Structure of stabilizer Lie algebras $\mathfrak{G}_W$
June 25, 2003

This article is an appendix to [E2] and is meant to be read with that article. Here W always denotes a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$. We recall that the elements A of $\mathrm{End}(\mathbb{R}^q)$ act on $\mathfrak{so}(q,\mathbb{R})$ by $A(Z) = AZ + ZA^t$. We note that if A is skew symmetric , then $A(Z) = \mathrm{ad}\, A(Z)$ for all $Z \in W$.

Let $\mathfrak{G}_W = \{A \in \mathrm{End}(\mathbb{R}^q) : A(W) \subseteq W$ and trace $A = 0$ on $\mathbb{R}^q\}$. We shall see in this section that imposing the condition of trace zero can be useful, and we lose no information with this restriction.

In the case that $\mathfrak{G}_W$ is self adjoint some of the results below become sharper and more relevant. In particular, $\mathfrak{G}_W$ contains the symmetric and skew symmetric parts of each of its elements. If $\mathfrak{G}_W$ is self adjoint , then $\mathfrak{G}_W = \mathfrak{G}_W{}^{\perp}$ and by the discussion in Proposition 3.4e of [E2] we obtain the following : if $X \in \mathrm{Der}(\mathfrak{N})$, then there exist unique elements $A \in \mathrm{End}(\mathbb{R}^q)$ and $B \in \mathrm{Hom}(\mathbb{R}^q,W)$ such that

1) $A \in \mathfrak{G}_W$; that is, $AZ + ZA^t \in W$ for all $Z \in W$
2) $X(v) = A(v) + B(v)$ for all $v \in \mathbb{R}^q$.
   $X(Z) = AZ + ZA^t$ for all $Z \in W$.

Section 1 Centralizer of W in $\mathfrak{gl}(q,\mathbb{R})$

Proposition 1.1 Let W act irreducibly on $\mathbb{R}^q$. Let $\mathfrak{Z}_W = \{ A \in \mathfrak{gl}(q,\mathbb{R}) : AZ = ZA$ for all $Z \in W\}$. If $\mathfrak{Z}_W$ is nonzero, then $\mathfrak{Z}_W$ is a division algebra over $\mathbb{R}$. Hence $\mathfrak{Z}_W$ is isomorphic as an algebra to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Proof Let A be a nonzero element of $\mathfrak{Z}_W$. Then the kernel and range of A in $\mathbb{R}^q$ are both invariant under W, and hence A is an isomorphism since W acts irreducibly on $\mathbb{R}^q$. It follows that $\mathfrak{Z}_W$ is a finite dimensional division algebra over $\mathbb{R}$, and by a theorem of Frobenius must be isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Remark Any element in $\mathfrak{Z}_W$ commutes with all elements in the Lie algebra $\mathfrak{H} \subseteq \mathfrak{so}(q,\mathbb{R})$ that is generated by W. For all W in a dense, open subset O of $G(p, \mathfrak{so}(q,\mathbb{R}))$ this subalgebra $\mathfrak{H}$ is $\mathfrak{so}(q,\mathbb{R})$. Hence $\mathfrak{Z}_W = \{0\}$ for all W in O since $\mathfrak{so}(q,\mathbb{R})$ acts irreducibly on $\mathbb{R}^q$.

Definition 1.2 We will say that W is of <u>real</u>, <u>complex</u> or <u>quaternionic</u> type respectively if $\mathfrak{Z}_W = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively.

We make some further general remarks before determining the type of some specific examples.

Proposition 1.3 For a subspace W of $\mathfrak{so}(q,\mathbb{R})$ let $\mathfrak{Z}^o_W = \{ A \in \mathfrak{sl}(q,\mathbb{R}) : AZ = ZA$ for all $Z \in W\}$. Then

1) $\mathfrak{Z}^o_W$ is a self adjoint subalgebra of $\mathfrak{sl}(q,\mathbb{R})$.

2) If W acts irreducibly on $\mathbb{R}^q$, then $\mathfrak{Z}^o_W \subseteq \mathfrak{so}(q,\mathbb{R})$. Moreover, exactly one of the following holds :

a) $\mathfrak{Z}^o_W = \{0\}$ (real type)

b) $\mathfrak{Z}^o_W = \text{span}\{I\}$, where $I^2 = -\text{Id}$ (complex type)

c) $\mathfrak{Z}^o_W = \text{span}\{I, J, K\}$, where $I^2 = J^2 = K^2 = -\text{Id}$; $IJ = -JI = K$; $KI = -IK = J$ and $JK = -KJ = I$. (quaternionic type)

Proof 1) Clearly $\mathfrak{Z}_W$ is a subalgebra of of $\mathfrak{sl}(q,\mathbb{R})$, and it is closed under the transpose operation since W is closed under transpose.

2) Let A be a symmetric transformation in $\mathfrak{Z}_W$ such that trace A = 0. Every eigenspace of A in $\mathbb{R}^q$ is invariant under W, and it follows that $A = \lambda\,\text{Id}$ for some real number λ since W acts irreducibly on $\mathbb{R}^q$. Hence $\lambda = 0$ since trace A = 0. By 1) every element of $\mathfrak{Z}_W$ is the sum of symmetric and skew symmetric elements of $\mathfrak{Z}_W$, and it is now immediate that $\mathfrak{Z}^o_W \subseteq \mathfrak{so}(q,\mathbb{R})$. It follows immediately from Proposition 1.1 that the cases a), b) and c) correspond to the cases that W is of real, complex or quaternionic type.

Section 2 Nonsingular subspaces W

Definition 2.1 We will say that W is of Clifford type if Z^2 is a negative multiple of the identity for every nonzero element Z of W. In this case it follows that $Z^2 = -(1 / \dim W) \langle Z, Z \rangle \,\text{Id}$ for all $Z \in W$ since the canonical inner product $\langle\ ,\ \rangle$ on $\mathfrak{so}(q,\mathbb{R})$ is defined by $\langle Z, Z^* \rangle = -\text{trace}\,(ZZ^*)$.

Remark A basic property of Clifford algebras says that $W = j(\mathbb{R}^p)$, where p = dim W and $j : C\ell(p) \to \text{End}(\mathbb{R}^q)$ is a representation of the real negative definite Clifford algebra $C\ell(p)$ determined by $\mathbb{R}^p$ with the standard inner product $\langle\ ,\ \rangle$. Conversely, it is also true that if $j : C\ell(p) \to \text{End}(\mathbb{R}^q)$ is a representation of the real Clifford algebra determined by $\{\mathbb{R}^p, \langle\ ,\ \rangle\}$, then $W = j(\mathbb{R}^p)$ is a subspace of Clifford type of $\mathfrak{so}(q,\mathbb{R})$ if $\mathbb{R}^q$ is endowed with a suitable inner product. Moreover, W is a Lie triple system with trivial center in $\mathfrak{so}(q,\mathbb{R})$. See section 3 of [E1], especially Proposition 3, for further discussion.

Definition 2.2 We say that W is <u>nonsingular</u> if every nonzero element Z of W is an invertible linear transformation of $\mathbb{R}^q$.

We will discuss two types of nonsingular subspaces W : 1) W is of Clifford type 2) W = $\mathfrak{su}(2) = \mathfrak{so}(3,\mathbb{R}) \subseteq \mathfrak{so}(q,\mathbb{R})$ and q is even. In the second case it is known that W is of quaternionic type and that $\mathbb{R}^q$ is the realification of a complex, irreducible $\mathfrak{su}(2)$ module.

Proposition 2.3 Let W be a nonsingular subspace of $\mathfrak{so}(q,\mathbb{R})$, and let W act irreducibly on $\mathbb{R}^q$.

a) Let W be of Clifford type. Then W has

Real type :	for p = 8k, 8k + 6 and 8k + 7.
Complex type :	for p = 4k + 1.
Quaternionic type :	for p = 8k + 2, 8k + 3 and 8k + 4.

b) Let W = $\mathfrak{su}(2) = \mathfrak{so}(3,\mathbb{R}) \subseteq \mathfrak{so}(q,\mathbb{R})$, where q is even. Then W has quaternionic type in all cases.

Proof

a) It is known that all of the Clifford algebras $C\ell(p)$ are isomorphic to K(n) or $K(n)\oplus K(n)$, where K = $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, K(n) denotes the n x n matrices with entries in the field K, and n is a power of 2 that depends on p. In each case the representation space is K^n and $C\ell(p)$ acts on K^n by matrix multiplication so that the action commutes with left multiplication by elements of K.

It remains only to determine the field K for each integer p. From the canonical isomorphisms and periodicity relations of Clifford algebras (cf.(1.4) of [LM]) and induction one obtains the following

Table of negative definite real Clifford algebras

$C\ell(8k) \cong \mathbb{R}(2^{4k})$	$C\ell(8k+1) \cong \mathbb{C}(2^{4k})$	$C\ell(8k+2) \cong \mathbb{H}(2^{4k})$
$C\ell(8k+3) \cong \mathbb{H}(2^{4k}) \oplus \mathbb{H}(2^{4k})$	$C\ell(8k+4) \cong \mathbb{H}(2^{4k+1})$	$C\ell(8k+5) \cong \mathbb{C}(2^{4k+2})$
$C\ell(8k+6) \cong \mathbb{R}(2^{4k+3})$	$C\ell(8k+7) \cong \mathbb{R}(2^{4k+3}) \oplus \mathbb{R}(2^{4k+3})$	

b) The complex, irreducible representation spaces of $\mathfrak{su}(2)$ are the spaces V_n, $n \geq 1$, of complex polynomials that are homogeneous of degree n. The complex dimension of V_n is n + 1. If n is even, then V_n is the complexification of a real, irreducible module U_n of odd dimension n + 1. Hence in this case each element of $\mathfrak{su}(2)$ has nontrivial kernel. If n is odd, then the realification U_n of V_n is an irreducible module of real dimension 2n + 2. In this case U_n is nonsingular, and it is well known that U_n is of quaternionic type. See sections 5 and 6 of Chapter II of [BtD] for further details. □

Examples of $\mathfrak{z}^o_W$ when W is nonsingular

Proposition 2.4

1) Let W be a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$ of Clifford type. Then

a) $\mathfrak{Z}^o_W = \{0\}$ if p = 8k, 8k+6 or 8k+7.

b) $\mathfrak{Z}^o_W = \text{span}\{I\}$, where $I^2 = -\text{Id}$, if p = 4k + 1.

c) $\mathfrak{Z}^o_W = \text{span}\{I, J, K\}$, where {I, J, K} satisfy the quaternion relations if p = 8k+2, 8k+3 or 8k+4,

2) If $W = \mathfrak{su}(2) = \mathfrak{so}(3,\mathbb{R}) \subseteq \mathfrak{so}(q,\mathbb{R})$ and q is even, then $\mathfrak{Z}^o_W = \text{span}\{I, J, K\}$, where {I, J, K} satisfy the quaternion relations.

The assertions above follow immediately from the determinations of real, complex or quaternionic type in Proposition 1.3.

Section 3 Skew symmetric elements of $\mathfrak{G}_W$

Recall that $X \in \text{End}(\mathbb{R}^q)$ acts on $\mathfrak{so}(q,\mathbb{R})$ by $X(Z) = XZ + ZX^t$. Hence If $X \in \mathfrak{so}(q,\mathbb{R})$, then $X(Z) = XZ + ZX^t = XZ - ZX = \text{ad } X(Z)$. Let $\mathfrak{K}_W$ denote the subalgebra of $\mathfrak{G}_W$ consisting of the skew symmetric elements of $\mathfrak{G}_W$. If $\mathfrak{G}_W$ is not self adjoint, then in general there is no reason to expect that $\mathfrak{K}_W$ is nonzero.

Note that $W \subseteq \mathfrak{K}_W \Leftrightarrow W \subseteq \mathfrak{G}_W \Leftrightarrow$ W is a subalgebra of $\mathfrak{so}(q,\mathbb{R})$, and $[W,W] \subseteq \mathfrak{K}_W \Leftrightarrow [W,W] \subseteq \mathfrak{G}_W \Leftrightarrow$ W is a Lie triple system. If $X \in \mathfrak{Z}^o_W$, then $X \in \mathfrak{K}_W$ since ad X $\equiv 0$ on W and $X \in \mathfrak{so}(q,\mathbb{R})$. Hence $\mathfrak{Z}^o_W + [W,W] \subseteq \mathfrak{K}_W$ if W is a Lie triple system. In fact, we show in Proposition 3.2 that equality holds if W has trivial center.

The simplest example of a Lie triple system is a Lie algebra. Although the next result is a corollary of the one that follows it, the proof is simpler in this special case and illustrates the main idea more clearly.

Structure of $\mathfrak{K}_W$

Proposition 3.1 Let $W \subseteq \mathfrak{so}(q,\mathbb{R})$ be a semisimple subalgebra that acts irreducibly on $\mathbb{R}^q$. Then $\mathfrak{K}_W = \mathfrak{Z}^o_W \oplus W$, where $\mathfrak{Z}^o_W = \{ A \in \mathfrak{so}(q,\mathbb{R}) : AZ = ZA \text{ for all } Z \in W\}$.

Proof We note that $\mathfrak{Z}^o_W \cap W = \{0\}$ since a semisimple Lie algebra has trivial center. Recall that by definition $\mathfrak{Z}^o_W = \{ A \in \mathfrak{sl}(q,\mathbb{R}) : AZ = ZA \text{ for all } Z \in W\}$. However, by 2) of Proposition 1.3 $\mathfrak{Z}^o_W, \subseteq \mathfrak{so}(q,\mathbb{R})$.

By the discussion above $\mathfrak{K}_W = \{ X \in \mathfrak{so}(q,\mathbb{R}) : \text{ad } X(W) \subseteq W\}$. If $X \in \mathfrak{K}_W$, then ad X is a derivation of the semisimple Lie algebra W, all of whose derivations are inner. It follows that ad X = ad X' on W for some element X' in W. Therefore X = X' + Y, where

$Y = X - X' \in \mathcal{SO}(q,\mathbb{R})$ commutes with all elements of W. Hence $\mathfrak{K}_W \subseteq \mathfrak{Z}^o_W \oplus W$, and the reverse inclusion has been noted already. □

Proposition 3.2 Let $W \subseteq \mathcal{SO}(q,\mathbb{R})$ be a Lie triple system with trivial center that acts irreducibly on $\mathbb{R}^q$. Then $\mathfrak{K}_W = \mathfrak{Z}^o_W \oplus [W,W]$, where $\mathfrak{Z}^o_W = \{ A \in \mathcal{SO}(q,\mathbb{R}) : AZ = ZA$ for all $Z \in W\}$.

