

THE RICCI TENSOR AND KILLING VECTOR FIELDS IN RIEMANNIAN SYMMETRIC SPACES

We recall the general framework of Riemannian symmetric spaces of the form G/K . Let G be a connected Lie group, and let $\sigma : G \rightarrow G$ be an automorphism of order 2 such that if $\tilde{K} = \text{Fix}(\sigma)$ then $\text{Ad}(\tilde{K})$ has compact closure in $\text{GL}(\mathfrak{G})$. Let K be a closed subgroup of G such that $\tilde{K}_0 \subset K \subset \tilde{K}$. Let \langle, \rangle be a G -invariant Riemannian structure on the coset manifold G/K . We showed elsewhere that G/K with the Riemannian structure \langle, \rangle is a Riemannian symmetric space.

Let $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{P}$ be the decomposition of \mathfrak{G} into $+1$ and -1 eigenspaces of $d\sigma : \mathfrak{G} \rightarrow \mathfrak{G}$. We have seen elsewhere that the tangent space $T_{eK} G/K$ may be identified with \mathfrak{P} as follows : for each $x \in T_{eK} G/K$ let X be the unique element of \mathfrak{P} such that $x = \alpha'(0)$, where $\alpha(t) = \exp(tX)K$ for all $t \in \mathbb{R}$. If $x, y, z \in T_{eK} G/K$ are identified with $X, Y, Z \in \mathfrak{P}$ in this manner, then we have seen that $R(x, y)z \in T_{eK} G/K$ is identified with $[X, Y], Z \in \mathfrak{P}$.

Proposition A Let G/K be a Riemannian symmetric space with a G -invariant Riemannian structure \langle, \rangle as defined above. Let $x, y \in T_{eK} G/K$ be identified with $X, Y \in \mathfrak{P}$. Then $\text{Ric}(x, y) = -\frac{1}{2} B(X, Y)$, where Ric denotes the Ricci tensor on G/K and B denotes the Killing form on \mathfrak{G} .

We shall need several steps.

I The metric \langle, \rangle^* on \mathfrak{P}

Let $\pi : G \rightarrow G/K$ be the projection map. We recall from the handout on G/H that $\pi_* : \mathfrak{P}(e) \rightarrow T_{eK} G/K$ is an isomorphism, where $\mathfrak{P}(e) = \{X(e) : X \in \mathfrak{P}\} \subset T_e G$. Let \langle, \rangle^* be the unique inner product on \mathfrak{P} such that $\langle X, Y \rangle^* = \langle \pi_* X(e), \pi_* Y(e) \rangle$ for all $X, Y \in \mathfrak{P}$.

Lemma 1. Let $x \in T_{eK} G/K$, and let $X \in \mathfrak{P}$ be the unique element that corresponds to x as above. Then $x = \pi_* X(e) = \tilde{X}(eK)$, where \tilde{X} is the vector field on G/K such with flow transformations $\{L_{\exp(tX)}\}$.

Proof. Note that $x = \alpha'(0)$, where $\alpha(t) = L_{\exp(tX)}(eK)$ is an integral curve of \tilde{X} . This shows that $\tilde{X}(eK) = \alpha'(0) = x$. Next, write $\alpha(t) = (\pi \circ \beta)(t)$, where $\beta(t) = \exp(tX)$. Then $x = \alpha'(0) = \pi_* \beta'(0) = \pi_* X(e)$. \square

Lemma 2. Let $x, y \in T_{eK} G/K$ correspond to $X, Y \in \mathfrak{P}$ as above. Let $T_X = (ad X)^2 : \mathfrak{G} \rightarrow \mathfrak{G}$ and let $R_x : T_{eK} G/K \rightarrow T_{eK} G/K$ be the curvature transformation given by $R_x(y) = R(x, y)x$. Then

- 1) T_X leaves invariant both \mathfrak{K} and \mathfrak{P} .
- 2) $R_x(y) \in T_{eK} G/K$ corresponds to $-T_X(Y) \in \mathfrak{P}$.
- 3) $T_X : \mathfrak{P} \rightarrow \mathfrak{P}$ is symmetric relative to \langle, \rangle^* .

