho_2(x) \circ T = T \circ \rho_1(x)$ for all $x \in C\ell(n)$.

For an algebra homomorphism $\rho : C\ell(n) \to \mathrm{End}(U)$ we define $Z(\rho) =$

$\{T \in GL(U) : T \circ \rho(x) = \rho(x) \circ T$ for all $x \in C\ell(n)\}$. If $T \in Z(\rho)$, then Ker(T) and Im(T) are subspaces of U left invariant by $\rho(C\ell(n))$. Hence if U is an irreducible $C\ell(n)$–module, then for every nonzero T in $Z(\rho)$ we must have Ker(T) = {0} and Im(T) = U; that is, T is invertible. This shows that if U is an irreducible $C\ell(n)$–module, then $Z(\rho)$ is a division algebra; that is, every nonzero element of $Z(\rho)$ is invertible. A famous theorem of Frobenius says that every division algebra that is finite dimensional over $\mathbb{R}$ must be isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Definition An irreducible representation $\rho : C\ell(n) \to End(U)$ is called of real, complex or quaternionic type if $Z(\rho)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively.

Clifford algebras as matrix algebras

We summarize some basic information about the Clifford algebras $C\ell(n)$. Further details may be found in previous handouts.

If $n \neq 4k+3$, then $C\ell(n)$ is isomorphic as an algebra to a matrix algebra M(p,K) for some positive integer p, where K = $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If n = 4k+3, then $C\ell(n) = A_1 \oplus A_2$, where A_1 and A_2 are two sided ideals isomorphic to $C\ell(n-1)$. We summarize this information in the following table, which has periodicity 8 :

$C\ell(8k) \cong M(2^{4k},\mathbb{R})$ $C\ell(8k+1) \cong M(2^{4k},\mathbb{C})$ $C\ell(8k+2) \cong M(2^{4k},\mathbb{H})$
$C\ell(8k+3) \cong M(2^{4k},\mathbb{H}) \oplus M(2^{4k},\mathbb{H})$ $C\ell(8k+4) \cong M(2^{4k+1},\mathbb{H})$
$C\ell(8k+5) \cong M(2^{4k+2},\mathbb{C})$ $C\ell(8k+6) \cong M(2^{4k+3},\mathbb{R})$
$C\ell(8k+7) \cong M(2^{4k+3},\mathbb{R}) \oplus M(2^{4k+3},\mathbb{R})$

It is known that a matrix algebra M(p,K), where K = $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, has exactly one irreducible module up to equivalence, namely K^p, the p-tuples of elements of K regarded as column vectors. The action of M(p,K) on K^p is by left multiplication. The matrix algebra M(p,K), $\oplus$M(p,K), has exactly two irreducible modules up to equivalence. In both cases the module itself is K^p but the actions of M(p,K), $\oplus$M(p,K) on K^p are inequivalent. In the first case $(X_1,X_2)(x) = X_1(x)$ and in the second case $(X_1,X_2)(x) = X_2(x)$ for $x \in K^p$ and $(X_1,X_2) \in$ M(p,K), $\oplus$M(p,K). Note that in each case one copy of M(p,K) acts trivially on K^p.

Each of these modules K^p is a real vector space of real dimension p, 2p or 4p depending on whether K = $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

From the table above we obtain the following classification of irreducible representations $\rho : C\ell(n) \to$ End(U into real, complex or quaternionic type :

Real type	n = 8k, 8k+6 and 8k+7
Complex type	n = 8k+1, and 8k+5 (i.e. n = 4k+1)
Quaternionic type	n = 8k+2, 8k+3 and 8k+4

Canonical isomorphisms $\rho : C\ell(n) \to M(p,K)$, $n \neq 4k+3$

If $n \neq 4k+3$, then the discussion above shows that we have an isomorphism $A : C\ell(n) \to M(p,K)$, where $p = \dim_K U$ and $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ according to whether $\rho : C\ell(n) \to End(U$ is of real, complex or quaternionic type. The isomorphism A is not unique since we may always compose A with an automorphism of $C\ell(n)$ or an automorphism of M(p,K). However, we show in the next result that one may always single out a special class of isomorphisms $A : C\ell(n) \to M(p,K)$ if $n \neq 4k+3$.

If $n = 4k+3$, then $C\ell(n) = A_1 \oplus A_2$, where A_1 and A_2 are two sided ideals isomorphic to $C\ell(n-1)$. In this case we also get a special isomorphism $C\ell(n) \to M(p,K) \oplus M(p,K)$ from the special isomorphism $C\ell(n-1) \to M(p,K)$.

Proposition Let $n \neq 4k+3$ be a positive integer, and let $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ denote the type of the unique, irreducible $C\ell(n)$–module U of K– dimension p. Then there exists an algebra isomorphism $A : C\ell(n) \to M(p,K)$ such that $A(\overline{x}) = A(x)^*$ for all $x \in C\ell(n)$.