Proof We recall from the discussion above that $\mathfrak{Z}^o_W + [W,W] \subseteq \mathfrak{K}_W$. We observe next that the sum is direct. We note that $\mathfrak{G}' = W + [W,W]$ is a semisimple Lie algebra by Proposition 4.3b of [E4]. If $Z \in \mathfrak{Z}^o_W \cap [W,W]$, then Z commutes with [W,W] since Z commutes with W. It follows that Z lies in the center of $\mathfrak{G}' = W + [W,W]$, and hence Z = 0 by the semisimplicity of $\mathfrak{G}'$.

We shall need the following

Lemma 3.3 Let $W \subseteq \mathcal{SO}(q,\mathbb{R})$ be a Lie triple system with trivial center. Let $X \in W$ be an element with $\mathrm{ad}\, X(W) \subseteq W$. Then $X \in \mathfrak{H} = W \cap [W,W]$.

For the moment we postpone the proof of the lemma and use it to prove the proposition. If $X \in \mathfrak{K}_W$, then $X \in \mathcal{SO}(q,\mathbb{R})$ and $\mathrm{ad}\, X(W) \subseteq W$. It follows that $\mathrm{ad}\, X([W,W]) \subseteq [W,W]$ by the Jacobi identity, and hence $\mathrm{ad}\, X(\mathfrak{G}') \subseteq \mathfrak{G}'$, where $\mathfrak{G}' = W + [W,W]$. The Lie algebra $\mathfrak{G}'$ is semisimple as we noted above, and the transformation ad X is a derivation of $\mathfrak{G}'$. Hence there exists an element X' in $\mathfrak{G}'$ such that ad X = ad X' on $\mathfrak{G}'$. If $Y = X - X'$, then $Y \in \mathfrak{Z}^o_W$ since $Y \in \mathcal{SO}(q,\mathbb{R})$ and ad Y is zero on $\mathfrak{G}'$ and hence on W.

We have proved that if $X \in \mathfrak{K}_W$, then $X = X' + Y$, where $X' \in W + [W,W]$ and $Y \in \mathfrak{Z}^o_W$. Now write $X' = X_1 + X_2$, where $X_1 \in W$ and $X_2 \in [W,W]$. Note that $\mathrm{ad}\, X'(W) = \mathrm{ad}\, X(W) \subseteq W$ and $\mathrm{ad}\, X_2(W) \subseteq W$ since $[W,W] \subseteq \mathfrak{K}_W$. It follows that $X_1 \in W$ and $\mathrm{ad}\, X_1(W) \subseteq W$. From the lemma we conclude that $X_1 \in \mathfrak{H} = W \cap [W,W] \subseteq [W,W]$. Therefore $X' = X_1 + X_2 \in [W,W]$, and $X = X' + Y \in [W,W] + \mathfrak{Z}^o_W$. This proves that $\mathfrak{K}_W \subseteq \mathfrak{Z}^o_W + [W,W]$, and we already noted the reverse inclusion. □

Proof of the lemma Let $\mathfrak{G}' = W + [W,W]$, and let $\mathfrak{H} = W \cap [W,W]$. The Lie algebra $\mathfrak{G}'$ is semisimple, and $\mathfrak{H}$ is an ideal of $\mathfrak{G}'$ since $\mathfrak{H}$ is invariant under ad ξ for $\xi \in W$ and $\xi \in [W,W]$. Let $\mathfrak{H}^\perp$ denote the orthogonal complement of $\mathfrak{H}$ in $\mathfrak{G}'$ in relative to the Killing form B' on $\mathfrak{G}'$. Then $\mathfrak{H}^\perp$ is also a semisimple ideal of $\mathfrak{G}'$ by standard facts (cf. [Hel,p.131-133]).

We first show

(*) If $X \in W$ and $\mathrm{ad}\, X(W) \subseteq W$, then $\mathrm{ad}\, X(\mathfrak{H}^\perp) = \{0\}$.

To verify this, note that if ad $X(W) \subseteq W$, then ad $X(W) \subseteq \mathfrak{H} = W \cap [W,W]$ since clearly ad $X(W) \subseteq [W,W]$. Hence ad $X(\mathfrak{G}') \subseteq$ ad $X(W)$ + ad $X([W, W]) \subseteq \mathfrak{H} + [\mathfrak{H}, \mathfrak{H}] \subseteq \mathfrak{H}$. It follows that ad $X(\mathfrak{H}^{\perp}) \subseteq \mathfrak{H}^{\perp} \cap$ ad $X(W) \subseteq \mathfrak{H}^{\perp} \cap \mathfrak{H} = \{0\}$ since both $\mathfrak{H}$ and $\mathfrak{H}^{\perp}$ are ideals of $\mathfrak{G}'$.

Now let $X \in W$ be an element such that ad $X(W) \subseteq W$. Then ad $X([W,W]) \subseteq [W,W]$, and it follows immediately that ad $X(\mathfrak{H}) \subseteq \mathfrak{H}$. Since $\mathfrak{H}$ is semisimple and ad X is a derivation of $\mathfrak{H}$ there exists $X_1 \in \mathfrak{H}$ such that ad X = ad X_1 on $\mathfrak{H}$. If $Y = X - X_1$, then $Y \in W$ since $X_1 \in \mathfrak{H} \subseteq W$. Moreover, ad $X_1(W) \subseteq W$ since $X_1 \in \mathfrak{H} \subseteq [W,W] \subseteq \mathfrak{G}_{\mathbf{W}}$. Therefore ad $Y(W) \subseteq W$ and ad $Y \equiv 0$ on $\mathfrak{H}^{\perp}$ by (*), On the other hand ad $Y \equiv 0$ on $\mathfrak{H}$ by the definition of Y. It follows that ad $Y \equiv 0$ on $\mathfrak{H} \oplus \mathfrak{H}^{\perp} = \mathfrak{G}' = W + [W,W] \supseteq W$. Hence Y lies in the center of W, which by hypothesis is $\{0\}$. We conclude that $X = X_1 \in \mathfrak{H}$. This completes the proof of the lemma. □

Examples of $\mathfrak{K}_{\mathbf{W}}$ when W is nonsingular

Proposition 3.4

1) Let W be a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$ of Clifford type. Then $\mathfrak{K}_{\mathbf{W}} = \mathfrak{Z}^{o}_{W} \oplus \mathfrak{so}(p,\mathbb{R})$ (direct sum), where
   a) $\mathfrak{Z}^{o}_{W} = \{0\}$ if p = 8k, 8k+6 or 8k+7.
   b) $\mathfrak{Z}^{o}_{W}$ = span{I}, where $I^2 = -\mathrm{Id}$, if p = 8k+1 or 8k+5
   c) $\mathfrak{Z}^{o}_{W}$ = span {I, J, K}, where {I, J, K} satisfy the quaternion relations, if p = 8k+2,8k+3 or 8k+4.

2) If $W = \mathfrak{su}(2) = \mathfrak{so}(3,\mathbb{R}) \subseteq \mathfrak{so}(q,\mathbb{R})$ and q is even, then $\mathfrak{K}_{\mathbf{W}} = \mathfrak{Z}^{o}_{W} \oplus \mathfrak{su}(2)$ (direct sum), where $\mathfrak{Z}^{o}_{W}$ = span {I, J, K}, and {I, J, K} satisfy the quaternion relations.

To verify 1) we consider first the case that $p \neq 3$. We note that by previous discussion $W = j(\mathbb{R}^p)$, where $j : C\ell(p) \to \mathrm{End}(\mathbb{R}^q)$ is a representation of the real negative definite Clifford algebra $C\ell(p)$ and $\mathbb{R}^q$ is equipped with a suitable inner product. It is well known that $[\mathbb{R}^p,\mathbb{R}^p]$ is isomorphic to $\mathfrak{so}(p,\mathbb{R})$, where [,] denotes the natural Lie algebra structure on $C\ell(p)$. Moreover, $j : C\ell(p) \to \mathrm{End}(\mathbb{R}^q)$ is injective if $p \neq 3$. For example, see Proposition 3 of [E1] for details. Hence $[W,W] = j[\mathbb{R}^p,\mathbb{R}^p]$ is isomorphic to $\mathfrak{so}(p,\mathbb{R})$. Note that the elements of $\mathfrak{Z}^{o}_{W}$ commute with [W,W] since they commute with W. It follows that $\mathfrak{Z}^{o}_{W} \cap [W,W] = \{0\}$ since the center of $[W, W] \cong \mathfrak{so}(p,\mathbb{R})$ is trivial. Therefore from the proposition above we obtain $\mathfrak{K}_{\mathbf{W}} = \mathfrak{Z}^{o}_{W} \oplus [W,W] = \mathfrak{Z}^{o}_{W} \oplus \mathfrak{so}(p,\mathbb{R})$. The possibilities for $\mathfrak{Z}^{o}_{W}$ are copied from the earlier list in (2.4).

Next suppose that p = 3. By Proposition 3 of [E1] either $j : C\ell(3) \to \mathrm{End}(\mathbb{R}^q)$ is injective and $[W, W] \cong \mathfrak{so}(3,\mathbb{R})$ or $[W, W] = W \cong \mathfrak{so}(3,\mathbb{R})$. In either case $\mathfrak{K}_{\mathbf{W}} = \mathfrak{Z}^{o}_{W} \oplus$

$[W, W] \cong \mathfrak{Z}^{o}_{W} \oplus \mathfrak{so}(3,\mathbb{R})$. Note that in the second of these cases $W \cong \mathfrak{su}(2)$ is a subalgebra of $\mathfrak{so}(q,\mathbb{R})$, and we are actually in the second case of this proposition.

The verification of 2) is similar. Note that $\mathfrak{Z}^{o}_{W} \cap W = \mathfrak{Z}^{o}_{W} \cap \mathfrak{su}(2) = \{0\}$ since $\mathfrak{su}(2)$ has trivial center. From the corollary to the proposition above $\mathfrak{K}_{\mathbf{W}} = \mathfrak{Z}^{o}_{W} \oplus W = \mathfrak{K}_{\mathbf{W}} = \mathfrak{Z}^{o}_{W} \oplus \mathfrak{su}(2)$, where the description of $\mathfrak{Z}^{o}_{W}$ also comes from the earlier list.

Section 4 Symmetric elements of $\mathfrak{G}_{\mathbf{W}}$

It is more difficult to determine the symmetric elements of $\mathfrak{G}_{W}$, even in the cases where $\mathfrak{G}_{W}$ is known to be self adjoint. We can give a complete description only in the self adjoint nonsingular cases where W is of Clifford type and where $W = \mathfrak{su}(2)$ and q is even, If W is a semisimple subalgebra of $\mathfrak{so}(q,\mathbb{R})$ other than $\mathfrak{su}(2)$, then we don't know much about the symmetric elements of $\mathfrak{G}_{W}$.

Let $\mathfrak{P}_{\mathbf{W}} = \{A \in \mathfrak{G}_{W} : A = A^{t}\}$. Note that the elements of $\mathfrak{P}_{\mathbf{W}}$ have trace zero since by definition $\mathfrak{G}_{W} \subseteq \mathfrak{sl}(q,\mathbb{R})$.

Structure of $\mathfrak{P}_{\mathbf{W}}$

We begin with some general observations and then proceed to the two examples of nonsingular W mentioned above.

Proposition 4.1 Let A be an element of $\mathfrak{P}_{\mathbf{W}}$. Let $\{v_1, \dots v_q\}$ be an orthonormal basis of $\mathbb{R}^q$ such that $Av_i = \lambda_i v_i$ for $1 \le i \le q$, where are $\{\lambda_1, \dots \lambda_q\}$ are the (real) eigenvalues of A.

1) If $[v_i, v_j]$ is nonzero in W, then $A\,[v_i, v_j] = (\lambda_i+\lambda_j)[v_i, v_j]$.

2) If $A(Z) = \lambda Z$ for some $Z \in W$ and $\lambda \in \mathbb{R}$, then $AZ^2 = Z^2A$.

3) Let $T = \frac{1}{2}\sum_{i=1}^{p} j(Z_i)^2$, where $\{Z_1, \dots, Z_p\}$ is any orthonormal basis of $\mathfrak{Z}$. Then T commutes with every element of $\mathfrak{H}_{\mathbf{W}} = \mathfrak{P}_{\mathbf{W}} \oplus [\mathfrak{P}_{\mathbf{W}}, \mathfrak{P}_{\mathbf{W}}]$.

Remarks

1)The transformation T in 3) describes the Ricci tensor on $\mathfrak{Z}^{\perp}$ by the proposition in (1.4) of [E2].

2) It is easy to show that the subspace $\mathfrak{H}_{\mathbf{W}}$ in 3) is a subalgebra of $\mathfrak{G}_{\mathbf{W}}$. We shall see below in (7.1) that if $\mathfrak{G}_{\mathbf{W}}$ is self adjoint, then $\mathfrak{H}_{\mathbf{W}}$ is an ideal in $\mathfrak{G}_{\mathbf{W}}$.

Proof We recall the discussion following (3.4e) in [E2] of symmetric and skew symmetric derivations of $\mathfrak{N} = \mathbb{R}^q \oplus W$. We showed there that A defines a derivation φ on $\mathfrak{N}$ by $\varphi(v) = A(v)$ for all v in $\mathbb{R}^q$ and $\varphi(Z) = A(Z) = AZ + ZA^t = AZ + ZA$ for all Z in W.

1) If $Av_i = \lambda_i v_i$ for $1 \le i \le q$, then $A\,[v_i, v_j] = [A(v_i), v_j] + [v_i, A(v_j)] = (\lambda_i+\lambda_j)[v_i, v_j]$.

2) If $\lambda Z = A(Z) = AZ + ZA$, then $AZ^2 = (\lambda Z - ZA)Z = \lambda Z^2 - Z(\lambda Z - ZA) = Z^2 A$.