Proof. 1) Since \mathfrak{K} and \mathfrak{P} are the $+1$ and -1 eigenspaces of $d\sigma : \mathfrak{G} \rightarrow \mathfrak{G}$ it follows that $[\mathfrak{K}, \mathfrak{K}] \subset \mathfrak{K}$, $[\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P}$ and $[\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}$. If $X \in \mathfrak{P}$, and $Y \in \mathfrak{K}$, then $[X, Y] \in [\mathfrak{P}, \mathfrak{K}] \subset \mathfrak{P}$ and $T_X(Y) = adX([X, Y]) \in [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}$. Hence T_X leaves \mathfrak{K} invariant, and a similar argument shows that T_X leaves \mathfrak{P} invariant.

2) By the formula above for the curvature tensor we have $R_x(y) = R(x,y)x$ corresponds to $[[X, Y], X] = -[X, [X, Y]] = -(ad X)^2(Y) = -T_X(Y)$.

3) Let $x, y, z \in T_{eK} G/K$ correspond to $X, Y, Z \in \mathfrak{P}$. From Lemma 1, 2) of Lemma 2 and the symmetry of R_x we obtain $\langle T_X(Y), Z \rangle^* = \langle \pi_* T_X(Y)(e), \pi_* Z(e) \rangle = \langle -R_x(y), z \rangle = \langle -y, R_x(z) \rangle = \langle Y, T_X(Z) \rangle^*$. The last equality comes from reversing the argument used in the previous equalities. \square

II The action of T_X on \mathfrak{K}

We saw in 1) of Lemma 2 that T_X leaves invariant \mathfrak{K} as well as \mathfrak{P} . By 3) of Lemma 2 it follows that T_X is diagonalizable over \mathbb{R} on \mathfrak{P} . The goal of this section is to show that T_X is also diagonalizable over \mathbb{R} on \mathfrak{K} , and has the same eigenvalues with the same multiplicities as on \mathfrak{P} .

Lemma 3. *Let $X \in \mathfrak{P}$ and consider the map $T_X : \mathfrak{K} \rightarrow \mathfrak{K}$. Then*

1) *The eigenvalues of T_X on \mathfrak{P} are the same as the eigenvalues of T_X on \mathfrak{K} .*

2) *Let $\lambda \in \mathbb{R}$ be an eigenvalue for T_X on both \mathfrak{P} and \mathfrak{K} , and let \mathfrak{P}_λ and \mathfrak{K}_λ be the corresponding eigenspaces in \mathfrak{P} and \mathfrak{K} for T_X . Then the maps $ad X : \mathfrak{P}_\lambda \rightarrow \mathfrak{K}_\lambda$ and $ad X : \mathfrak{K}_\lambda \rightarrow \mathfrak{P}_\lambda$ are isomorphisms.*

Proof. Let $\Lambda_{\mathfrak{P}} \subset \mathbb{R}$ be the set of nonzero eigenvalues of $T_X|_{\mathfrak{P}}$ and let $\Lambda_{\mathfrak{K}} \subset \mathbb{C}$ be the set of nonzero eigenvalues of $T_X|_{\mathfrak{K}}$. Let $\lambda \in \Lambda_{\mathfrak{P}}$ be given, and let $Y \in \mathfrak{P}$ be a nonzero vector with $T_X(Y) = \lambda Y$. Since $ad X$ commutes with $T_X = (ad X)^2$ we have $T_X(ad X(Y)) = ad X(T_X(Y)) = \lambda(ad X(Y))$. Note $ad X(Y)$ is nonzero since $0 \neq \lambda Y = T_X(Y) = ad X(ad X(Y))$. Since $ad X(Y) \in [\mathfrak{P}, \mathfrak{P}] \subset \mathfrak{K}$ it follows that λ is an eigenvalue for T_X acting on \mathfrak{K} with eigenvector $ad X(Y)$. We have proved

(a) $\Lambda_{\mathfrak{P}} \subset \Lambda_{\mathfrak{K}}$ and $ad X(\mathfrak{P}_\lambda) \subset \mathfrak{K}_\lambda$.