Remark The isomorphism A with the properties stated above is not unique. If g is an element of M(p,K) such that $gg^* = g^*g = Id$, then $X \to gXg^* = \varphi(g)$ is an automorphism of M(p,K) such that $\varphi(X)^* = \varphi(X^*)$ for all $X \in M(p,K)$. If $B = \varphi \circ A$, then it is routine to show that $B(\overline{x}) = B(x)^*$ for all $x \in C\ell(n)$. Nevertheless, the existence of such an isomorphism $A : C\ell(n) \to M(p,K)$ is useful. In the discussion at the end of this section we apply the Proposition above to identify the group $G = \{x \in C\ell(n) : x\overline{x} = \overline{x}x = 1\}$.

Proof of the Proposition We shall prove this only in the most difficult case, where U is of quaternionic type. The proofs in the other cases are analogous but easier.

Let $\rho : C\ell(n) \to End(U)$ be an irreducible representation of quaternionic type, and let I,J,K denote the elements of End(U) such that $Z(\rho) = \mathbb{R}$–span {Id, I, J, K}. We need some preliminary results.

Lemma 1 There exists an inner product < , > on U such that the elements of $\rho(\mathbb{R}^n)$ are skew symmetric and the elements {I,J,K} are both skew symmetric and orthogonal relative to < , >.

Lemma 2 There exists an integer p such that $\dim_{\mathbb{R}} U = 4p$. Let < , > be an inner product as in Lemma 1. Then there exists an orthonormal basis $\mathfrak{B} = \{u_1, \ldots u_{4p}\}$ such that for

$1 \le r \le p$ we have $u_{p+r} = I(u_r)$, $u_{2p+r} = J(u_r)$ and $u_{3p+r} = K(u_r)$.

The $\mathbb{H}$–matrix of an element of End(U)

Let $\mathfrak{B} = \{u_1, \ldots u_{4p}\}$ be an orthonormal $\mathbb{R}$–basis as in Lemma 2. Then $\{u_1, \ldots u_p\}$ becomes an $\mathbb{H}$– basis for U regarded as a free $\mathbb{H}$–module U. Specifically, if $u \in U$ is written $u = \sum_{r=1}^{p} \alpha_r u_r + \sum_{r=1}^{p} \beta_r I(u_r) + \sum_{r=1}^{p} \gamma_r J(u_r) + \sum_{r=1}^{p} \delta_r K(u_r)$, where $\{\alpha_r , \beta_r , \gamma_r , \delta_r \}$ are real numbers, then we may write $u = \sum_{r=1}^{p} h_r u_r$, where $h_r = \alpha_r + \beta_r i + \gamma_r j + \delta_r k \in \mathbb{H}$ for $1 \le r \le p$.

The elements of $\rho(C\ell(n))$ may be regarded as elements of $End_{\mathbb{H}}(U)$ since they commute with the linear maps {I,J,K}, which define an $\mathbb{H}$–module structure on U. For an element T in $End_{\mathbb{H}}(U)$ we let A(T) denote the matrix in $M(n, \mathbb{H})$ determined by the $\mathbb{H}$–basis $\{u_1, \ldots u_p\}$; that is, $T(u_i) = \sum_{r=1}^{p} A(T)_{ri} u_r$.

For the next result we recall that $C^*_{ij} = -\overline{C_{ji}}$ for all $C \in M(p, \mathbb{H})$.

Lemma 3 Let < , > and $\mathfrak{B} = \{u_1, \ldots u_{4p}\}$ be as in Lemmas 1 and 2. For every $x \in C\ell(n)$ let A(x) denote the $\mathbb{H}$– matrix of $\rho(x)$ relative to the $\mathbb{H}$–basis $\{u_1, \ldots u_p\}$. Then $A(x)^* = -A(x)$ for all $x \in \mathbb{R}^n$.

Proof of the Proposition We postpone for the moment the proofs of the lemmas, and we complete the proof of the Proposition. Let $\rho : C\ell(n) \to End(U)$ be an irreducible representation of quaternionic type. If n = 8k+2 , then by the discussion above we know that $C\ell(8k+2) \cong M(2^{4k}, \mathbb{H})$ and $U = \mathbb{H}^{2^{4k}}$. Hence $\dim_{\mathbb{R}} U = 4p$, where $p = 2^{4k} = 2^{(n-2)/2}$ If n = 8k+4 then $C\ell(8k+4) \cong M(2^{4k+1}, \mathbb{H})$, $U = \mathbb{H}^{2^{4k+1}}$ and $\dim_{\mathbb{R}} U = 4p$, where $p = 2^{4k+1} = 2^{(n-2)/2}$. In either case $\dim_{\mathbb{R}} C\ell(n) = 2^n = 4p^2 = \dim_{\mathbb{R}} M(p, \mathbb{H})$.

