Lie triple systems in compact semisimple Lie algebras

The elementary theory of Lie triple systems is not easily found in the literature, and for this reason we present a brief self contained treatment of some parts of the theory.

Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite. It is known that if G is any connected Lie group with Lie algebra $\mathfrak{G}$, then G is compact (See for example Proposition 6.6 and Theorem 6.9 of ([He, pp. 132-133]). A subspace W of $\mathfrak{G}$ is called a <u>Lie triple system</u> in $\mathfrak{G}$ if $[X, [Y, Z]] \in W$ for all $X, Y, Z \in W$. Fix a bi-invariant metric on any compact connected Lie group G with Lie algebra $\mathfrak{G}$. If X = exp(W), where exp denotes the matrix exponential map, then it is a well known fact from the theory of Riemannian symmetric spaces that X is a totally geodesic submanifold of G $\Leftrightarrow$ W is a Lie triple system in $\mathfrak{G}$.

We are interested in the case that $\mathfrak{G} = \mathcal{O}(m)$, the Lie algebra of m x m skew symmetric matrices, and G is the special orthogonal group SO(m) for $m \geq 3$. Our motivation for studying Lie triple systems here is somewhat different. We have already seen that if Cl V is the Clifford algebra determined by a finite dimensional real inner product space V and if $j : \mathrm{Cl}\, V \to \mathrm{End}\,(U)$ is a representation on a finite dimensional real vector space U, then there exists an inner product $\langle\, , \rangle$ on V such that W = j(Cl V) is a Lie triple system in $\mathcal{O}(U, \langle\, , \rangle)$. Moreover, if $m = \dim V \geq 2$, then $W + [W,W] = \mathcal{O}(m+1)$ if $m \neq 3$ and either $W + [W,W] = \mathcal{O}(4)$ or $W = \mathcal{O}(3)$ if m = 3. These facts are generalized by the following result, which is relevant for us in the case that $\mathfrak{G} = \mathcal{O}(m)$.

Proposition

Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite. Let W be a Lie triple system in $\mathfrak{G}$. Then

1) The subspaces [W, W] and $\mathfrak{H} = W + [W,W]$ are subalgebras of $\mathfrak{G}$.

2) Let $W_o = \{X \in W : [X, Y] = 0 \text{ for all } Y \in W\}$. Then W_o is the center of $\mathfrak{H}$ and $\mathfrak{H} = W_o \oplus \mathfrak{H}_o$, where $\mathfrak{H}_o = [\mathfrak{H}, \mathfrak{H}]$ is a semisimple ideal of $\mathfrak{H}$.

Moreover, the direct sum is orthogonal for any $\mathrm{ad}_{\mathfrak{H}}$-invariant inner product $\langle\, , \rangle$ on $\mathfrak{H}$.

Remark

An inner product $\langle\, , \rangle$ on a Lie algebra $\mathfrak{H}$ is called $\mathrm{ad}_{\mathfrak{H}}$-invariant if $\mathrm{ad}\, X : \mathfrak{H} \to \mathfrak{H}$ is skew symmetric relative to $\langle\, , \rangle$ for every element X in $\mathfrak{H}$. If $\mathfrak{G}$ is a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite, then by well known properties of the Killing form the inner product $\langle\, , \rangle = -B_{\mathfrak{G}}$ is $\mathrm{ad}_{\mathfrak{G}}$-invariant. See for example section 6 of [He , p. --].

Proof

For the remainder of the proof we set $\mathfrak{P} = W$ and $\mathfrak{K} = [W,W] = [\mathfrak{P}, \mathfrak{P}]$, a notation that is familiar when discussing Cartan decompositions of semisimple Lie algebras. We note that from the definitions of $\mathfrak{K}$, $\mathfrak{P}$ and Lie triple system we obtain the bracket relations $[\mathfrak{K}, \mathfrak{P}] \subseteq \mathfrak{P}$ and $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$.

Lemma 1

The subspaces $\mathfrak{K}$ and $\mathfrak{H} = \mathfrak{K} + \mathfrak{P}$ are Lie subalgebras of $\mathfrak{G}$.