3) Let A be an element of $\mathfrak{P}_W$. By the discussion in (3.2) of [E2] the element A acts as a symmetric transformation of $\mathcal{SO}(q,\mathbb{R})$ since A is a symmetric transformation of $\mathbb{R}^q$. Since $A(W) \subseteq W$ we can find an orthonormal basis $\{Z_1, \ldots, Z_p\}$ of W such that $\lambda_i Z_i = A(Z_i) = AZ_i + Z_i A$ for all i. By 2) A commutes with each Z_i^2, and hence A commutes with $T = \frac{1}{2} \sum_{i=1}^{p} j(Z_i)^2$. Since T commutes with $\mathfrak{P}_W$ it follows immediately that T commutes with $\mathfrak{H}_W = \mathfrak{P}_W \oplus [\mathfrak{P}_W, \mathfrak{P}_W]$. □

Proposition 4.2 $[\mathfrak{K}_W, \mathfrak{P}_W] \subseteq \mathfrak{P}_W$.

Proof Note that $\mathfrak{G}_W$ is a Lie subalgebra of End($\mathbb{R}^q$), and $\mathfrak{K}_W$, $\mathfrak{P}_W$ are subspaces of $\mathfrak{G}_W$. It follows that $[\mathfrak{K}_W, \mathfrak{P}_W] \subseteq \mathfrak{G}_W$. If A is skew symmetric and B is symmetric, then $[A,B] = AB - BA$ is symmetric. □

It follows from Propositions 5.2, 7.3 and 7.5 below that equality holds in the result above if W is the Lie algebra of a compact, connected subgroup of SO(q,$\mathbb{R}$) that acts irreducibly on $\mathbb{R}^q$.

If W is nonsingular, then we can say a great deal more.

Proposition 4.3 Let W be a nonsingular p-dimensional subspace of $\mathcal{SO}(q,\mathbb{R})$, and let A be a nonzero element of $\mathfrak{P}_W$. Let W act irreducibly on $\mathbb{R}^q$. Then

1) $AZ = - ZA$ for all $Z \in W$.

2) $A^2 = \lambda^2 \,\mathrm{Id}$ for some positive number λ.
$\mathbb{R}^q = V_\lambda \oplus V_{-\lambda}$ where $V_\lambda, V_{-\lambda}$ denote the λ, $-\lambda$ eigenspaces of A on $\mathbb{R}^q$.
The eigenspaces V_λ and $V_{-\lambda}$ have the same dimension.

3) Let A_1, A_2 be elements of $\mathfrak{P}_W$ such that $\langle A_1, A_2 \rangle = -\,\mathrm{trace}\, A_1 A_2 = 0$. Then $A_1 A_2 = - A_2 A_1$.

4) $\dim \mathfrak{P}_W \le 3$.

We shall see below in Proposition 6.4d that equality is achieved in 4) if W is of Clifford type and has dimension 8k + 2 for some integer $k \ge 0$.

Proof 1) By the proposition in (3.2) of [E2] the transformation $A(Z) = AZ + ZA^t = AZ + ZA$ is symmetric on W with the canonical inner product $\langle\, ,\, \rangle$ from $\mathcal{SO}(q,\mathbb{R})$. It suffices to show that zero is the only eigenvalue of A acting on W. Let $\lambda Z = A(Z) = AZ + ZA$ for some nonzero element Z of W. Since Z is invertible we obtain $\lambda\, \mathrm{Id} = Z^{-1}AZ + A$, and hence $q\lambda = \mathrm{trace}\,(\lambda\, \mathrm{Id}) = \mathrm{trace}\,(Z^{-1}AZ + A) = 2\, \mathrm{trace}\, A = 0$.

2) If λ is an eigenvalue of A on $\mathbb{R}^q$, then it follows from 1) that $-\lambda$ is also an eigenvalue and $Z(V_\lambda) \subseteq V_{-\lambda}$, $Z(V_{-\lambda}) \subseteq V_\lambda$ for all Z in W. In particular W leaves invariant the subspace $V_\lambda \oplus V_{-\lambda}$, which must equal $\mathbb{R}^q$ by the irreducibility of the W action. It

follows immediately that $A^2 = \lambda^2$ Id. Finally, since each nonzero element Z of W is invertible and interchanges the eigenspaces V_λ and $V_{-\lambda}$ it follows that V_λ and $V_{-\lambda}$ have the same dimension.

3) The transformation $A_1 A_2 + A_2 A_1$ is symmetric and commutes with the elements of W by 1). Since W acts irreducibly on $\mathbb{R}^q$ it follows that $A_1 A_2 + A_2 A_1 = \lambda$ Id for some real number λ. Finally, $q \lambda$ = trace (λ Id) = 2 trace($A_1 A_2$) = 0.

4) We suppose that dim $\mathfrak{P}_W \geq 4$ and obtain a contradiction. Let $\{A_1 , A_2 , A_3 , A_4\}$ be part of an orthogonal basis of $\mathfrak{P}_W$ relative to the canonical inner product < , > on $\mathfrak{so}(q,\mathbb{R})$. In particular, trace $A_i A_j = 0$ if $i \neq j$. Without loss of generality we may also assume that A_i^2 = Id for all i by 2). By 3) and the fact that each A_i is symmetric it follows that $A_i A_j$ is skew symmetric for all $i \neq j$. By 1) $A_i A_j$ commutes with the elements of W if $i \neq j$, and each $A_i A_j$ is nonzero since A_i^2 = Id. By Proposition 1.3 we know that 3 ≥ dim $\mathfrak{Z}^o_W = \{ A \in \mathfrak{sl}(q,\mathbb{R}) : AZ = ZA \text{ for all } Z \in W\}$.

Choose real numbers a,b,c,d , not all zero, so that $0 = a A_1 A_2 + b A_1 A_3 + c A_1 A_4 + d A_2 A_3$. We consider separately the cases d = 0 and d ≠ 0.

If d = 0, then by multiplying the equation above on the left by A_1 we obtain $0 = a A_2 + b A_3 + c A_4$. Since the set $\{A_2 , A_3 , A_4\}$ is linearly independent it follows that a = b = c = 0. Since d = 0 we obtain a contradiction.

If d ≠ 0, then $\xi = A_2 A_3 = a' A_1 A_2 + b' A_1 A_3 + c' A_1 A_4$ for suitable real numbers a' , b' , c'. By 3) it follows that $A_1 \xi = \xi A_1$. However, $A_1 \xi = a' A_2 + b' A_3 + c' A_4$ while $\xi A_1 = (a' A_1 A_2 + b' A_1 A_3 + c' A_1 A_4) A_1 = -(a' A_1^2 A_2 + b' A_1^2 A_3 + c' A_1^2 A_4)$ (by 3)) $= -(a' A_2 + b' A_3 + c' A_4) = -A_1 \xi$. Hence $A_1 \xi = \xi A_1 = -A_1 \xi$. It follows that $A_1 \xi = 0$, which contradicts the facts that A_1^2 = Id and $\xi \neq 0$.

The contradictions in the cases d = 0 and d ≠ 0 prove that dim $\mathfrak{P}_W \leq 3$.□

Examples of $\mathfrak{P}_W$ when W is nonsingular

We now determine the structure and dimension of $\mathfrak{P}_W$ in the nonsingular cases that W is of Clifford type or W = $\mathfrak{su}(2) = \mathfrak{so}(3,\mathbb{R})$.

Proposition 4.4 Let W be a p-dimensional subspace of $\mathfrak{so}(q,\mathbb{R})$ of Clifford type that acts irreducibly on $\mathbb{R}^q$. Then

a) If p = 4k, then dim $\mathfrak{P}_W = 1$.
b) If p = 8k + 2, then dim $\mathfrak{P}_W = 3$.
c) If p = 8k + 6, then $\mathfrak{P}_W = \{0\}$.
d) If p = 4k + 3, then $\mathfrak{P}_W = \{0\}$.
e) If p = 8k + 1, then dim $\mathfrak{P}_W = 2$.

f) If p = 8k + 5, then $\mathfrak{P}_W = \{0\}$.

Proposition 4.5 Let W = $\mathcal{SU}(2) = \mathcal{SO}(3,\mathbb{R}) \subseteq \mathcal{SO}(q,\mathbb{R})$, where q is even and W acts irreducibly on $\mathbb{R}^q$. Then $\mathfrak{P}_W = \{0\}$.

Proof of proposition 4.4) For the discussion below we fix an orthonormal basis $\{e_1, e_2, \ldots e_p\}$ for $\mathbb{R}^p$, and we set $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$ in the real negative definite Clifford algebra $C\ell(p)$. We also let z denote the corresponding element in $W = j(\mathbb{R}^p)$, where $j : C\ell(p) \to \mathrm{End}(\mathbb{R}^q)$ is an irreducible representation and $\mathbb{R}^q$ is equipped with an inner product so that $W \subseteq \mathcal{SO}(q,\mathbb{R})$. The properties of the element z determine the structure of $\mathfrak{P}_W$.

We note that each element e_i preserves the inner product on $\mathbb{R}^q$ since $e_i\, e_i^t = -\, e_i^2 = \mathrm{Id}$. Hence $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$ preserves the inner product on $\mathbb{R}^q$. It follows that if $z^2 = 1$ (respectively $z^2 = -1$), then z is symmetric (respectively skew symmetric) as an element of $\mathrm{End}(\mathbb{R}^q)$.

We shall need the following standard information. See for example Lemma 1b in the appendix of [E1] for details.

Lemma 4.6

1) If p = 4k, then $z^2 = 1$ and $z\, e_i = -\, e_i\, z$ for all i. As an element of $\mathrm{End}(\mathbb{R}^q)$ the transformation z is symmetric.

2) If p = 4k + 1, then z lies in the center of $C\ell(p)$ and $z^2 = -1$. As an element of $\mathrm{End}(\mathbb{R}^q)$ the transformation z is skew symmetric and commutes with the elements of W.

3) If p = 4k + 2, then $z^2 = -1$ and $z\, e_i = -\, e_i\, z$ for all i. As an element of $\mathrm{End}(\mathbb{R}^q)$ the transformation z is skew symmetric.

4) If p = 4k + 3, then z lies in the center of $C\ell(p)$ and $z^2 = 1$. As an element of $\mathrm{End}(\mathbb{R}^q)$ the transformation z is symmetric and commutes with the elements of W.

We now prove the proposition.

a) Let p = 4k for some integer $k \geq 1$. By case 1) of the lemma above, the nonzero element $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$ is symmetric, $z^2 = 1$, and z anticommutes with the elements of W. Hence $z \in \mathfrak{P}_W$ since z acts as the zero transformation on W.

Suppose now that $\dim \mathfrak{P}_W \geq 2$ and let A be an element of $\mathfrak{P}_W$ such that $< A, z > = -\,\mathrm{trace}\, Az = 0$. We know that $Az = -\, zA$ by 3) of Proposition 4.3, and A anticommutes with the elements of W by 1) of Proposition 4.3. Hence Az commutes with the elements of W since both A and z anticommute, and in particular Az commutes with $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$. However, $(Az)z = A$ and $z(Az) = -\, Azz = -\, A$. This implies that A = 0, contradicting the assumption on A. Hence $\dim \mathfrak{P}_W = 1$.

b) Let $p = 8k + 2$ for some integer $k \geq 0$. If $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$, then by case 3) of the lemma above z is skew symmetric, $z^2 = -\mathrm{Id}$ and z anticommutes with the elements of W. By Proposition 2.3 W is of quaternionic type, and from Proposition 2.4 it follws that $\mathfrak{Z}^o_W = \{ A \in \mathfrak{sl}(q,\mathbb{R}) : AZ = ZA \text{ for all } Z \in W\} = \mathrm{span}\,\{I, J, K\}$, where {I, J, K} satisfy the standard quaternion relations.

It suffices to show that the set {z I, z J, z K} is a basis for $\mathfrak{P}_W$. We observe

(*) Let ξ be a nonzero element of $\mathfrak{Z}^o_W$. Then $z\,\xi \in \mathfrak{P}_W$.

Assuming for the moment that (*) has been proved, it follows that $\{z I, z J, z K\} \subseteq \mathfrak{P}_W$. Since $\dim \mathfrak{P}_W \leq 3$ by 4) of Proposition 4.3 it suffices to show that the set {z I, z J, z K} is linearly independent in $\mathfrak{P}_W$. If $0 = a\,(z I) + b\,(z J) + c\,(z K)$ for some real numbers a,b and c, then by multiplying this equation on the left by z we obtain $0 = a I + b J + c K$, which implies that $a = b = c = 0$ since the set {I, J, K} is a basis of $\mathfrak{Z}^o_W$.

We prove (*). Let $\xi = a I + b J + c K$ be a nonzero element of $\mathfrak{Z}^o_W$. By the quaternion relations that govern {I, J, K} it follows that ξ^2 is a negative multiple of the identity, and in particular that $z\,\xi \neq 0$. The elements z and ξ are skew symmetric, and they commute since the elements of $\mathfrak{Z}^o_W$ commute with W and hence with $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$. Hence $(z\,\xi)^t = \xi^t z^t = (-\xi)(-z) = z\,\xi$, and $z\,\xi$ anticommutes with the elements of W since z anticommutes and ξ commutes. We conclude that $z\,\xi \in \mathfrak{P}_W$. □

Example Let k = 0, Then $p = 8k + 2 = 2$, and $C\ell(\mathbb{R}^2)$ can be identified with the quaternions $\mathbb{H}$. Let {1, i, j, k} be the standard basis for $\mathbb{H}$. Let L_x and R_x denote left and right quaternion multiplication respectively by an element x of $\mathbb{H}$. It is obvious that L_x and R_x commute for any x in $\mathbb{H}$. If $x \in \mathrm{Im}\,\mathbb{H} = \mathrm{span}\,\{i, j, k\}$, then it is routine to check that L_x and R_x are skew symmetric relative to the inner product on $\mathbb{H}$ that makes {1, i, j, k} an orthonormal basis. Hence $L_x \circ R_y$ is symmetric for any elements $x,y \in \mathrm{Im}\,\mathbb{H}$.

If $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathbb{R}^2$, then we define an action of W = span $\{e_1,e_2\}$ on $\mathbb{H}$ by setting $e_1 = R_j$ and $e_2 = R_i$. Fix an element $x \in \mathrm{Im}\,\mathbb{H}$. Then $L_x \circ R_y$ anticommutes with e_1 and e_2 for $y \in \mathrm{Im}\,\mathbb{H} \Leftrightarrow$ y is a multiple of k. If $P_x = L_x \circ R_k$, then P_x is symmetric, anticommutes with the elements of W and has trace zero. It follows that $\mathfrak{P}_W = \{P_x : x \in \mathrm{Im}\,\mathbb{H}\}$.

c) Let $p = 8k + 6$, where $k \geq 0$ is an integer. If $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p$, then by 3) of the lemma above z is skew symmetric, $z^2 = -1$ and z anticommutes with the elements of W. Suppose that A is a nonzero element of $\mathfrak{P}_W$. Then A anticommutes with the elements of W by 1) of Proposition 4.3, and it follows that A commutes with z since z is the product of an even number of elements of W. In particular Az is skew symmetric since $(Az)^t = z^t A^t = (-z)A = -Az$. Moreover, Az commutes with the elements of W since A and z both anticommute. By definition $Az \in \mathfrak{Z}^o_W = \{ A \in \mathfrak{sl}(q,\mathbb{R}) : AZ = ZA \text{ for all } Z \in W\}$.