By the definitions of $A : C\ell(n) \to M(p, \mathbb{H})$ in Lemma 3 and matrix multiplication in $M(p, \mathbb{H})$ it is clear that the map A is an injective algebra homomorphism. In fact, A is an isomorphism since $\dim_{\mathbb{R}} C\ell(n) = \dim_{\mathbb{R}} M(p, \mathbb{H})$. Let $X = \{x \in C\ell(n) : A(\overline{x}) = A(x)^*\}$. Since the maps $x \to \overline{x}$ and $C \to C^*$ are anti-automorphisms of $C\ell(n)$ and $M(p, \mathbb{H})$ respectively it follows that X is a subalgebra of $C\ell(n)$. By Lemma 3 X contains $\mathbb{R}^n$ since $\overline{x} = -x$ for all $x \in \mathbb{R}^n$. We conclude that $X = C\ell(n)$ since $\mathbb{R}^n$ generates $C\ell(n)$.$\square$

Proof of Lemma 1 Recall that $Pin(n) = \{x \in C\ell(n) : x = x_1 x_2 \ldots x_m \in C\ell(n)$, where each x_i is a unit vector in $\mathbb{R}^n$ and m is an arbitrary positive integer}. The group Pin(n) is compact and $x\overline{x} = 1$ for all $x \in Pin(n)$. Since Pin(n) is compact there exists an inner product < , >* on U that is preserved by the elements of $\rho(Pin(n))$. If x is a unit vector in $\mathbb{R}^n$, then $x^2 = -1$ and hence $\rho(x)^2 = -Id$. The map $\rho(x)$ is orthogonal relative to < , >*, and hence it is also skew symmetric since $< \rho(x)u , v >^* = < \rho(x)^2 u , \rho(x)v >^* = - < u, \rho(x)v >^*$

for all u,v $\in$ U. It follows that $\rho(x)$ is skew symmetric relative to < , >* for all $x \in \mathbb{R}^n$ since this holds for all unit vectors x.

Let $Q = \{\pm \text{Id}, \pm I, \pm J, \pm K\}$. If we define $< u , v > = \sum_{\varphi\in Q} < \varphi(u), \varphi(v) >^*$, then since $\rho(C\ell(n))$ commutes with Q we obtain an inner product < , > that is invariant under both $\rho(\text{Pin}(n))$ and Q. Since $I^2 = J^2 = K^2 = -\text{Id}$ it follows as above that I,J and K are skew symmetric as well as orthogonal relative to < , >.

Proof of Lemma 2 Let < , > be an inner product on U as in Lemma 1, and let u_1 be a unit vector in U. Let $U_1 = \mathbb{R}$–span $\{u_1, I(u_1), J(u_1), K(u_1)\}$. It suffices to show that $\{u_1, I(u_1), J(u_1), K(u_1)\}$ is an orthonormal basis of U_1. If this has been established, then we consider the orthogonal complement $U_1^{\perp}$ of U_1 in U. The elements of $Q = \{\pm \text{Id}, \pm I, \pm J, \pm K\}$ leave invariant $U_1^{\perp}$ by Lemma 1, so we may split off another 4-dimensional subspace $U_2 = \mathbb{R}$–span $\{u_2, I(u_2), J(u_2), K(u_2)\} \subseteq U_1^{\perp}$. The proof of Lemma 2 then follows by induction.

The vectors $\{u_1, I(u_1), J(u_1), K(u_1)\}$ all have length one since u_1 has length one and the transformations I, J, K are orthogonal relative to < , >. Moreover, u_1 is othogonal to $I(u_1)$, $J(u_1)$ and $K(u_1)$ since I, J and K are skew symmetric relative to < , > by Lemma 1. Using the skew symmetry of I, J, K and the fact that these elements anti-commute we obtain $< I(u_1) , J(u_1) > = - < u_1, IJ(u_1) > = < u_1, JI(u_1) > = - < J(u_1) , I(u_1) >$, which proves that $< I(u_1) , J(u_1) > = 0$. Similar arguments show that any two of the vectors $\{u_1, I(u_1), J(u_1), K(u_1)\}$ are orthogonal.$\square$