Proof of Lemma 1

Let P_1, P_2, P_1' and P_2' be arbitrary elements of $\mathfrak{P}$. To show that $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$ it suffices to show that $[[P_1, P_2], [P_1', P_2']] \in \mathfrak{K}$. If $X = [P_1', P_2']$, then $[[P_1, P_2], X] = \text{ad}([P_1, P_2])(X) = [\text{ad } P_1, \text{ad } P_2](X) = \text{ad } P_1 (\text{ad } P_2 (X)) - \text{ad } P_2 (\text{ad } P_1 (X))$ by the Jacobi identity. Now $\text{ad } P_2 (X) \in [\mathfrak{P}, \mathfrak{K}] \subseteq \mathfrak{P}$ and hence $\text{ad } P_1 (\text{ad } P_2 (X)) \in [\mathfrak{P}, \mathfrak{P}] \subseteq \mathfrak{K}$. Similarly $\text{ad } P_2 (\text{ad } P_1 (X)) \in \mathfrak{K}$, which proves that $\mathfrak{K}$ is a Lie subalgebra of $\mathfrak{G}$. From the bracket relations $[\mathfrak{K}, \mathfrak{P}] \subseteq \mathfrak{P}$ $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$ and $[\mathfrak{K}, \mathfrak{K}] \subseteq \mathfrak{K}$ it follows immediately that $\mathfrak{H} = \mathfrak{K} + \mathfrak{P}$ is a Lie subalgebra of $\mathfrak{G}$. □

Lemma 2

Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite. Let $\mathfrak{H}$ be a subalgebra of $\mathfrak{G}$, and let $\mathfrak{Z}$ denote the center of $\mathfrak{H}$. Then

1) $B_{\mathfrak{H}}(X, X) \le 0$ for all X in $\mathfrak{H}$ with equality $\Leftrightarrow X \in \mathfrak{Z}$.
2) $\mathfrak{H} = \mathfrak{Z} \oplus [\mathfrak{H}, \mathfrak{H}]$, an orthogonal direct sum for any $\text{ad}_{\mathfrak{H}}$-invariant inner product $\langle\, ,\, \rangle$ on $\mathfrak{H}$.
3) The Killing form of $[\mathfrak{H}, \mathfrak{H}]$ is negative definite and in particular $[\mathfrak{H}, \mathfrak{H}]$ is semisimple.

Corollary 2

Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite. Let $\mathfrak{H}$ be any subalgebra of $\mathfrak{G}$. Then

a) $\mathfrak{H}$ is semisimple $\Leftrightarrow \mathfrak{H} = [\mathfrak{H}, \mathfrak{H}] \Leftrightarrow \mathfrak{H}$ has trivial center.
b) There exists a Lie triple system W in $\mathfrak{G}$ such that $\mathfrak{H} = W + [W, W]$.

Proof of Corollary 2

The assertion a) follows immediately from 2) and 3). To prove b) we start by writing $\mathfrak{H} = \mathfrak{Z} \oplus \mathfrak{H}_o$, where $\mathfrak{Z}$ is the center of $\mathfrak{H}$ and $\mathfrak{H}_o = [\mathfrak{H}, \mathfrak{H}]$ has negative definite Killing form by 3) of Lemma 2. Next, we use the well known fact that for any Lie algebra $\mathfrak{H}_o$ whose Killing form B is negative definite there exists a Lie algebra automorphism $\sigma : \mathfrak{H}_o \to \mathfrak{H}_o$ of order two. (See for example [He, pp.451-455] for the case that $\mathfrak{H}_o$ is simple.) Now let $\mathfrak{P}$ and $\mathfrak{K}$ be the eigenspaces of σ in $\mathfrak{H}_o$ corresponding to the eigenvalues -1 and 1 respectively. Then $\mathfrak{P}$ is a Lie triple system, $\mathfrak{K} = [\mathfrak{P}, \mathfrak{P}]$ and $\mathfrak{H}_o = \mathfrak{K} \oplus \mathfrak{P}$. Finally, if $W = \mathfrak{Z} \oplus \mathfrak{P}$, then $[W, W] = [\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$ and $W + [W, W] =$

$\mathfrak{Z} \oplus \{\mathfrak{K} \oplus \mathfrak{P}\} = \mathfrak{Z} \oplus \mathfrak{H}_o = \mathfrak{H}$. The subspace W is a Lie triple system since [W, [W, W]] $= [\mathfrak{P}, \mathfrak{K}] \subseteq \mathfrak{P} \subseteq W$.

Proof of Lemma 2

1) Let < , > be an $ad_{\mathfrak{H}}$-invariant inner product on $\mathfrak{H}$. Then $B_{\mathfrak{H}}(X, X) = trace_{\mathfrak{H}}(ad\ X)^2 \leq 0$, and equality holds $\Leftrightarrow$ ad X = 0 on $\mathfrak{H}$ since $(ad\ X)^2$ is symmetric and negative semidefinite on $\mathfrak{H}$.