However, $\mathfrak{Z}_W^o = \{0\}$ by Proposition 2.4 since W is of real type by Proposition 2.3. It follows that $0 = (Az)z = -A$, a contrradiction to the hypothesis that $\mathfrak{P}_W \neq \{0\}$.

d) Let $p = 4k + 3$, where $k \geq 0$ is an integer. If $z = e_1 \cdot e_2 \cdot ... \cdot e_p$, then by 4) of the lemma above z is symmetric, $z^2 = 1$ and z commutes with the elements of W. Hence W leaves invariant each eigenspace of z , and we conclude that $z = \lambda$ Id for some $\lambda \in \mathbb{R}$ since W acts irreducibly on $\mathbb{R}^q$. It follows that $\lambda = \pm 1$ since $z^2 = 1$, and we conclude that $z = \pm$ Id on $\mathbb{R}^q$.

If A is a nonzero element of $\mathfrak{P}_W$, then by 1) of Proposition 4.3 A anticommutes with the elements of W. Hence A anticommutes with z, which is the product of an odd number of elements of W. However, A also commutes with z since $z = \pm$ Id on $\mathbb{R}^q$, which is impossible.

e) We shall need the following

Lemma 4.7 Let $W \subseteq \mathcal{SO}(q,\mathbb{R})$ be a (4k+1)-dimensional subspace of Clifford type that acts irreducibly on $\mathbb{R}^q$. Then dim $\mathfrak{P}_W = 0$ or 2.

Proof We recall that W is of complex type by Proposition 6.2c, and $\mathfrak{Z}_W^o = \text{span}\{I\}$, where $I^2 = -$ Id, by Proposition 2.4. Moreover, if $p = 4k+1$ and $\{e_1, ... , e_p\}$ is the standard orthonormal basis of $\mathbb{R}^p$, then by assertion 2) of Lemma 4.6 we know that I = z = $e_1 \cdot e_2 ... \cdot e_p$, where $\cdot$ denotes multiplication in the Clifford algebra $C\ell(p)$. Note also that ad z acts skew symmetrically on $\mathcal{SO}(q,\mathbb{R})$ relative to the canonical inner product < , > since z acts skew symmetrically on $\mathbb{R}^q$.

Now assume that $\mathfrak{P}_W \neq \{0\}$ and let A be a nonzero element of $\mathfrak{P}_W$. Without loss of generality we may assume that $A^2 =$ Id by 2) of Proposition 4.3. Observe that A anticommutes with W by 1) of Proposition 4.3. Hence A anticommutes with z = $e_1 \cdot e_2 ... \cdot e_p$ since p is odd, and it follows that B = zA anticommutes with W since z commutes with W. We conclude that $B \in \mathfrak{P}_W$ since $B^t = A^t z^t = -Az = zA = B$. Moreover, B is orthogonal to A in $\mathfrak{P}_W$ since $\langle B, A \rangle = -\text{trace}(zA^2) = -\text{trace}(z) = 0$ by the skew symmetry of z on $\mathbb{R}^q$.

We have proved that dim $\mathfrak{P}_W \geq 2$ if $\mathfrak{P}_W \neq \{0\}$ and dim W = 4k+1. Recall that $z \in \mathfrak{Z}_W^o \subseteq \mathfrak{K}_W$, by Proposition 3.2. Hence ad z $(\mathfrak{P}_W) \subseteq [\mathfrak{K}_W, \mathfrak{P}_W] \subseteq \mathfrak{P}_W$ by Proposition 4.2. By the remarks above ad z acting on $\mathfrak{P}_W$ is skew symmetric and nonsingular since ad z(A) = zA – Az = 2 zA is nonzero if $A \in \mathfrak{P}_W$ is nonzero. Hence dim $\mathfrak{P}_W \neq 3$ and it follows from 4) of Proposition 4.3 that dim $\mathfrak{P}_W \leq 2$. We conclude that dim $\mathfrak{P}_W = 2$ if $\mathfrak{P}_W \neq \{0\}$. □

Remark For future reference we note that the discussion above shows that

$[\mathfrak{P}_W, \mathfrak{P}_W]$ = span {[A, B]} = span{z} if $\mathfrak{P}_W \neq \{0\}$ and dim W = 4k+1.

To complete the proof of e) we need to recall some facts about the real Clifford algebras, both positive definite and negative definite.

For any integer $p \geq 2$ we fix an inner product < , > on $\mathbb{R}^p$, and we let $\{e_1, \ldots, e_p\}$ denote an orthonormal basis of $\mathbb{R}^p$ relative to < , >. We let $C\ell^*(p)$ denote the real positive definite Clifford algebra determined by $\mathbb{R}^p$ with the inner product < , > and multiplication relations defined by the conditions $e_i^2 = 1$ for $1 \leq i \leq p$ and $e_i e_j = - e_j e_i$ for $i \neq j$. We let $C\ell(p)$ denote the real negative definite Clifford algebra determined by $\mathbb{R}^p$ with the inner product < , > and multiplication relations defined by the conditions $e_i^2 = -1$ for $1 \leq i \leq p$ and $e_i e_j = - e_j e_i$ for $i \neq j$. We let $\mathbb{R}(p)$, $\mathbb{C}(p)$, or $\mathbb{H}(p)$ denote the p x p matrices with entries in $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

The Clifford algebras satisfy the periodicity relations $C\ell^*(p+8) \cong C\ell^*(p) \otimes \mathbb{R}(16)$ and $C\ell(p+8) \cong C\ell(p) \otimes \mathbb{R}(16)$. See [LM, pp.27-28] for details. From Table I in [LM, p.28] one easily proves by induction the following isomorphisms

Lemma 4.8 $C\ell^*(8k+1) \cong \mathbb{R}(2^{4k}) \oplus \mathbb{R}(2^{4k})$ and $C\ell^*(8k+5) \cong \mathbb{H}(2^{4k+1}) \oplus \mathbb{H}(2^{4k+1})$
$C\ell(8k+1) \cong \mathbb{C}(2^{4k})$ and $C\ell(8k+5) \cong \mathbb{C}(2^{4k+2})$

We are going to use an irreducible representation of $C\ell^*(8k+1)$ on $\mathbb{R}^{2^{4k}}$ to construct an irreducible representation of $C\ell(8k+1)$ on $\mathbb{C}^{2^{4k}}$ that will allow us to prove e). This method fails in the case p = 8k+5, and in fact the outcomes are different for the cases p = 8k+1 and p = 8k+5, as we shall see.

There are exactly two irreducible $C\ell^*(8k+1)$ modules up to equivalence. In each case $C\ell^*(8k+1)$ acts on the Euclidean space $\mathbb{R}^{2^{4k}}$. The first action is given by (A,B)(v) = Av and the second (inequivalent) action is given by (A,B)(v) = Bv.

Fix one of these two actions of $C\ell^*(p)$ on $\mathbb{R}^{2^{4k}}$, where p = 8k+1. Let $\{e_1, \ldots, e_p\}$ denote matrices in $\mathbb{R}(2^{4k})$ that satisfy the relations

(1) $e_j^2 = \mathrm{Id}$ for $1 \leq j \leq p$ and $e_j e_r = - e_r e_j$ for $r \neq j$.

Let $G = \mathfrak{B} \cup - \mathfrak{B}$ be the group of order $2 \cdot 2^p$ in $C\ell^*(p)$, where $\mathfrak{B}$ is the natural vector space basis of $C\ell^*(p)$ that consists of all finite products $e_{i_1} \cdot e_{i_2} \cdot \ldots e_{i_k}$ for $1 \leq k \leq p$ and $1 \leq i_1 < i_2 < \ldots < i_k \leq p$. Choose a G-invariant inner product < , > on $\mathbb{R}^{2^{4k}}$. Note that each element e_r, $1 \leq r \leq p$, is symmetric on $\mathbb{R}^{2^{4k}}$ since $\mathrm{Id} = e_r e_r^t = e_r^2$.

Extend < , > to a real, positive definite inner product on $\mathbb{R}^{2^{4k+1}} \cong \mathbb{C}^{2^{4k}}$ by < z, w > = < Re z, Re w > + < Im z, Im w > for all z,w $\in \mathbb{C}^{2^{4k}}$. Let $z_r = i e_r$ act on $\mathbb{C}^{2^{4k}} = \mathbb{R}^{2^{4k}} \oplus i\mathbb{R}^{2^{4k}}$ by $z_r(u + i v) = z_r(u, v) = (- e_r(v), e_r(u))$. Then z_r is skew symmetric relative to < , > on $\mathbb{C}^{2^{4k}} \cong \mathbb{R}^{2^{4k+1}}$ since e_r is symmetric relative to < , > on $\mathbb{R}^{2^{4k+1}}$. Moreover, from (1) we obtain

(2) $z_j^2 = -\mathrm{Id}$ for $1 \le j \le p$ and $z_j z_r = -z_r z_j$ for $r \ne j$.
If $W = \mathrm{span}\{z_1, \ldots, z_p\}$, then W consists of skew symmetric transformations acting on $\mathbb{R}^{2^{4k+1}}$. The relations in (2) now define a representation of the entire negative definite Clifford algebra $C\ell(p)$ on $\mathbb{R}^{2^{4k+1}}$. This representation of $C\ell(p)$ is irreducible since the dimension of an irreducible $C\ell(p)$ module is 2^{4k+1} by the lemma above, which shows that $\mathbb{R}^{2^{4k+1}} \cong \mathbb{C}^{2^{4k}}$ is the unique irreducible $C\ell(8k+1)$ module, up to equivalence. Hence W acts irreducibly on $\mathbb{R}^{2^{4k+1}}$ since W generates $C\ell(8k+1)$.

Define $A : \mathbb{C}^{2^{4k}} \to \mathbb{C}^{2^{4k}} = \mathbb{R}^{2^{4k}} \oplus i\mathbb{R}^{2^{4k}}$ to be the conjugation map given by $A(u + i v) = u - i v$. From (2) it is easy to verify that $Az_r = -z_r A$ for $1 \le r \le p$, and it is also obvious from the definitions that $A^2 = \mathrm{Id}$ and A is symmetric relative to $<\,,\,>$. Hence A acts as the zero transformation on W since $A(Z) = AZ + ZA^t = AZ + ZA = 0$ for all $Z \in W$. We conclude that A is a nonzero element of $\mathfrak{P}_W$, which proves that $\dim \mathfrak{P}_W = 2$ by Lemma 4.7.

f) Before beginning the proof we indicate why the method used in e) for the case p = 8k+1 does not produce the same outcome here in the case p = 8k+5. If p = 8k+5, then $C\ell^*(p) \cong \mathbb{H}(2^{4k+1}) \oplus \mathbb{H}(2^{4k+1})$ by Lemma 4.8, and $C\ell^*(p)$ acts irreducibly on $\mathbb{H}^{2^{4k+1}} \cong \mathbb{C}^{2^{4k+2}} \cong \mathbb{R}^{2^{4k+3}}$. As above, we obtain a representation of $C\ell(p)$ on $\mathbb{R}^{2^{4k+4}} \cong \mathbb{C}^{2^{4k+3}} = \mathbb{R}^{2^{4k+3}} \oplus i\mathbb{R}^{2^{4k+3}}$, but this representation is not irreducible. If p = 8k+5, then by Lemma 6.4h the algebra $C\ell(p) \cong \mathbb{C}(2^{4k+2})$ acts irreducibly on $\mathbb{C}^{2^{4k+2}} \cong \mathbb{R}^{2^{4k+3}}$. Hence the representation of $C\ell(p)$ on $\mathbb{R}^{2^{4k+4}}$, which is constructed from the representation of $C\ell^*(p)$ on $\mathbb{R}^{2^{4k+3}}$, is a direct sum of two irreducible $C\ell(p)$ modules.

We now prove f) by showing that if $\mathfrak{P}_W \ne \{0\}$, then we obtain a nontivial representation of $C\ell^*(8k+5)$ on $\mathbb{R}^{2^{4k+2}}$. This will contradict the fact that 2^{4k+3} is the smallest dimension for a nontrivial, irreducible $C\ell^*(8k+5)$ module.

Note that $C\ell(8k+5) \cong \mathbb{C}(2^{4k+2})$ by Lemma 4.8, and hence $\mathbb{C}^{2^{4k+2}} \cong \mathbb{R}^{2^{4k+3}}$ is the unique irreducible $C\ell(8k+5)$ module up to equivalence. If p = 8k+5, $q = 2^{4k+3}$ and $W = j(\mathbb{R}^p) \subseteq j(C\ell(p))$, then W acts irreducibly on $\mathbb{R}^q$.

We suppose that $\mathfrak{P}_W$ contains a nonzero element A, and we derive a contradiction. By 2) of Proposition 4.3 we may assume that $A^2 = \mathrm{Id}$ and $\mathbb{R}^q = V_1 \oplus V_{-1}$, where V_1 and V_{-1} are the +1 and −1 eigenspaces of A. In addition, both V_1 and V_{-1} have dimension $q / 2 = 2^{4k+2}$.

Now let $\{e_1, \ldots, e_p\}$, p = 8k+5, denote an orthonormal basis of $\mathbb{R}^p$ relative to $<\,,\,>$ and let $z = e_1 \cdot e_2 \cdot \ldots \cdot e_p \in C\ell(p)$. By Lemma 4.6, z commutes with the elements of W, $z^2 = -1$ and z is skew symmetric on $\mathbb{R}^q$. Hence $\mathrm{span}\{z\} = \mathfrak{Z}_W^o$ by 1b) of Proposition 2.4. By 1) of Proposition 4.3 A anticommutes with the elements of W, and hence A anticommutes with z, which is the product of an odd number of elements of W.

Now define $f_i = z\, e_i$ for $1 \le i \le p$. From the remarks above and the facts that $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$ one easily verifies the following :

(*) A commutes with each f_i , $f_i^2 = 1$ and $f_i f_j = -f_j f_i$ for $i \neq j$.