Next, we show that $\Lambda_{\mathfrak{K}} \subset \Lambda_{\mathfrak{P}} \subset \mathbb{R}$. A priori, the elements of $\Lambda_{\mathfrak{K}}$ might be complex numbers, but the inclusion assertion above will show they are real numbers. Let $\mu \in \Lambda_{\mathfrak{K}}$ be given, and let $\xi \in \mathfrak{K}^{\mathbb{C}}$ be a nonzero vector such that $T_X(\xi) = \mu \xi$. Note that $ad X(\mathfrak{K}^{\mathbb{C}}) \subset \{ad X(\mathfrak{K})\}^{\mathbb{C}} \subset [\mathfrak{P}, \mathfrak{K}]^{\mathbb{C}} \subset \mathfrak{P}^{\mathbb{C}}$. In particular, $\xi' = ad X(\xi) \in \mathfrak{P}^{\mathbb{C}}$ and $T_X(\xi') = (T_X \circ ad X)(\xi) = (ad X \circ T_X)(\xi) = \mu ad X(\xi) = \mu \xi'$. Again, $\xi' = ad X(\xi)$ is nonzero since $0 \neq \mu \xi = T_X(\xi) = ad X(ad X(\xi)) = ad X(\xi')$.

The arguments above show that μ is a nonzero eigenvalue of T_X on $\mathfrak{P}^{\mathbb{C}}$. However, the eigenvalues of T_X on \mathfrak{P} are the same as the eigenvalues of T_X on $\mathfrak{P}^{\mathbb{C}}$. Hence $\mu \in \Lambda_{\mathfrak{P}}$, and since $\mu \in \Lambda_{\mathfrak{K}}$ was arbitrary we have proved that $\Lambda_{\mathfrak{K}} \subset \Lambda_{\mathfrak{P}}$. Combining this statement with (a) yields

(b) $\Lambda_{\mathfrak{P}} = \Lambda_{\mathfrak{K}} = \Lambda \subset \mathbb{R}$.

We have completed the proof of 1), and we now complete the proof of 2). An argument almost identical to that used in the proof of (a) shows

(c) $ad X(\mathfrak{K}_\lambda) \subset \mathfrak{P}_\lambda$.

To prove 2) we show that $ad X : \mathfrak{K}_\lambda \rightarrow \mathfrak{P}_\lambda$ and $ad X : \mathfrak{P}_\lambda \rightarrow \mathfrak{K}_\lambda$ are injective. It will then follow that $\dim \mathfrak{P}_\lambda \geq \dim \mathfrak{K}_\lambda$ and $\dim \mathfrak{K}_\lambda \geq \dim \mathfrak{P}_\lambda$ respectively. We conclude that $\dim \mathfrak{P}_\lambda = \dim \mathfrak{K}_\lambda$, and 2) will now follow from the injectivity of $ad X$.

Let $Z \in \mathfrak{K}_\lambda$ be an element such that $ad X(Z) = 0$. Then $Z \in Ker ad X \subset Ker (ad X)^2 = Ker T_X$, which implies that $Z \in \mathfrak{K}_0 \cap \mathfrak{K}_\lambda = \{0\}$. Hence $ad X : \mathfrak{K}_\lambda \rightarrow \mathfrak{P}_\lambda$ is injective, and a similar argument shows that $ad X : \mathfrak{P}_\lambda \rightarrow \mathfrak{K}_\lambda$ is injective. \square

Lemma 4. *$T_X : \mathfrak{K} \rightarrow \mathfrak{K}$ is diagonalizable over \mathbb{R} .*

Proof. For $\lambda \in \Lambda$ let $\mathfrak{K}_\lambda^* = \{Z \in \mathfrak{K} : (T_X - \lambda I)^N(Z) = 0 \text{ for some positive integer } N\}$ and let $\mathfrak{P}_\lambda^* = \{Z \in \mathfrak{P} : (T_X - \lambda I)^N(Z) = 0 \text{ for some positive integer } N\}$. Let $\mathfrak{K}_0^* = \{Z \in \mathfrak{K} : T_X^N(Z) = 0 \text{ for some positive integer } N\}$ and let $\mathfrak{P}_0^* = \{Z \in \mathfrak{P} : T_X^N(Z) = 0 \text{ for some positive integer } N\}$. The eigenvalues of T_X on \mathfrak{K} and \mathfrak{P} are real, and hence by the