2) We must show that $\mathfrak{Z} = [\mathfrak{H}, \mathfrak{H}]^{\perp}$, the orthogonal complement of $[\mathfrak{H}, \mathfrak{H}]$ in $\mathfrak{H}$ relative to < , >. If $Z \in [\mathfrak{H}, \mathfrak{H}]^{\perp}$ and $X, Y \in \mathfrak{H}$ are any elements, then 0 = < Z, [X, Y] > = < [Z, X], Y > since ad X is skew symmetric. Hence [Z, X] = 0 and $Z \in \mathfrak{Z}$ since X and Y were arbitrary. Reversing the argument above shows that $[\mathfrak{H}, \mathfrak{H}]^{\perp} \supseteq \mathfrak{Z}$ and completes the proof of 2).

3) Any element Z in the center of $[\mathfrak{H}, \mathfrak{H}]$ would also lie in the center of $\mathfrak{H}$, which is ruled out by 2) unless Z = 0. Hence $[\mathfrak{H}, \mathfrak{H}]$ has trivial center and the Killing form of $[\mathfrak{H}, \mathfrak{H}]$ is negative definite by 1) applied to $[\mathfrak{H}, \mathfrak{H}]$.□

Lemma 3

Let $W_o = \{X \in W : [X, Y] = 0 \text{ for all } Y \in W\}$, and let $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{Z}$ be as in Lemma 2. Then

a) $W_o \subseteq \mathfrak{Z}$.

b) If $\mathfrak{Z} \neq \{0\}$, then $W_o \neq \{0\}$.

Proof of Lemma 3

a) Since $\mathfrak{K} = [W, W]$ it follows from the Jacobi identity and the definition of W_o that ad $\mathfrak{K}(W_o) = \{0\}$. The proof is now complete since ad $\mathfrak{P}(W_o) = \{0\}$ by the definition of W_o.

b) If $\mathfrak{Z} \neq \{0\}$, then by Lemma 2 $[\mathfrak{H}, \mathfrak{H}] = \mathfrak{K} + [\mathfrak{K}, \mathfrak{P}]$ is a proper subspace of $\mathfrak{H}$, which implies that $[\mathfrak{K}, \mathfrak{P}]$ is a proper subspace of $\mathfrak{P}$. Let < , > be any $ad_{\mathfrak{H}}$-invariant inner product on $\mathfrak{H}$, and let $[\mathfrak{K}, \mathfrak{P}]^{\perp}$ denote the orthogonal complement of $[\mathfrak{K}, \mathfrak{P}]$ in $\mathfrak{H}$. Then $\mathfrak{P} \cap \{[\mathfrak{K}, \mathfrak{P}]^{\perp}\}$ is nonzero, and it suffices to show that $\mathfrak{P} \cap \{[\mathfrak{K}, \mathfrak{P}]^{\perp}\} \subseteq W_o$. If P_1, P_2 and P_3 are elements of $\mathfrak{P}$ and Z is an element of $\mathfrak{P} \cap \{[\mathfrak{K}, \mathfrak{P}]^{\perp}\}$, then $[P_1, [P_2, P_3]] \in [\mathfrak{K}, \mathfrak{P}]$ and $0 = \langle Z, [P_1, [P_2, P_3]] \rangle = \langle [Z, P_1], [P_2, P_3] \rangle$ since ad P_1 is skew symmetric. This shows that $[Z, \mathfrak{P}]$ is orthogonal to $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$. Since $[Z, \mathfrak{P}] \subseteq [\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$ we conclude that $[Z, \mathfrak{P}] = \{0\}$.□

Corollary 3

Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite. Let W be a Lie triple system in $\mathfrak{G}$, and let $\mathfrak{H} = W + [W,W]$. Let $W_o = \{X \in W : [X, Y] = 0 \text{ for all } Y \in W\}$, and suppose that $W_o = \{0\}$. Then $\mathfrak{H}$ is semisimple.

Proof of Corollary 3

By b) of Lemma 3 the center of $\mathfrak{H}$ is trivial, and the result now follows from Corollary 2.□

Irreducible Lie triple systems

To complete the proof of the Proposition we discuss a natural decomposition of Lie triple systems. A Lie triple system W of $\mathfrak{G}$ is called <u>irreducible</u> if W cannot be written as $W = W_1 \oplus W_2$, where W_1 and W_2 are Lie triple systems such that $[W_1, W_2] = \{0\}$.