The fact that A commutes with each f_i implies that each f_i leaves invariant the +1 eigenspace V_1 of A. Identifying V_1 with $\mathbb{R}^{2^{4k+2}}$ we obtain from (*) a nontrivial representation of $C\ell^*(8k+5)$ on $\mathbb{R}^{2^{4k+2}}$. However, $C\ell^*(8k+5) \cong \mathbb{H}(2^{4k+1}) \oplus \mathbb{H}(2^{4k+1})$ by Lemma 4.8, and hence $\mathbb{H}^{2^{4k+1}} \cong \mathbb{C}^{2^{4k+2}} \cong \mathbb{R}^{2^{4k+3}}$ is the representation space, up to equivalence, for an irreducible $C\ell^*(8k+5)$ module. In particular, the dimension of an irreducible $C\ell^*(8k+5)$ module is 2^{4k+3}, which contradicts the fact that we constructed a nontrivial $C\ell^*(8k+5)$ module of dimension 2^{4k+2}. This contradiction shows that $\mathfrak{P}_W = \{0\}$ if dim W = p = 8k+5. □

Proof of proposition 4.5 In the case that $W = \mathcal{SU}(2) = \mathcal{SO}(3,\mathbb{R}) \subseteq \mathcal{SO}(q,\mathbb{R})$, where q is even, it is well known that q = 4n + 4 for an integer $n \ge 0$. Moreover, $\mathbb{R}^q$ is the realification of the complex, irreducible $\mathcal{SU}(2)$ module V_{2n+1}, which consists of the complex polynomials of degree 2n+1 that are homogeneous. See, for example, sections 5 and 6 of Chapter II of [BtD].

If $n \ge 1$, then for each nonzero element Z of W, the symmetric element Z^2 has n+1 distinct eigenvalues, each of multiplicity 4 : $-c^2(2n+1)^2, -c^2(2n-1)^2, -c^2(2n-3)^2, \ldots, -c^2$, where c is a positive constant that depends on Z. If n = 0 and q = 4, then $\mathbb{R}^q = \mathbb{H}$ and $\mathcal{SU}(2)$ is the subspace of purely imaginary quaternions acting on $\mathbb{H}$ by left multiplication.

We consider first the case that $n \ge 1$, or equivalently that $q \ge 5$. We assume that $\mathfrak{P}_W$ contains a nonzero element A, and we derive a contradiction. By 2) of Proposition 4.3 we may assume furthermore that $A^2 = \mathrm{Id}$. Note that $W \subseteq \mathfrak{K}_W$ by Proposition 3.1 since W is a semisimple subalgebra of $\mathcal{SO}(q,\mathbb{R})$. Hence $\mathrm{ad}\, A(W) \subseteq [\mathfrak{P}_W, \mathfrak{K}_W] \subseteq \mathfrak{P}_W$ by Proposition 4.2.

We shall need the following

Lemma ad A : $W \to \mathfrak{P}_W$ is a linear isomorphism.

Proof It suffices to show that ad A is injective on W since dim W = 3 and dim $\mathfrak{P}_W \le 3$ by 5) of Proposition 4.3. By the definition of $\mathfrak{P}_W$, A anticommutes with all elements of W. Hence for $Z \in W$ we have ad A (Z) = AZ – ZA = 2 AZ. If $Z \neq 0$, then $AZ \neq 0$ since $A^2 = \mathrm{Id}$. □

To complete the proof of the proposition for the case $n \ge 1$ it suffices to show that Z^2 is a negative multiple of the identity for every nonzero Z in W. By the discussion above this is impossible if $q \ge 5$, for then Z^2 must have at least two distinct eigenvalues.

If Z is any nonzero element of W, then B = ad A(Z) is a nonzero element of $\mathfrak{P}_W$ by the lemma above. Since A anticommutes with the elements of W it follows that B = AZ – ZA = 2 AZ. By 3) of Proposition 4.3 we know that $B^2 = c^2$ Id for some positive number c. However, $B^2 = 4\,AZAZ = 4\,A(-AZ)Z = -4\,Z^2$, and we conclude that $Z^2 = -(c^2/4)$ Id. This contradiction shows that $\mathfrak{P}_W$ contains no nonzero element A.

Finally, we consider the case that n = 0, q = 4, $\mathbb{R}^q = \mathbb{H}$ and $\mathfrak{su}(2)$ is the subspace of purely imaginary quaternions acting on $\mathbb{H}$ by left multiplication. Suppose that A is a nonzero element of $\mathfrak{P}_W$. We may assume that $A^2 = \mathrm{Id}$ by 2) of Proposition 4.3. By 2) of Proposition 4.3 we know that $\mathbb{H} = \mathbb{R}^q = V_1 \oplus V_{-1}$, where V_1 and V_{-1} denote the +1 and –1 eigenspaces of A. In addition, both V_1 and V_{-1} have the same dimension, which in this case is 2.

The elements of W interchange the eigenspaces of A by 1) of Proposition 4.3. Fix a nonzero element v of V_1 and define a linear map $T : W \to V_{-1}$ by T(Z) = Z(v) for all $Z \in W$. The map T must have nonzero kernel since dim W = 3 and dim V_{-1} = 2. Therefore Z(v) = 0 for some nonzero element Z of W, but this contradicts the invertibility of Z and the fact that v is nonzero. The contradiction shows that $\mathfrak{P}_W = \{0\}$ and completes the proof of Proposition 4.5. □

Section 5 Criteria for $\mathfrak{G}_W$ to be self adjoint

We begin with some simple conditions that are equivalent to the self adjointness of $\mathfrak{G}_W$ with respect to the metric transpose on $\mathbb{R}^q$. Because of their roles in (3.4) of [E2] in describing Aut($\mathfrak{N}$), $\mathfrak{N} = \mathbb{R}^q \oplus W$, we also need to consider the group $\overline{G}_W = \{g \in GL(q,\mathbb{R}) : g(W) = W\}$ and its Lie algebra $\overline{\mathfrak{G}}_W = \{X \in \mathrm{End}(\mathbb{R}^q) : X(W) \subseteq W\}$.

Proposition 5.1 The following statements are equivalent

1) $\overline{\mathfrak{G}}_W$ is self adjoint
2) $\mathfrak{G}_W = \overline{\mathfrak{G}}_W \cap \mathfrak{sl}(q,\mathbb{R})$ is self adjoint.
3) $\overline{\mathfrak{G}}_W = \overline{\mathfrak{G}}_{W^\perp}$.
4) $(\overline{G}_W)_o = (\overline{G}_{W^\perp})_o$.

Proof Let $\sigma : GL(q,\mathbb{R}) \to GL(q,\mathbb{R})$ be the involutive automorphism given by $\sigma(g) = (g^t)^{-1}$, where g^t denotes the metric transpose of g on $\mathbb{R}^q$. For $g \in GL(q,\mathbb{R})$ it is easy to see that g^t is also the metric transpose of g on $\mathfrak{so}(q,\mathbb{R})$ with respect to the standard action of End($\mathbb{R}^q$) on $\mathfrak{so}(q,\mathbb{R})$ defined in (3.2) of [E2] and the canonical inner product on $\mathfrak{so}(q,\mathbb{R})$. Hence $\sigma(\overline{G}_W) = \overline{G}_{W^\perp}$, $\sigma(\overline{G}_{W^\perp}) = \overline{G}_W$ and the corresponding statements are true for the identity components. At the Lie algebra level it follows that $d\sigma(\overline{\mathfrak{G}}_W) = \overline{\mathfrak{G}}_{W^\perp}$ and

$d\sigma(\overline{\mathfrak{G}}_{W^\perp}) = \overline{\mathfrak{G}}_W$. The derivative map $d\sigma : \mathrm{End}(\mathbb{R}^q) \to \mathrm{End}(\mathbb{R}^q)$ is given by $d\sigma(X) = -X^t$. Now $\overline{G}_W$ is self adjoint $\Leftrightarrow \sigma(\overline{G}_W) = \overline{G}_W \Leftrightarrow d\sigma(\overline{\mathfrak{G}}_W) = \overline{\mathfrak{G}}_W$. The same argument shows that $\mathfrak{G}_W$ is self adjoint $\Leftrightarrow d\sigma(\mathfrak{G}_W) = \mathfrak{G}_W$. The equivalence of 1) and 2) is evident, and the remaining assertions of the proposition now follow immediately. □

The main goal of this section is the next result.

Proposition 5.2 Let $\mathfrak{N} = \mathbb{R}^q \oplus W$. Then $\mathfrak{G}_W$ is self adjoint if either of the following conditions holds :

a) $W = \mathfrak{G}$, the Lie algebra of a compact connected subgroup of $SO(q,\mathbb{R})$.

b) W is a subspace of $\mathfrak{so}(q,\mathbb{R})$ such that Z^2 is a negative multiple of the identity for each nonzero element Z of W.

Remark Examples of a) arise when G is a compact connected Lie group and $\rho : G \to \mathrm{End}(\mathbb{R}^q)$ is a representation of G. In this case $W = d\rho(\mathfrak{G})$ is the Lie algebra of $\rho(G)$ and W is a subalgebra of $\mathfrak{so}(q,\mathbb{R})$ if $\mathbb{R}^q$ is endowed with a $\rho(G)$-invariant metric.

Examples of b) arise $\Leftrightarrow W = j(\mathbb{R}^p)$, where $j : C\ell(p) \to \mathrm{End}(\mathbb{R}^q)$ is a representation of the real negative definite Clifford algebra $C\ell(p)$determined by $\mathbb{R}^p$. In this case q must be divisible by a certain integer d(p), where $d(8k) = 2^{4k}$, $d(8k+1) = 2^{4k+1}$, $d(8k+2) = d(8k+3) = 2^{4k+2}$ and $d(8k+a) = 2^{4k+3}$ if $4 \le a \le 7$. See the proof of Theorem 2.4 of [E1].

Proof of the proposition We break the proof into several steps listed below. It is clear that the proposition follows immediately from these steps.

Step 1 We extend $\mathfrak{N} = \mathbb{R}^q \oplus W$ to a metric, 3-step solvable Lie algebra $\{\mathfrak{S}, < , >^*\}$, where $\mathfrak{S} = \mathfrak{N} \oplus \mathbb{R}$ with a suitable Lie bracket.

Step 2 The Lie algebra $\{\mathfrak{S}, < , >^*\}$ is Einstein of standard type under either condition a) or b).

Step 3 If a solvable, metric Lie algebra$\{\mathfrak{S}, < , >^*\}$ is Einstein of standard type, then $\overline{\mathfrak{H}} = \{D \in \mathrm{Der}(\mathfrak{S}) : D(\mathbb{R}) = \{0\}\}$ is self adjoint with respect to $< , >^*$.

Step 4 $\overline{\mathfrak{H}}$ is isomorphic to $\overline{\mathfrak{H}}_o = \{D \in \mathrm{Der}(\mathfrak{N}) : D(\mathbb{R}^q) \subseteq \mathbb{R}^q\}$. Moreover, $\overline{\mathfrak{H}}_o$ is the Lie algebra of $\overline{H} = \{\varphi \in \mathrm{Aut}(\mathfrak{N}) : \varphi(\mathbb{R}^q) = \mathbb{R}^q\} = T(\overline{G}_{W^\perp})$ (cf. (3.4)).

Remark We recall from (3.4) of [E2] that $T : (\overline{G}_{W^\perp}) \to \mathrm{Aut}(\mathfrak{N})$ is the injective group homomorphism given by $T(g)(v) = g(v)$ for all $v \in \mathbb{R}^q$ and $T(g)(Z) = \pi(gZg^t)$ for all $Z \in W$, where $\pi : \mathfrak{F}_2(q) \to \mathfrak{N}$ is the projection homomorphism with $\mathrm{Ker}\,\pi = W^\perp$.

Step 5 $\mathfrak{H}$ is self adjoint with respect to the metric transpose on $\{\mathfrak{S}, < , >^*\} \Leftrightarrow \mathfrak{G}_W$ is self adjoint with respect to the metric transpose on $\mathbb{R}^q$.

We now fill in the details of the steps listed above.

Step 1) In either case a) or b) one begins by defining a solvable Lie algebra $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N} = \mathbb{R} \oplus \mathbb{R}^q \oplus W$, where the bracket of $\mathfrak{S}$ extends that of $\mathfrak{N}$ by setting A = 1 and requiring that ad A = Id on $\mathbb{R}^q$ and ad A = 2 Id on W. Let $< , >^*$ be the inner product on $\mathfrak{S}$ such that $< , >^*$ is the standard inner product on $\mathbb{R}$ and $\mathbb{R}^q$, $< \mathbb{R} , \mathfrak{N} >^* = 0$ and $< Z, Z' >^* = -(1/4q)$ trace (ZZ') for all Z,Z' in W. Note that $< , >^* = (1 / 4q) < , >$ on W, where $< , >$ is the canonical inner product on $\mathcal{SO}(q,\mathbb{R})$. We call $\{\mathfrak{S}, < , >^*\}$ the 3-step Carnot solvmanifold determined by $\mathfrak{N}$. See section 3 of [EH] for further discussion of the geometry of 3-step Carnot solvmanifolds,

Remark Let $< , >^*_c$ denote the inner product on $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N} = \mathbb{R} \oplus \mathbb{R}^q \oplus W$ such that $< , >^*_c = < , >^*$ on $\mathbb{R}$, $< , >^*_c = c^2 < , >^*$ on $\mathbb{R}^q$ and $< , >^*_c = < , >^*$ on $\mathfrak{N}$. It is not difficult to show that the solvmanifolds $\{\mathfrak{S}, < , >^*\}$ and $\{\mathfrak{S}, < , >^*_c\}$ are both isometric and isomorphic for any positive constant c. See, for example, the third paragraph of the proof of Proposition 3.21A of [EH].

Step 2) If a) or b) holds, then $\{\mathfrak{S}, < , >^*\}$ defines an Einstein structure on the simply connected solvable Lie group S with Lie algebra $\mathfrak{S}$ that is equipped with the left invariant Riemannian metric induced from the inner product $< , >^*$ on $\mathfrak{S}$. In case a) this follows from Lemma 3.21A and Proposition 3.19 of [EH]. In case b) it follows from Proposition 3.19 of [EH] and the fact that $< Z, Z >^* = 1 \Leftrightarrow Z^2 = -4$ Id for Z in W.