Proof of Lemma 3 Given $x \in \mathbb{R}^n$ and $1 \le s \le p$ we let $\rho(x)(u_s) = \sum_{r=1}^{p} \alpha_{rs} u_r + \sum_{r=1}^{p} \beta_{rs} I(u_r) + \sum_{r=1}^{p} \gamma_{rs} J(u_r) + \sum_{r=1}^{p} \delta_{rs} K(u_r)$ for suitable real numbers α_{rs}, β_{rs}, γ_{rs} and δ_{rs}. By the definition of A(x) we obtain $\rho(x)(u_s) = \sum_{r=1}^{p} A(x)_{rs} u_r$, where $A(x)_{rs} = \alpha_{rs} + \beta_{rs} i + \gamma_{rs} j + \delta_{rs} k$. We compute $A(X)^*_{rs} = \overline{A(X)_{sr}} = \alpha_{sr} - \beta_{sr} i - \gamma_{sr} j - \delta_{sr} k$ and $-A(X)_{rs} = -\alpha_{rs} - \beta_{rs} i - \gamma_{rs} j - \delta_{rs} k$. The proof of the lemma will now follow from the assertions

(*) $\alpha_{rs} = -\alpha_{sr}$; $\beta_{rs} = \beta_{sr}$; $\gamma_{rs} = \gamma_{sr}$; $\delta_{rs} = \delta_{sr}$

By Lemma 1 the transformations I, J, K and $\rho(x)$, $x \in \mathbb{R}^n$, are all skew symmetric relative to < , >. From the expression above for $\rho(x)(u_s)$ we see that $\alpha_{rs} = < \rho(x)(u_s), u_r > = - < u_s , \rho(x)(u_r) > = - \alpha_{sr}$. Moreover, $\beta_{rs} = < \rho(x)(u_s), I(u_r) > = - < u_s , \rho(x) I(u_r) > = - < u_s , I \rho(x)(u_r) > = < I(u_s), \rho(x)(u_r) > = \beta_{sr}$. Similarly, we see that $\gamma_{rs} = \gamma_{sr}$ and $\delta_{rs} = \delta_{sr}$.$\square$

The group $G_n = \{x \in C\ell(n) : x \bar{x} = \bar{x} x = 1\}$

For $n \neq 4k+3$ we identify $C\ell(n)$ with $M(p,K)$ for a suitable integer p and $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We show that G_n is isomorphic to the group of isomorphisms of K^p that preserve the canonical inner product of K^p (real, Hermitian or quaternionic). Equivalently, G_n is isomorphic to $\{A \in M(p,K) : AA^* = A^*A = Id\}$.

In K^p the canonical inner product < , > is given by $< x, y > = \sum_{r=1}^{p} x_i \overline{y_i}$ for all $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in K^p$. We state some basic properties of < , > whose proofs are routine and omitted. The peculiar appearance of these statements is made necessary by the noncommutativity of $\mathbb{H}$.

The action of $M(p,K)$ by K-endomorphisms on K^p is given by $Ae_i = \sum_{r=1}^{p} A_{ji}\, e_j$.

Lemma The canonical inner product < , > on has the following properties :

1) $\overline{< x , y >} = < y , x >$ for all $x,y \in K^p$.
2) $< cx , y > = c < x , y >$ for all $x,y \in K^p$, $c \in K$.
3) $< x , cy > = < x , y > \overline{c}$ for all $x,y \in K^p$, $c \in K$.
4) $< Ax , y > = < x , A^*y >$ for all $x,y \in K^p$, $A \in M(p,K)$.
5) $< Ax , Ay > = < x , y >$ for all $x,y \in K^p, A \in M(p,K) \Leftrightarrow AA^* = A^*A = Id$.

Notation

$O(n) = \{A \in M(n,\mathbb{R}) : AA^* = A^*A = Id\}$

$U(n) = \{A \in M(n,\mathbb{C}) : AA^* = A^*A = Id\}$

$Sp(n) = \{A \in M(n,\mathbb{H}) : AA^* = A^*A = Id\}$

Proposition Let $A : C\ell(n) \to M(p,K)$ be an algebra isomorphism such that $A(\overline{x}) = A(x)^*$ for all $x \in C\ell(n)$, where $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and p is a suitable positive integer. Let $G_n = \{x \in C\ell(n) : x\overline{x} = \overline{x}x = 1\}$. Then

1) $A(G_n) = O(p)$ if $K = \mathbb{R}$.
2) $A(G_n) = U(p)$ if $K = \mathbb{C}$.
3) $A(G_n) = Sp(p)$ if $K = \mathbb{H}$.

Proof In view of the lemma above it suffices to prove that $x\overline{x} = \overline{x}x = 1$ for $x \in C\ell(n) \Leftrightarrow A(x)A(x)^* = A(x)^*A(x) = Id$. If $1 = x\overline{x}$, then $Id = A(x\overline{x}) = A(x)A(\overline{x}) = A(x)A(x)^*$, and similarly if $1 = \overline{x}x = 1$, then $Id = A(x)^*A(x)$. Conversely, if $Id = A(x)A(x)^* = A(x)A(\overline{x}) = A(x\overline{x})$, then $1 = \overline{x}x$ and similarly if $Id = A(x)^*A(x) = A(\overline{x})A(x) = A(\overline{x}x)$, then $\overline{x}x = 1$. □