primary decomposition theorem it follows that $\mathfrak{K} = \mathfrak{K}_0^* \oplus \sum_{\lambda \in \Lambda} \mathfrak{K}_\lambda^*$ and $\mathfrak{P} = \mathfrak{P}_0^* \oplus \sum_{\lambda \in \Lambda} \mathfrak{P}_\lambda^*$. Clearly, $\mathfrak{P}_0 \subset \mathfrak{P}_0^*$ and $\mathfrak{P}_\lambda \subset \mathfrak{P}_\lambda^*$, where \mathfrak{P}_0 and \mathfrak{P}_λ denote the kernel and λ -eigenspace of T_X in \mathfrak{P} . By 3) of Lemma 2 $\mathfrak{P} = \mathfrak{P}_0 \oplus \sum_{\lambda \in \Lambda} \mathfrak{P}_\lambda$, and it follows that $\mathfrak{P}_0^* = \mathfrak{P}_0$ and $\mathfrak{P}_\lambda^* = \mathfrak{P}_\lambda$ for all $\lambda \in \Lambda$.

Similarly, $\mathfrak{K}_0 \subset \mathfrak{K}_0^*$ and $\mathfrak{K}_\lambda \subset \mathfrak{K}_\lambda^*$. It suffices to show that the reverse inclusions hold. Let $Z \in \mathfrak{K}_\lambda^*$ be given. Then $(T_X - \lambda I)^N(Z) = 0$ for some positive integer N . If $Y = \text{ad } X(Z)$, then $Y \in [\mathfrak{P}, \mathfrak{K}] \subset \mathfrak{P}$ and $(T_X - \lambda I)^N(Y) = (T_X - \lambda I)^N \text{ad } X(Z) = \text{ad } X\{(T_X - \lambda I)^N(Z)\} = 0$. This shows that $Y \in \mathfrak{P}_\lambda^* = \mathfrak{P}_\lambda$. Hence $(T_X - \lambda I)(Z)$ lies in the kernel of $\text{ad } X$ since $\text{ad } X\{(T_X - \lambda I)(Z)\} = (T_X - \lambda I)(\text{ad } X(Z)) = (T_X - \lambda I)(Y) = 0$. Note that $\text{Ker}(\text{ad } X) \subset \text{Ker}(\text{ad } X)^2 = \text{Ker}(T_X) = \mathfrak{K}_0$. Hence $(T_X - \lambda I)(Z) \in \mathfrak{K}_\lambda^* \cap \mathfrak{K}_0 \subset \mathfrak{K}_\lambda^* \cap \mathfrak{K}_0^* = \{0\}$, and $Z \in \mathfrak{K}_\lambda$. This proves that $\mathfrak{K}_\lambda^* \subset \mathfrak{K}_\lambda$, and hence equality must hold. A similar argument shows that $\mathfrak{K}_0^* = \mathfrak{K}_0$. Hence $\mathfrak{K} = \mathfrak{K}_0 \oplus \sum_{\lambda \in \Lambda} \mathfrak{K}_\lambda$. \square

III Proof of Proposition A

Proof. Let $x, y \in T_{eK} G/K$ be identified with $X, Y \in \mathfrak{P}$ as described above. We wish to show that $\text{Ric}(x, y) = -\frac{1}{2}B(X, Y)$. Since both Ric and B are symmetric bilinear forms it suffices by polarization to show that $\text{Ric}(x, x) = -\frac{1}{2}B(X, X)$. By 2) of Lemma 2 it follows that $\text{Ric}(x, x) = \text{trace } R_x = -\text{trace}(T_X|_{\mathfrak{P}}) = -\text{trace}(\text{ad } X)^2|_{\mathfrak{P}}$. Note that $B(X, X) = \text{trace}(\text{ad } X)^2 = \text{trace}(\text{ad } X)^2|_{\mathfrak{K}} + \text{trace}(\text{ad } X)^2|_{\mathfrak{P}}$. Hence it suffices to prove that $\text{trace}(\text{ad } X)^2|_{\mathfrak{K}} = \text{trace}(\text{ad } X)^2|_{\mathfrak{P}}$.