Step 3) The Einstein structure is <u>standard</u> in the terminology of [Heb] ; that is, $[\mathfrak{S}, \mathfrak{S}]^\perp$ is abelian. In this case $[\mathfrak{S}, \mathfrak{S}]^\perp$ is abelian since $[\mathfrak{S}, \mathfrak{S}] = \mathfrak{N}$ and $\mathfrak{N}$ has codimension 1 in $\mathfrak{S}$. It now follows from Theorem B of [Heb] that the subalgebra $\mathfrak{H}$ of derivations of $\mathfrak{S}$ that vanish on $\mathbb{R} = [\mathfrak{S}, \mathfrak{S}]^\perp$ is self adjoint with respect to the inner product $< , >^*$.

Step 4) Note that the elements of Der($\mathfrak{N}$) always leave W invariant since W = $[\mathfrak{N},\mathfrak{N}]$. Hence $\bar{\mathfrak{H}}_o = \{D \in \mathrm{Der}(\mathfrak{N}) : D(\mathbb{R}^q) \subseteq \mathbb{R}^q\}$ is isomorphic to the Lie algebra of $\bar{H} = T(\bar{G}_{W^\perp}) = \{\varphi \in \mathrm{Aut}(\mathfrak{N}) : \varphi(\mathbb{R}^q) = \mathbb{R}^q\}$ by the discussion in (3.4) and Remark 3.4c of [E2].

We now construct an explicit isomorphism between $\bar{\mathfrak{H}} = \{D \in \mathrm{Der}(\mathfrak{S}) : D(\mathbb{R}) = \{0\}\}$ and $\bar{\mathfrak{H}}_o$.

Lemma 5.3 Let $\bar{D}$ be an element of Der($\mathfrak{S}$) such that $\bar{D}(\mathbb{R}) = \{0\}$, and let D denote the restriction of $\bar{D}$ to $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$. Then D is an element of Der($\mathfrak{N}$) such that $D(\mathbb{R}^q) \subseteq \mathbb{R}^q$. Conversely, let D be an element of Der($\mathfrak{N}$) such that $D(\mathbb{R}^q) \subseteq \mathbb{R}^q$. If we define $\bar{D} \in$ End($\mathfrak{S}$) by $\bar{D} = D$ on $\mathfrak{N}$ and $\bar{D}(\mathbb{R}) = \{0\}$, then $\bar{D} \in \mathrm{Der}(\mathfrak{S})$.

Proof Let $\bar{D}$ be an element of Der($\mathfrak{S}$) such that $\bar{D}(\mathbb{R}) = \{0\}$, and let D denote the restriction of $\bar{D}$ to $\mathfrak{N}$. Define $C_1 \in \mathrm{End}(\mathbb{R}^q)$ and $C_2 \in \mathrm{Hom}(\mathbb{R}^q, W)$ by setting D(v) =

$C_1(v) + C_2(v)$ for all $v \in \mathbb{R}^q$. If we let A denote 1 in $\mathbb{R} \subseteq \mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}$, then ad A = Id on $\mathbb{R}^q$ and ad A = 2 Id on W by the definition of $\mathfrak{S}$. For $v \in \mathbb{R}^q$ it follows that $\bar{D}[A,v] = \bar{D}(v) = D(v) = C_1(v) + C_2(v)$. However, since $\bar{D}$ is a derivation of $\mathfrak{S}$ we also have $\bar{D}[A,v] = [\bar{D}(A),v] + [A, \bar{D}(v)] = [A, D(v)] = [A, C_1(v)] + [A, C_2(v)] = C_1(v) + 2\,C_2(v)$. It follows that $C_2 \equiv 0$ and $D(\mathbb{R}^q) \subseteq \mathbb{R}^q$. Conversely, if $D(\mathbb{R}^q) \subseteq \mathbb{R}^q$ for some $D \in \mathrm{Der}(\mathfrak{N})$, then a similar argument shows that if D is extended to an element $\bar{D}$ of End($\mathfrak{S}$) such that $\bar{D}(\mathbb{R}) = \{0\}$, then $\bar{D} \in \mathrm{Der}(\mathfrak{S})$.□

Step 5) The assertion of step 5 follows immediately from Proposition 5.1 and the equivalence of 1) and 3) in the next result.

Lemma 5.4 Let $\{\mathfrak{N} = \mathbb{R}^q \oplus W, < , >\}$ be a standard, metric 2-step nilpotent Lie algebra of type (p,q), and let $\{\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}, < , >^*\}$ be the corresponding 3-step Carnot solvmanifold. Let $\bar{\mathfrak{H}} = \{ \bar{D} \in \mathrm{Der}(\mathfrak{S}) ; \bar{D}(\mathbb{R}) = \{0\}\}$. Let $\bar{H} = T(\bar{G}_{W^\perp}) = \{\varphi \in \mathrm{Aut}(\mathfrak{N}) : \varphi(\mathbb{R}^q) = \mathbb{R}^q\}$. Then the following assertions are equivalent :

1) $\bar{\mathfrak{H}}$ is self adjoint with respect to the metric transpose on $\{\mathfrak{S}, < , >^*\}$.

2) $\bar{H}_o$ is self adjoint with respect to the metric transpose on $\{\mathfrak{N}, < , >\}$.

3) $(\bar{G}_W)_o$ is self adjoint with respect to the metric transpose on $\mathbb{R}^q$.

Proof 1) ⇒ 2). Let $\bar{\mathfrak{H}}_o = L(\bar{H}_o) = \{ D \in \mathrm{Der}(\mathfrak{N}) : D(\mathbb{R}^q) \subseteq \mathbb{R}^q\}$. Relative to the metric transpose on $\{\mathfrak{N}, < , >\}$ it is evident that $\bar{H}_o$ is self adjoint ⇔ $\bar{\mathfrak{H}}_o$ is selfadjoint. We show that $\bar{\mathfrak{H}}_o$ is selfadjoint. Let $D \in \bar{\mathfrak{H}}_o$ be given and let $\bar{D}$ denote its extension to an element of $\bar{\mathfrak{H}}$. If $\bar{E}$ denotes the metric adjoint of $\bar{D}$ in End($\mathfrak{S}$), then $\bar{E} \in \bar{\mathfrak{H}}$ by 1). Hence the restriction E of $\bar{E}$ to $\mathfrak{N}$ is an element of $\bar{\mathfrak{H}}_o$ by Lemma 5.3.

By definition E is the metric adjoint of D on $\mathfrak{N}$ relative to $< , >^*$. Moreover, $E(\mathbb{R}^q) \subseteq \mathbb{R}^q$ and $E(W) \subseteq W$ since D has these properties. It follows that E is the metric adjoint of D on $\mathfrak{N}$ relative to the canonical inner product $< , >$ since $< , >^* = < , >$ on $\mathbb{R}^q$, and $< , >^* = (1 / 4q) < , >$ on W.

The arguments above show that $\bar{\mathfrak{H}}_o$ is self adjoint relative to $< , >$ if $\bar{\mathfrak{H}}$ is self adjoint relative to $< , >^*$. This proves 1) ⇒ 2) and reversing the argument shows that 2) ⇒ 1).

We prove 2) ⇒ 3). Let $g \in (\bar{G}_W)_o$ be given. Then $g^t \in (\bar{G}_{W^\perp})_o$ and $T(g^t) \in \bar{H}_o$ by Proposition 3.4b of [E2]. By 2) there exists $g' \in (\bar{G}_{W^\perp})_o$ such that $T(g') = T(g^t)^*$, the metric adjoint of $T(g^t)$ in $\mathfrak{N}$. It follows that $g = g' \in (\bar{G}_{W^\perp})_o$ since $< g'(v), w > = < T(g')(v), w > = < v, T(g^t)(w) > = < v, g^t(w) > = < g(v), w >$ for all v,w in $\mathbb{R}^q$. Since $g \in (\bar{G}_{W^\perp})_o$ it follows that $g^t \in (\bar{G}_W)_o$. Hence $(g^t)^{-1} = \sigma(g) \in (\bar{G}_W)_o$, which proves that $\bar{G}_W)_o$ is self adjoint.

We prove 3) ⇒ 2). Let $h = T(g) \in \bar{H}_o$ for some $g \in (\bar{G}_W)_o$. By hypothesis $(\bar{G}_W)_o = (\bar{G}_{W^\perp})_o$ and $g^t \in (\bar{G}_W)_o$. Hence it suffices to show that $T(g^t) = T(g)^*$. If v, v' are any elements of $\mathbb{R}^q$ and Z,Z' are any elements of W, then $< T(g)^*(v + Z), v' + Z' > =$

$< v, g(v') > + < Z, gZ'g^t > = < v, g(v') > - \text{trace}(ZgZ'g^t) = < g^t(v), v' > - \text{trace}(g^t ZgZ') =$
$< g^t(v), v' > + < g^t Zg, Z' > = < T(g^t)(v + Z), v' + Z' >.$ □

Remark For the discussion in the remainder of this article we consider stabilizers in $SL(q,\mathbb{R})$ rather than $GL(q,\mathbb{R})$. For a p-dimensional subspace W of $\mathcal{SO}(q,\mathbb{R})$ $G_W = \{g \in SL(q,\mathbb{R}) : g(W) = W\}$, where $g(W) = gWg^t$. As usual, $\mathfrak{G}_W$ denotes the Lie algebra of G_W.

Section 6 Relation between self adjoint and reductive groups G_W

Definition 6.1 A Lie group G is <u>reductive</u> if the Lie algebra $\mathfrak{G}$ is the direct sum of the center $\mathfrak{Z}$ and a semisimple subalgebra $\mathfrak{G}'$.

If a Lie group $G \subseteq GL(q,\mathbb{R})$ is self adjoint, then it is easy to see that G is reductive. We omit the details. Clearly, if G is reductive, then any conjugate group $G' = gGg^{-1}$ is also reductive since G' is isomorphic to G for any $g \in GL(q,\mathbb{R})$. Mostow in [Mo] has shown a converse, stated below.

Proposition 6.2 Let $\mathfrak{N} = \mathbb{R}^q \oplus W$, and suppose that $G_W \subseteq SL(q,\mathbb{R})$ is reductive. Then there exists $g \in GL(q,\mathbb{R})$ such that $gG_Wg^{-1} = G_{g(W)} \subseteq SL(q,\mathbb{R})$ is self adjoint with respect to the metric transpose on $\mathbb{R}^q$.

Proof By hypothesis the real algebraic group G_W is reductive. The main theorem of [Mo] says that there exists an inner product $< , >^*$ on $\mathbb{R}^q$ such that G_W is self adjoint with respect to the metric transpose of $< , >^*$. Let g be the element of $GL(q,\mathbb{R})$ such that $< v, w >^* = < g(v), g(w) >$ for all v,w in $\mathbb{R}^q$, where $< , >$ denotes the standard inner product on $\mathbb{R}^q$. Then $G_{g(W)} = g\, G_W\, g^{-1}$ is self adjoint with respect to $< , >$. □

Remark The standard metric 2-step nilpotent Lie algebras $\mathfrak{N} = \mathbb{R}^q \oplus W$ and $\mathfrak{N}' = \mathbb{R}^q \oplus g(W)$ are isomorphic by Proposition 3.1 of [E3]. One effect of the result above is that in some cases the purely algebraic condition of G_W being reductive is just as satisfactory as the metric condition of G_W being self adjoint.

In the proposition of (4.2) in [E2] we showed that the isotropy group G_W is isomorphic to the isotropy group $\{SL(q,\mathbb{R}) \times GL(p,\mathbb{R})\}_S$ on the real vector space $V = \mathcal{SO}(q,\mathbb{R})^p$, where $S = (S_1, \ldots, S_p) \in V$ is a point such that $\{S_1, \ldots, S_p\}$ is a basis of W. Hence we can pass from the problem of determining the isotropy groups of a nonlinear action of $SL(q,\mathbb{R})$ on the Grassmann manifold $G(p, \mathcal{SO}(q,\mathbb{R}))$ to the more tractable problem of computing the isotropy groups of a linear action of $SL(q,\mathbb{R}) \times GL(p,\mathbb{R})$ on a vector space V.

The result above becomes applicable if we can find conditions under which $\{SL(q,\mathbb{R}) \times GL(p,\mathbb{R})\}_S$ is reductive.

Section 7 Structure of self adjoint Lie algebras $\mathfrak{G}_W$

We begin with some general results about the structure of self adjoint Lie algebras $\mathfrak{G}_W$.

A distinguished ideal

Proposition 7.1 Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ and suppose that $\mathfrak{G}_W$ is self adjoint. Write $\mathfrak{G}_W = \mathfrak{K}_W \oplus \mathfrak{P}_W$, where $\mathfrak{K}_W$ and $\mathfrak{P}_W$ denote the skew symmetric and symmetric elements of $\mathfrak{G}_W$ respectively. Then

1) $[\mathfrak{P}_W, \mathfrak{P}_W]$ is an ideal in $\mathfrak{K}_W$.

2) $\mathfrak{H}_W = [\mathfrak{P}_W, \mathfrak{P}_W] \oplus \mathfrak{P}_W$ is an ideal in $\mathfrak{G}_W$.

Proof 1) Clearly $[\mathfrak{P}_W, \mathfrak{P}_W] \subseteq \mathfrak{K}_W$ since the bracket of two symmetric transformations is skew symmetric. The subspace $[\mathfrak{P}_W, \mathfrak{P}_W]$ is invariant under ad ξ for $\xi \in \mathfrak{K}_W$ by Proposition 4.2 and the Jacobi identity.

2) Note that $\mathfrak{H}_W$ is invariant under ad ξ for $\xi \in \mathfrak{K}_W$ by Proposition 4.2 and the Jacobi identity. It follows from 1) and Proposition 4.2 that $\mathfrak{H}_W$ is invariant under ad ξ for $\xi \in \mathfrak{P}_W$. Hence $\mathfrak{H}_W$ is invariant under ad ξ for $\xi \in \mathfrak{K}_W \oplus \mathfrak{P}_W = \mathfrak{G}_W$.

Examples

We give some examples of the ideal $[\mathfrak{P}_W, \mathfrak{P}_W]$ in $\mathfrak{S}_W$ and the ideal $\mathfrak{H}_W$ in $\mathfrak{G}_W$.