Let λ be an eigenvalue of \mathfrak{P} and hence also of \mathfrak{K} by Lemma 3. Lemma 3 also shows that $\dim(\mathfrak{P}_\lambda) = \dim(\mathfrak{K}_\lambda)$. Hence $\text{trace}(\text{ad } X)^2|_{\mathfrak{P}} = \sum_{\lambda \in \Lambda} \lambda \dim(\mathfrak{P}_\lambda) = \sum_{\lambda \in \Lambda} \lambda \dim(\mathfrak{K}_\lambda) = \text{trace}(\text{ad } X)^2|_{\mathfrak{K}}$. \square

IV Killing vector fields on γ_X and eigenvectors of T_X

The eigenvectors of T_X give rise to special Jacobi vector fields on the geodesic $\gamma_X(s) = \exp(sX)K$ in G/K . Before making a precise statement in Proposition C below we derive a result that is interesting in its own right.

For $A \in \mathfrak{G}$ let \tilde{A} denote the vector field on G/K with flow transformations $\{L_{\exp(tA)}\}$. We recall from Lemma 1 that for each $a \in T_{eK}G/K$ there exists a unique element A of \mathfrak{P} such that $a = \tilde{A}(eK)$. The vector field \tilde{A} is a Killing vector field since its flow transformations are isometries of G/K , and hence the restriction of \tilde{A} to any geodesic $\gamma(s)$ of G/K is a Jacobi vector field on $\gamma(s)$.

Proposition B Let $\gamma_X(s) = \exp(sX)K$, a geodesic in G/K . Let $Z \in \mathfrak{G}$ be given, and consider the geodesic variation $f(s, t) = \exp(tZ)\gamma_X(s) = \exp(tZ)\exp(sX)K$. Let $Y(s)$ be the variation vector field on $\gamma_X(s)$.

1) If $Z \in \mathfrak{K}$, then $Y(s)$ is the unique Jacobi vector field on $\gamma_X(s)$ such that $Y(0) = 0$ and $Y'(0) = [Z, X](eK)$.

2) If $Z \in \mathfrak{P}$, then $Y(s)$ is the unique Jacobi vector field on $\gamma_X(s)$ such that $Y(0) = \tilde{Z}(eK)$ and $Y'(0) = 0$.

Remark Note that if $Z \in \mathfrak{K}$ and $X \in \mathfrak{P}$, then $[Z, X] \in [\mathfrak{K}, \mathfrak{P}] \subset \mathfrak{P}$.

Proof. Let $f(s, t)$ be the geodesic variation given by $f(s, t) = \exp(tZ)\exp(sX)K$ of $\gamma_X(s) = f(s, 0)$, and let $Y(s)$ be the variation vector field. By definition $Y(s) = f_t(s, 0) = \frac{d}{dt}|_{t=0} f(s, t)$, and $Y'(s) = \nabla_{f_s} f_t(s, 0)$. By the first variation lemma $Y'(0) = \nabla_{f_t} f_s(0, 0) =$ the covariant derivative at $t = 0$ of the vector field $f_s(0, t)$ along the curve $f(0, t)$.

1) In this case $\exp(tZ) \in K$ for all $t \in \mathbb{R}$ since $Z \in \mathfrak{K} = LK$. Hence $f(0, t) = \exp(tZ)K = eK$ for all t , and it follows that $Y(0) = \frac{d}{dt}|_{t=0} f(0, t) = 0$.

Since $\exp(tZ) \in K$ for all t we may write $f(s, t) = \exp(tZ)\exp(sX)\exp(-tZ)K = \exp(sX(t))K$, where $X(t) = \text{Ad}(\exp(tZ))(X)$. Hence $f_s(0, t) = \widetilde{X}(t)(eK) \in T_{eK}G/K$ for all t . Since $f(0, t) \equiv eK$ is a constant curve it follows that $Y'(0) = \lim_{t \rightarrow 0} \left\{ \frac{f_s(0, t) - f_s(0, 0)}{t} \right\} = \lim_{t \rightarrow 0} \left\{ \frac{\widetilde{X}(t)(eK) - \widetilde{X}(0)(eK)}{t} \right\} = \tilde{Y}(eK)$, where $Y = \lim_{t \rightarrow 0} \left\{ \frac{X(t) - X(0)}{t} \right\}$. We use the fact that $A \rightarrow \tilde{A}$ is a linear map between \mathfrak{G} and $\tilde{\mathfrak{X}}(G/K)$. Finally, $Y = \lim_{t \rightarrow 0} \left\{ \frac{\text{Ad}(\exp(tZ))X - X}{t} \right\} = \text{ad}Z(X) = [Z, X]$. This completes the proof of 1).

2) In this case $f(0, t) = \exp(tZ)K = \gamma_Z(t)$, a geodesic of G/K that starts at eK . It follows that $Y(0) = \left. \frac{d}{dt} \right|_{t=0} f(0, t) = \tilde{Z}(eK)$.

To compute $Y'(0)$ we first compute $f_s(0, t) = \left. \frac{d}{ds} \right|_{s=0} f(s, t) = \left. \frac{d}{ds} \right|_{s=0} \exp(tZ)\gamma_X(s) = \exp(tZ)_* \gamma'_X(0)$. Since G/K is a symmetric space and $Z \in \mathfrak{P}$ we know that $\exp(tZ)_* : T_{eK}G/K \rightarrow T_{\gamma_Z(t)}G/K$ is parallel translation along the geodesic $\gamma_Z(t)$. Hence $f_s(0, t)$ is a parallel vector field $X(t)$ along $\gamma_Z(t)$. Since $Y'(0) =$ the covariant derivative at $t = 0$ of the vector field $f_s(0, t)$ along the curve $f(0, t) = \gamma_Z(t)$, we conclude that $Y'(0) = 0$. \square

We recall from Lemma 3 that for $X \in \mathfrak{P}$ the nonzero eigenvalues for T_X on \mathfrak{P} and \mathfrak{K} are the same real numbers with the same multiplicities. Moreover, $\text{ad } X : \mathfrak{K}_\lambda \rightarrow \mathfrak{P}_\lambda$ and $\text{ad } X : \mathfrak{P}_\lambda \rightarrow \mathfrak{K}_\lambda$ are isomorphisms. As before we let \mathfrak{K}_λ and \mathfrak{P}_λ denote the eigenspaces in \mathfrak{K} and \mathfrak{P} respectively for T_X .

Proposition C For $X \in \mathfrak{P}$ we let $\gamma(s) = \exp(sX)K$, a geodesic in G/K . Let $Z \in \mathfrak{G}$ be given, and consider the geodesic variation $f(s, t) = \exp(tZ)\gamma(s) = \exp(tZ)\exp(sX)K$. Let $Y(s)$ be the variation vector field on $\gamma(s) = \exp(sX)K$. Let $\lambda \in \mathbb{R}$ be an eigenvalue of T_X .

1) If $Z \in \mathfrak{K}_\lambda$, then $Y(s) = y(s)E(s)$, where $E(s)$ is a parallel vector field on $\gamma(s)$, and $y(s)$ is the C^∞ function such that $y(0) = 0, y'(0) = 1$ and $y''(s) - \lambda y(s) = 0$.

2) If $Z \in \mathfrak{P}_\lambda$, then $Y(s) = y(s)E(s)$, where $E(s)$ is a parallel vector field on $\gamma(s)$, and $y(s)$ is the C^∞ function such that $y(0) = 1, y'(0) = 0$ and $y''(s) - \lambda y(s) = 0$.

To prove 1) we first derive a preliminary result. Recall that for $x \in T_{eK}$ the curvature transformation $R_x : T_{eK}G/K \rightarrow T_{eK}G/K$ is given by $R_x(y) = R(x, y)x$.

Lemma. Let $X \in \mathfrak{P}$ be given, and let $\lambda \in \mathbb{R}$ be an eigenvalue of T_X . Let $\gamma(s) = \exp(sX)K$, a geodesic in G/K , and let $E(s)$ be a parallel vector field on $\gamma(s)$ such that $E(0) = \tilde{Z}(eK)$ for some $Z \in \mathfrak{P}_\lambda$. Then $R_{\gamma'(s)}E(s) = -\lambda E(s)$ for all s .

Proof. Clearly $-\lambda E(s)$ is a parallel vector field on $\gamma(s)$ and $R_{\gamma'(s)}E(s)$ is a parallel vector field on $\gamma(s)$ since G/K is a symmetric space. It remains only to prove that $-\lambda E(s)$ and $R_{\gamma'(s)}E(s)$ have the same value at $s = 0$.