1) Subspaces $W \subseteq \mathfrak{so}(q,\mathbb{R})$ of Clifford type acting irreducibly on $\mathbb{R}^q$

$[\mathfrak{P}_W, \mathfrak{P}_W] = \{0\}$	if p = 4k, 4k+3 or 8k+6
$[\mathfrak{P}_W, \mathfrak{P}_W] = \mathfrak{Z}^o_W \cong \mathfrak{so}(3,\mathbb{R})$	if p = 8k+2
$[\mathfrak{P}_W, \mathfrak{P}_W]$ is 1-dimensional	if p = 8k+1
$[\mathfrak{P}_W, \mathfrak{P}_W] = \{0\}$	if p = 8k+5

$\mathfrak{H}_W = \{0\}$	if p = 4k+3 or 8k+6
$\mathfrak{H}_W = \mathfrak{P}_W$ is the 1-dimensional center of $\mathfrak{G}_W$	if p = 4k
$\mathfrak{H}_W \cong \mathfrak{so}(3,1)$	if p = 8k+2
$\mathfrak{H}_W \cong \mathfrak{so}(2,1)$	if p = 8k+1
$\mathfrak{H}_W = \{0\}$	if p = 8k+5

2) $W = \mathfrak{so}(3,\mathbb{R}) \subseteq \mathfrak{so}(q,\mathbb{R})$, where q is even and W acts irreducibly on $\mathbb{R}^q$.

$\mathfrak{H}_W = \{0\}$

3) Subalgebras $W \subseteq \mathfrak{so}(q,\mathbb{R})$ acting irreducibly on $\mathbb{R}^q$.

In this situation we have the following partial result.

Proposition 7.2 Let $W \subseteq \mathfrak{so}(q,\mathbb{R})$ be a subalgebra acting irreducibly on $\mathbb{R}^q$. Then either

a) $[\mathfrak{P}_W, \mathfrak{P}_W] \subseteq \mathfrak{Z}_W^o$

or b) $[\mathfrak{P}_W, \mathfrak{P}_W] \cap W$ is a nonzero ideal in W.

Remarks i) $\dim \mathfrak{Z}_W^o \leq 3$ by Proposition 6.1c

ii) If W is simple, then either $[\mathfrak{P}_W, \mathfrak{P}_W] \subseteq \mathfrak{Z}_W^o$ or $W \subseteq [\mathfrak{P}_W, \mathfrak{P}_W]$ by b).

Proof of the examples The proof of 2) follows immediately from Proposition 4.5. To justify the assertions in 1) we use the results of Proposition 4.4. The assertions for p = 4k+3 or 8k+6 follow immediately from that result.

If p = 4k, then $\dim \mathfrak{P}_W = 1$ by a) of Proposition 4.4, and the proof of that result shows that $\mathfrak{P}_W = \text{span}\{z\}$, where $z^2 = \text{Id}$, z is a product of elements in W and z anticommutes with the elements in W. Hence $[\mathfrak{P}_W, \mathfrak{P}_W] = \{0\}$ and $\mathfrak{H}_W = \mathfrak{P}_W = \text{span}\{z\}$. Now z commutes with $\mathfrak{Z}_W^o$ since z is a product of elements in W. Moreover, z commutes with the elements in [W,W] since z anticommutes with the elements in W. It follows that z commmutes with $\mathfrak{K}_W = \mathfrak{Z}_W^o \oplus [W,W]$ by Proposition 3.2. Hence z is a central element of $\mathfrak{G}_W = \mathfrak{K}_W \oplus \mathfrak{P}_W$. Conversely, the center of $\mathfrak{G}_W$ must lie in $\mathfrak{Z}_W^o \oplus \mathfrak{P}_W$ since $\mathfrak{P}_W = \text{span}\{z\}$ lies in the center of $\mathfrak{G}_W$ and $[W,W] \cong \mathfrak{so}(p,\mathbb{R})$ has trivial center. However, by Proposition 3.4 $\mathfrak{Z}_W^o = \{0\}$ if p = 8k and $\mathfrak{Z}_W^o$ is isomorphic to the imaginary quaternions and has trivial center if p = 8k+4. In either case it follows that $\mathfrak{P}_W$ is the 1-dimensional center of $\mathfrak{G}_W$.

If p = 8k+2, then by b) of Proposition 4.4 and the proof of b) in that result we know that $\dim \mathfrak{P}_W = 3$ and $[\mathfrak{P}_W, \mathfrak{P}_W] = \text{span}\{I, J, K\} = \mathfrak{Z}_W^o \cong \mathfrak{so}(3,\mathbb{R})$, where {I, J, K} satisfy the standard quaternion relations. By consulting the list of É. Cartan (cf. [Hel, pp.444-455 and 515-520]) it becomes clear that $\mathfrak{H}_W = [\mathfrak{P}_W, \mathfrak{P}_W] \oplus \mathfrak{P}_W = \mathfrak{so}(3,1)$, the Lie algebra of SO(3,1), which is the identity component of the isometry group of real hyperbolic 3-space.

If p = 8k+1, then $\dim \mathfrak{P}_W = 2$ by e) of Proposition 4.4. By the remark in the proof of e), 2) of Lemma 4.6 and Proposition 2.4 it follows that $[\mathfrak{P}_W, \mathfrak{P}_W] = \text{span}\{z\} = \mathfrak{Z}_W^o$, where $z^2 = -\text{Id}$. From the list of É. Cartan it now becomes clear that $\mathfrak{H}_W = [\mathfrak{P}_W, \mathfrak{P}_W] \oplus \mathfrak{P}_W = \mathfrak{so}(2,1)$, the Lie algebra of SO(2,1), which is the identity component of the isometry group of real hyperbolic 2-space.

If p = 8k+5, then $\mathfrak{H}_W = \{0\}$ since $\mathfrak{P}_W = \{0\}$ by f) of Proposition 4.4. This completes the discussion of the examples W of Clifford type.

We prove Proposition 7.2. By Propositions 7.1 and 3.1 $\mathfrak{K}'_W = [\mathfrak{P}_W, \mathfrak{P}_W]$ is an ideal in $\mathfrak{K}_W = \mathfrak{Z}^o_W \oplus W$. Let ξ be any element of $\mathfrak{K}'_W \subseteq \mathfrak{K}_W$ and write $\xi = z + w$, where $z \in \mathfrak{Z}^o_W$ and $w \in W$. Now ad $W(\xi) \subseteq \mathfrak{K}'_W$ since $W \subseteq \mathfrak{K}_W$. On the other hand, ad $W(\xi) =$ ad $W(w) \subseteq [W, W] \subseteq W$. This proves that ad $W(\mathfrak{K}'_W \cap W) \subseteq$ ad $W(\mathfrak{K}'_W) \subseteq \mathfrak{K}'_W \cap W$; that is, $\mathfrak{K}'_W \cap W$ is an ideal in W.

If $\mathfrak{K}'_W \cap W = \{0\}$, then ad $W(\xi) = 0$ for all $\xi \in \mathfrak{K}'_W$, and by definition this means that $\mathfrak{K}'_W \subseteq \mathfrak{Z}^o_W$, which completes the proof of Proposition 7.2.

The center $\mathfrak{Z}(\mathfrak{G}_W)$ of $\mathfrak{G}_W$

If $\mathfrak{G}_W$ is self adjoint, then its center $\mathfrak{Z}(\mathfrak{G}_W)$ is also self adjoint, and we may write $\mathfrak{Z}(\mathfrak{G}_W) = (\mathfrak{Z}(\mathfrak{G}_W) \cap \mathfrak{K}_W) \oplus (\mathfrak{Z}(\mathfrak{G}_W) \cap \mathfrak{P}_W)$. We have seen that $\mathfrak{Z}(\mathfrak{G}_W) \cap \mathfrak{P}_W$ may be nonzero, which happens, for example, if W is of Clifford type with dimension 4k. However, this does not occur if W is a subalgebra of $\mathfrak{so}(q,\mathbb{R})$ that acts irreducibly on $\mathbb{R}^q$.

Proposition 7.3 Let W be a subalgebra of $\mathfrak{so}(q,\mathbb{R})$ that acts irreducibly on $\mathbb{R}^q$. Then $\mathfrak{Z}(\mathfrak{G}_W) \subseteq (\text{center of } \mathfrak{Z}^o_W) \oplus (\text{center of W}) \subseteq \mathfrak{so}(q,\mathbb{R})$.

Remark The center of $\mathfrak{Z}^o_W$ has dimension at most 1 by Proposition 1.3. If W is semisimple, then W has trivial center. Hence dim $\mathfrak{Z}(\mathfrak{G}_W) \leq 1$ if W is semisimple.

Proof of the proposition We show first that $\mathfrak{Z}(\mathfrak{G}_W) \cap \mathfrak{P}_W = \{0\}$. If $A \in \mathfrak{Z}(\mathfrak{G}_W) \cap \mathfrak{P}_W$, then A commutes with W since $W \subseteq \mathfrak{K}_W \subseteq \mathfrak{G}_W$. Since W leaves invariant each eigenspace of A and W acts irreducibly on $\mathbb{R}^q$ it follows that $A = \lambda$ Id for some real number λ. Finally, $\lambda = 0$ since the elements of $\mathfrak{P}_W$ have trace zero.

From the previous paragraph and Proposition 3.1 we know that $\mathfrak{Z}(\mathfrak{G}_W) \subseteq \mathfrak{G}_W \cap \mathfrak{so}(q,\mathbb{R}) = \mathfrak{K}_W = \mathfrak{Z}^o_W \oplus W$. Hence $\mathfrak{Z}(\mathfrak{G}_W) \subseteq$ center $(\mathfrak{Z}^o_W \oplus W) = (\text{center of } \mathfrak{Z}^o_W) \oplus$ (center of W).□

The structure of $\mathfrak{G}_W$ and its center when W has Clifford type

We first derive a general result about the structure of $\mathfrak{G}_W$ when W has Clifford type.

Proposition 7.4 Let $W \subseteq \mathfrak{so}(q,\mathbb{R})$ be a p-dimensional subspace of Clifford type that acts irreducibly on $\mathbb{R}^q$. Then

(*) $\mathfrak{G}_W = \mathfrak{Z}^o_W \oplus \mathfrak{P}_W \oplus [W, W]$ (direct sum)

and the subspaces $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ and [W, W] are ideals in $\mathfrak{G}_W$. Moreover, [W, W] is isomorphic to $\mathfrak{so}(p,\mathbb{R})$.

Proof By Proposition 5.2 $\mathfrak{G}_W$ is self adjoint, and hence $\mathfrak{G}_W = \mathfrak{K}_W \oplus \mathfrak{P}_W$. By the remark following (2.1) W is a Lie triple system with trivial center in $\mathfrak{so}(q,\mathbb{R})$. Hence $\mathfrak{K}_W = \mathfrak{Z}^o_W \oplus [W, W]$ by Proposition 3.2. This proves (*).

Next we show that the elements of $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ commute with the elements of [W, W]. By definition $\mathfrak{Z}^o_W$ commutes with W, and hence $\mathfrak{Z}^o_W$ also commutes with [W, W]. By 1) of Proposition 4.3 $\mathfrak{P}_W$ anticommutes with W, and hence $\mathfrak{P}_W$ commutes with [W, W]. We conclude that $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ commutes with [W, W].

From the theory of Clifford algebras it follows that [W, W] is a subalgebra of $\mathfrak{K}_W$ isomorphic to $\mathfrak{so}(p,\mathbb{R})$, where p = dim W. See, for example, Lemma 2 and Proposition 3 in section 3 of [E2] for a proof. If $\mathfrak{C} = \{\xi \in \mathfrak{G}_W : \mathrm{ad}\,\xi([W, W]) = \{0\}\}$, then $\mathfrak{C}$ is a subalgebra of $\mathfrak{G}_W$ that contains $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W$. Equality holds by (*) since the semisimple subalgebra [W, W] has trivial center. In particular, $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W = \mathfrak{C}$ is a subalgebra of $\mathfrak{G}_W$.

Since $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ and [W, W] are commuting subalgebras of $\mathfrak{G}_W$ whose direct sum is $\mathfrak{G}_W$ it follows that $\mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ and [W, W] are ideals in $\mathfrak{G}_W$. This completes the proof of Proposition 7.4. □

We now determine the center $\mathfrak{Z}(\mathfrak{G}_W)$ of $\mathfrak{G}_W$ in the case that W has Clifford type. It is simplest just to list the various possibilities for $\mathfrak{G}_W = \mathfrak{K}_W \oplus \mathfrak{P}_W$, using the result above and information previously obtained about the structure of $\mathfrak{K}_W$ and $\mathfrak{P}_W$. One can then easily determine the center $\mathfrak{Z}(\mathfrak{G}_W)$.

Description of $\mathfrak{G}_W$

It follows from the list below that $\mathfrak{G}_W$ is semisimple and has trivial center unless p = 4k or 8k+5, in which cases $\mathfrak{G}_W$ has 1-dimensional center. In the list below all of the direct summands commute with each other.

If p = 8k	$\mathfrak{G}_W \cong \mathbb{R} \oplus \mathfrak{so}(8k,\mathbb{R})$
If p = 8k+1	$\mathfrak{G}_W \cong \mathfrak{so}(2,1) \oplus \mathfrak{so}(8k+1,\mathbb{R})$
If p = 8k+2	$\mathfrak{G}_W \cong \mathfrak{so}(3,1) \oplus \mathfrak{so}(8k+2,\mathbb{R})$
If p = 8k+3	$\mathfrak{G}_W \cong \mathfrak{so}(3,\mathbb{R}) \oplus \mathfrak{so}(8k+3,\mathbb{R})$
If p = 8k+4	$\mathfrak{G}_W \cong \mathbb{R} \oplus \mathfrak{so}(3,\mathbb{R}) \oplus \mathfrak{so}(8k+4,\mathbb{R})$
If p = 8k+5	$\mathfrak{G}_W \cong \mathbb{R} \oplus \mathfrak{so}(8k+5,\mathbb{R})$
If p = 8k+6	$\mathfrak{G}_W \cong \mathfrak{so}(8k+6,\mathbb{R})$
If p = 8k+7	$\mathfrak{G}_W \cong \mathfrak{so}(8k+7,\mathbb{R})$

To verify the details of the assertions above one uses Proposition 7.4 , which asserts that $\mathfrak{G}_W = \mathfrak{G}_1 \oplus \mathfrak{G}_2$, where $\mathfrak{G}_1 = \mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ and $\mathfrak{G}_2 = [W,W] \cong \mathfrak{so}(p,\mathbb{R})$ are ideals in $\mathfrak{G}_W$. We now use Propositions 2.4 and 4.4 to evaluate $\mathfrak{Z}^o_W$, $\mathfrak{P}_W$ and $\mathfrak{G}_1 = \mathfrak{Z}^o_W \oplus \mathfrak{P}_W$.

We also recall from the examples following Proposition 7.1 that if p = 4k, then $\mathfrak{P}_W$ is the 1-dimensional center of $\mathfrak{G}_W$. The assertions above now follow immediately except in the cases p = 8k+1 or 8k+2.

If p = 8k+1, then $\mathfrak{G}_1 = \mathfrak{Z}^o_W \oplus \mathfrak{P}_W$ = span{I} $\oplus\, \mathfrak{P}_W \cong \mathcal{SO}(2,1)$ by the proof of Proposition 7.1 in this case.

If p = 8k+2, then $\mathfrak{G}_1 = \mathfrak{Z}^o_W \oplus \mathfrak{P}_W \cong \mathcal{SO}(3,1)$ by the proof of Proposition 7.1 in this case.□

Consequences of the condition that $\mathfrak{Z}(\mathfrak{G}_W) \subseteq \mathcal{SO}(q,\mathbb{R})$

Before stating the next result we recall some notation. Let $\rho : GL(q,\mathbb{R}) \rightarrow GL(\mathcal{SO}(q,\mathbb{R}))$ be the Lie group homomorphism such that $\rho(g)(Z) = gZg^t$ for all $g \in GL(q,\mathbb{R})$ and all $Z \in \mathcal{SO}(q,\mathbb{R})$. Let $d\rho : End(\mathbb{R}^q) \rightarrow End(\mathcal{SO}(q,\mathbb{R}))$ be the induced Lie algebra homomorphism given by $d\rho(X)(Z) = XZ + ZX^t$ for all $X \in End(\mathbb{R}^q)$ and all $Z \in \mathcal{SO}(q,\mathbb{R})$.

Proposition 7.5 Let W be a subpace of $\mathcal{SO}(q,\mathbb{R})$ such that $\mathfrak{G}_W$ is self adjoint and $\mathfrak{Z}(\mathfrak{G}_W) \subseteq \mathcal{SO}(q,\mathbb{R})$. Then

1) The elements of $d\rho(\mathfrak{G}_W)$ have trace zero as elements of End(W).
2) $\rho(G_W)_o \subseteq SL(W)$; that is, each element of $\rho(G_W)_o$ has determinant 1 in W.
3) $[\mathfrak{K}_W , \mathfrak{P}_W] = \mathfrak{P}_W$.

Proof We observe that $\mathfrak{G}_W$ is reductive since it is self adjoint , and it follows that $\mathfrak{G}_W = \mathfrak{Z}(\mathfrak{G}_W) \oplus [\mathfrak{G}_W , \mathfrak{G}_W]$. Hence $d\rho(\mathfrak{G}_W) = d\rho(\mathfrak{Z}(\mathfrak{G}_W)) + [d\rho(\mathfrak{G}_W) , d\rho(\mathfrak{G}_W)]$. The elements of $d\rho(\mathfrak{Z}(\mathfrak{G}_W))$ act skew symmetrically on W by (3.2) of [E2] since $\mathfrak{Z}(\mathfrak{G}_W) \subseteq \mathcal{SO}(q,\mathbb{R})$. In particular the elements of $d\rho(\mathfrak{Z}(\mathfrak{G}_W))$ have trace zero on W. Clearly, every element of $[d\rho(\mathfrak{G}_W , d\rho(\mathfrak{G}_W]$ has trace zero in W, which proves 1). Assertion 2) follows immediately from 1).

We prove 3). We know that $[\mathfrak{K}_W , \mathfrak{P}_W] \subseteq \mathfrak{P}_W$ by Proposition 4.2, so we must prove the reverse inclusion. Note that $\mathfrak{G}_W \subseteq \mathcal{SO}(q,\mathbb{R}) + [\mathfrak{G}_W , \mathfrak{G}_W]$ by the discussion above, and $[\mathfrak{G}_W , \mathfrak{G}_W] = [\mathfrak{K}_W , \mathfrak{K}_W] + [\mathfrak{P}_W , \mathfrak{P}_W] + [\mathfrak{K}_W , \mathfrak{P}_W] \subseteq \mathcal{SO}(q,\mathbb{R}) + [\mathfrak{K}_W , \mathfrak{P}_W]$. We conclude that $\mathfrak{P}_W \subseteq \mathfrak{G}_W \subseteq \mathcal{SO}(q,\mathbb{R}) + [\mathfrak{K}_W , \mathfrak{P}_W]$. Since $[\mathfrak{K}_W , \mathfrak{P}_W] \subseteq \mathfrak{P}_W$ and $\mathcal{SO}(q,\mathbb{R}) \cap \mathfrak{P}_W = \{0\}$ it follows that $\mathfrak{P}_W \subseteq [\mathfrak{K}_W , \mathfrak{P}_W]$.□

Section 8 Consequences of the self adjointness of $\mathfrak{G}_W$

Structure of Aut($\mathfrak{N}$) and Der($\mathfrak{N}$)

We return to a consideration of the full stabilizer $\bar{G}_W$ of W in GL(q,$\mathbb{R}$). If $(\bar{G}_W)_o \subseteq$ GL(q,$\mathbb{R}$) is self adjoint, then we can sharpen the statements of the canonical decompositions of Aut($\mathfrak{N}$)$_o$ and Der($\mathfrak{N}$) in (3.4) of [E2]. In the next result we suppress the notation ρ and dρ used above for the actions of GL(q,$\mathbb{R}$) and End($\mathbb{R}^q$) on $\mathcal{SO}$(q,$\mathbb{R}$).

Proposition 8.1 Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ and suppose that $(\bar{G}_W)_o$ is self adjoint.

a) Let $\varphi \in$ Aut($\mathfrak{N}$)$_o$. Then there exist unique elements g of GL(q,$\mathbb{R}$) and A of Hom($\mathbb{R}^q$,W) such that

1) g(W) ⊆ W and det g > 0 on $\mathbb{R}^q$ where g ∈ GL($\mathcal{SO}$(q,$\mathbb{R}$))
2) $\varphi(v) = g(v) + A(g(v))$ for all v ∈ $\mathbb{R}^q$.
$\varphi(Z) = gZg^t$ for all Z ∈ W.

Conversely, every element φ of GL($\mathfrak{N}$) that has the form above lies in Aut($\mathfrak{N}$)$_o$.

b) Let D be a derivation of $\mathfrak{N}$. Then there exist unique elements A of End($\mathbb{R}^q$) and B of Hom($\mathbb{R}^q$,W) such that

1) A(W) ⊆ W where A ∈ End($\mathcal{SO}$(q,$\mathbb{R}$))
2) D(v) = A(v) + B(v) for all v ∈ $\mathbb{R}^q$.
$D(Z) = AZ + ZA^t$ for all Z ∈ W.

Conversely, every element D of End($\mathfrak{N}$) that has the form above is a derivation of $\mathfrak{N}$.

Proof For an arbitrary subspace W of $\mathcal{SO}$(q,$\mathbb{R}$) an element g of GL(q,$\mathbb{R}$) lies in $(\bar{G}_W)_o \Leftrightarrow g^t$ lies in $(\bar{G}_{W^\perp})_o$. Hence when $(\bar{G}_W)_o$ is self adjoint it follows that $(\bar{G}_W)_o = (\bar{G}_{W^\perp})_o$. This means that in the canonical decompositions of Aut($\mathfrak{N}$) and Der($\mathfrak{N}$), described above in Propositions 3.4b, 3.4e of [E2] and their proofs, we may omit the projection homomorphism $\pi : \mathfrak{F}_2(q) \to \mathfrak{N}$. Proposition 3.4b of [E2] shows that Aut($\mathfrak{N}$)$_o = U \cdot \bar{H}_o$ (semidirect product), where $\bar{H}_o = T((\bar{G}_{W^\perp})_o) = T((\bar{G}_W)_o)$. The result now follows immediately.□

Compatibility of self adjointness with the KP decomposition

The next result (Proposition 8.2) is the group analogue of the fact that if $\bar{\mathfrak{G}}_W$ is self adjoint, then $\bar{\mathfrak{G}}_W$ contains the symmetric and skew symmetric parts of each of its elements. The last result in this section (Proposition 8.4) is the group analogue of Proposition 4.3.

Proposition 8.2 Let $\mathfrak{N} = \mathbb{R}^q \oplus W$ and suppose that $(\bar{G}_W)_o$ is self adjoint. For every g ∈ $(\bar{G}_W)_o$ let k,p be the unique elements of GL(q,$\mathbb{R}$) such that g = kp, where k ∈ O(n,$\mathbb{R}$) and p is a positive definite symmetric element of GL(q,$\mathbb{R}$). Then k ∈ $(\bar{G}_W)_o$ and p ∈ $(\bar{G}_W)_o$.

Proof We shall need the following elementary observation

Lemma 8.3 Let V be a real, finite dimensional inner product space, and let h be a positive definite, symmetric linear transformation on V. Then

1) h and h^2 have the same eigenvectors in V.

2) If W is a subspace of V such that $h^2(W) = W$, then $h(W) = W$.

Proof of the lemma Clearly any eigenvector of h is an eigenvector of h^2. If $h^2(v) = \lambda v$ for some positive number λ, then write $v = v_1 + ... + v_N$ where $h(v_i) = \lambda_i v_i$ for real numbers $\{\lambda_1, ... \lambda_N\}$. It follows that v is the sum of those v_i such that $\lambda_i = \sqrt{\lambda}$. This proves 1).

If W is a subspace of V invariant under h^2, then W admits an orthonormal basis $\{w_1, ... , w_p\}$ of eigenvectors of h^2. By 1) this is also a basis of eigenvectors of h, which proves 2).

Proof of the proposition Let g be an element of $(\bar{G}_W)_o$ and write g = kp as above. Then $g^t g = p^2$, and $p^2 \in (\bar{G}_W)_o$ since $(\bar{G}_W)_o$ is self adjoint. Since p and p^2 are positive definite and symmetric on $\mathbb{R}^q$ they are also positive definite and symmetric on $\mathfrak{so}(q,\mathbb{R})$ by the proposition in (3.2) of [E2]. Since $p^2(W) \subseteq W$ it follows from 2) of the lemma above that $p(W) \subseteq W$; that is, $p \in \bar{G}_W$. It remains only to show that $p \in (\bar{G}_W)_o$, for then $k = gp^{-1} \in (\bar{G}_W)_o$.

Since $g \in (\bar{G}_W)_o$ there exists a continuous curve g(t) in $(\bar{G}_W)_o$ such that g(0) = Id and g(1) = g. Let g(t) = k(t)p(t) be the corresponding KP decomposition of g(t). The argument above shows that $p(t) \in \bar{G}_W$ for all t. Since p(0) = Id and p(1) = p it follows that $p \in (\bar{G}_W)_o$. □

The structure of symmetric elements of $(G_W)_o$

Proposition 8.4 Let $\mathfrak{N} = \mathbb{R}^q \oplus W$, where $(\bar{G}_W)_o$ is self adjoint. Suppose that every nonzero element of W is invertible. Let p be a symmetric positive definite element of $(\bar{G}_W)_o$ such that det(p) = 1. Then

1) p is the identity on W, where $p(Z) = pZp^t = pZp$ for all Z in W.

2) If W acts irreducibly on $\mathbb{R}^q$, then there exists a positive real number λ such that $\mathbb{R}^q = V_\lambda \oplus V_{1/\lambda}$, where V_λ and $V_{1/\lambda}$ denote the λ and $1/\lambda$ eigenspaces of p in $\mathbb{R}^q$.

Proof 1) By the proposition in (3.2) of [E2] the action of p on W is positive definite and symmetric relative to the canonical inner product < , > on $\mathfrak{so}(q,\mathbb{R})$. Since $p(W) \subseteq W$ it follows that there exists an orthonormal basis $\{Z_1, ... , Z_p\}$ of W and positive real numbers $\{\lambda_1, ... , \lambda_p\}$ such that $p(Z_i) = \lambda_i Z_i$ for each i. It suffices to prove that $\lambda_i = 1$ for each i.

We are given that $\lambda_i Z_i = p(Z_i) = pZ_ip^t = pZ_ip$ for each i. Since Z_i is invertible it follows that $\lambda_i \mathrm{Id} = Z_i^{-1}pZ_ip$ and taking determinants yields $\lambda_i^q = (\det p)^2 = 1$. Hence $\lambda_i = 1$ for each i since λ_i is positive.

2) Let λ be an eigenvalue for p in $\mathbb{R}^q$, and let v be a nonzero vector in $\mathbb{R}^q$ such that $p(v) = \lambda v$. If Z is any element of W, then by 1) we know that $p(Z(v)) = Zp^{-1}(v) = Z(\frac{1}{\lambda} v) = \frac{1}{\lambda} Z(v)$. Hence $Z(V_\lambda) \subseteq V_{1/\lambda}$ and $Z(V_{1/\lambda}) \subseteq V_\lambda$ for all Z in W. The subspace $V_\lambda \oplus V_{1/\lambda}$ is invariant under W and therefore must be equal to $\mathbb{R}^q$ since W acts irreducibly on $\mathbb{R}^q$. $\square$

References

[BtD] T. Broecker and T. tom Dieck, Representations of Compact Lie Groups, GTM 98, Springer, New York, 1985.

[E1] P. Eberlein, " Clifford algebras and Lie triple systems ", authors website, www.math.unc.edu

[E2] -------, " Geometry of 2-step nilpotent Lie groups ", to appear.

[E3] -------, " The moduli space of 2-step nilpotent Lie algebras of type (p,q) ", to appear in Contemporary Math.

[E4] -------, " Riemannian submersions and lattices in 2-step nilpotent Lie groups ", to appear in Commun. in Analysis and Geom.

[EH] P. Eberlein and J. Heber, " Quarter pinched homogeneous spaces of negative curvature ", Int. Jour. Math. 7, (1996), 441-500.

[Heb] J. Heber, " Noncompact homogeneous Einstein spaces ", Invent. Math. 133, (1998), 279-352.

[Hel] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.

[LM] H.B. Lawson and M-L. Michelsohn, Spin Geometry, Princeton University Press, , Princeton,1989.

[Mo] G.D. Mostow, " Self adjoint groups ", Ann. of Math. (2) 62 (1955), 44-55.