Evidently, $-\lambda E(0) = -\lambda \tilde{Z}(eK) \in T_{eK}G/K$, which corresponds to $-\lambda Z \in \mathfrak{P}$. Since $\gamma'(0), E(0) \in T_{eK}G/K$ correspond to $X, Z \in \mathfrak{P}$ respectively, it follows from 2) of Lemma 2 that $R_{\gamma'(0)}E(0) \in T_{eK}G/K$ corresponds to $-T_X Z = -\lambda Z \in \mathfrak{P}$. \square

We now prove 1). From 1) of Proposition B we see that $Y(s)$ is the unique Jacobi vector field on $\gamma(s)$ such that $Y(0) = 0$ and $Y'(0) \in T_{eK}G/K$ corresponds to $[Z, X] \in \mathfrak{P}$. From Lemma 3 it follows that $[Z, X] = -\text{ad}X(Z) \in \mathfrak{P}_\lambda$ since $Z \in \mathfrak{K}_\lambda$.

Now let $\tilde{Y}(s) = y(s)E(s)$ where $E(s)$ is the parallel vector field on $\gamma(s)$ such that $E(0)$ corresponds to $[Z, X] \in \mathfrak{P}_\lambda$ and $y(s)$ is the unique function such that $y(0) = 0, y'(0) = 1$ and $y''(s) - \lambda y(s) = 0$. It suffices to prove that $\tilde{Y}(s)$ is a Jacobi vector field on $\gamma(s)$ with the same initial conditions as $Y(s)$.

Since $E(s)$ is parallel we compute $\tilde{Y}''(s) = y''(s)E(s)$ for all s . Next, $R_{\gamma'(t)}\tilde{Y}(s) = y(s)R_{\gamma'(t)}E(s) = -\lambda y(s)E(s)$ by the lemma above. Hence $\tilde{Y}''(s) + R_{\gamma'(t)}\tilde{Y}(s) = \{y''(s) - \lambda y(s)\}E(s) = 0$, which proves that $\tilde{Y}(s)$ is a Jacobi vector field on $\gamma(s)$.

By the definition of $y(s)$ we have $\tilde{Y}(0) = y(0)E(0) = 0$ and $\tilde{Y}'(0) = y'(0)E(0) = E(0)$, which corresponds to $[Z, X] \in \mathfrak{P}_\lambda$. This shows that the variation vector field $Y(s)$ and $\tilde{Y}(s)$ have the same initial conditions at $s = 0$.

We prove 2). From 2) of Proposition B we see that $Y(s)$ is the unique Jacobi vector field on $\gamma(s)$ such that $Y(0) \in T_{eK} G/K$ corresponds to $Z \in \mathfrak{P}_\lambda$ and $Y'(0) = 0$.

Let $Z \in \mathfrak{P}_\lambda$ be given, and let $Y(s)$ be the variation vector field of the geodesic variation of $\gamma(s)$ determined by Z as in the statement of the proposition. As in the proof of 1) we define $\tilde{Y}(s) = y(s)E(s)$, where $E(s)$ is the parallel vector field on $\gamma(s)$ such that $E(0) \in T_{eK} G/K$ corresponds to $Z \in \mathfrak{P}_\lambda$ and $y(s)$ is the unique function such that $y(0) = 1, y'(0) = 0$ and $y''(s) - \lambda y(s) = 0$. As above, it suffices to show that $\tilde{Y}(s)$ is a Jacobi vector field on $\gamma(s)$ with the same initial conditions as $Y(s)$.

Since $E(s)$ is parallel we obtain $\tilde{Y}''(s) = y''(s)E(s)$ for all s . By the lemma used in the proof of 1) we compute $R_{\gamma'(s)}\tilde{Y}(s) = y(s)R_{\gamma'(s)}E(s) = -\lambda y(s)E(s)$. Hence $\tilde{Y}''(s) + R_{\gamma'(s)}\tilde{Y}(s) = \{y''(s) - \lambda y(s)\}E(s) = 0$, which shows that $\tilde{Y}(s)$ is a Jacobi vector field on $\gamma(s)$. We compute $\tilde{Y}(0) = y(0)E(0) = E(0)$, which corresponds to $Z \in \mathfrak{P}_\lambda$, and $\tilde{Y}'(0) = y'(0)E(0) = 0$. Hence $Y(s)$ and $\tilde{Y}(s)$ have the same initial conditions, which completes the proof of 2) and of the proposition.