Remark

By using arguments similar to those that have been used above one can show that a Lie triple system $W = \mathfrak{P}$ is irreducible $\Leftrightarrow$ ad $\mathfrak{K}$ acts irreducibly on $\mathfrak{P}$. We shall not need the full strength of this statement.

Lemma 4

Let $\mathfrak{G}$, W, W_o and $\mathfrak{H}$ be as in Corollary 3, and let W be nonabelian. Let < , > be an $ad_{\mathfrak{H}}$-invariant inner product on $\mathfrak{H}$.

a) If $W_o \neq \{0\}$ and W_1 is the orthogonal complement of W_o in W, then W_1 is a nonabelian Lie triple system and $W = W_o \oplus W_1$.

b) $W = W_o \oplus \{\bigoplus_{i=1}^{N} W_i\}$, where for $1 \le i \le N$ each W_i is a nonabelian irreducible Lie triple system with $[W_i, W_j] = 0$ for $i \neq j$.

Proof

a) If P_1, P_2, P_3, are elements of W_1 and X is an element of W_o, then $< [P_1, [P_2, P_3]], X > = - < [P_2, P_3], [P_1, X] > = 0$ since ad P_1 is skew symmetric. Hence $[W_1,[W_1, W_1]] \subseteq W_o^{\perp} = W_1$.

b) This follows immediately from a) by decomposing W_1 into irreducible pieces.□

Lemma 5

Let $\mathfrak{G}$ be a finite dimensional Lie algebra over $\mathbb{R}$ such that the Killing form $B_{\mathfrak{G}}$ is negative definite. Let W be an irreducible nonabelian Lie triple system in $\mathfrak{G}$, and let $\mathfrak{H} = W + [W,W]$. Then

a) $\mathfrak{H}$ is semisimple. Either $\mathfrak{H} = W = [W, W]$ or $W \cap [W, W] = \{0\}$ and $\mathfrak{H} = W \oplus [W, W]$.

b) If $\mathfrak{H} = W \oplus [W, W]$, then $B_{\mathfrak{H}}(W, [W, W]) = \{0\}$, where $B_{\mathfrak{H}}$ denotes the Killing form of $\mathfrak{H}$.

Proof

a) Let < , > be an $ad_{\mathfrak{H}}$-invariant inner product on $\mathfrak{H}$. Let $\mathfrak{P}_1 = \mathfrak{K} \cap \mathfrak{P}$ and let $\mathfrak{P}_2$ denote the orthogonal complement of $\mathfrak{P}_1$ in $\mathfrak{P} = W$ relative to < , >. We shall show that $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are Lie triple systems such that $[\mathfrak{P}_1, \mathfrak{P}_2] = \{0\}$ and then apply the irreducibility hypothesis to complete the proof of a).

It is easy to check that $\mathfrak{P}_1$ is ad $\mathfrak{K}$-invariant (in fact, $\mathfrak{P}_1$ is an ideal in $\mathfrak{H}$), and hence $\mathfrak{P}_2$ is also ad $\mathfrak{K}$-invariant. Both $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are Lie triple systems since $[[\mathfrak{P}_i, \mathfrak{P}_i], \mathfrak{P}_i] \subseteq [[\mathfrak{P}, \mathfrak{P}], \mathfrak{P}_i] = [\mathfrak{K}, \mathfrak{P}_i] \subseteq \mathfrak{P}_i$ for i = 1,2. Note that $< \mathfrak{K}, [\mathfrak{P}_1, \mathfrak{P}_2] > = < [\mathfrak{K}, \mathfrak{P}_1], \mathfrak{P}_2 > = 0$ since $[\mathfrak{K}, \mathfrak{P}_1] \subseteq \mathfrak{P}_1$ and ad $\mathfrak{P}_1$ acts on $\mathfrak{G}$ by skew symmetric transformations. Hence $[\mathfrak{P}_1, \mathfrak{P}_2] = \{0\}$ since $[\mathfrak{P}_1, \mathfrak{P}_2]$ is a subspace of $\mathfrak{K}$ that is orthogonal to $\mathfrak{K}$.

By the irreducibility hypothesis either $\mathfrak{P}_1 = \mathfrak{K} \cap \mathfrak{P} = \{0\}$ or $\mathfrak{P}_1 = \mathfrak{P}$. In the first case $\mathfrak{K} \cap \mathfrak{P} = W \cap [W, W] = \{0\}$ and $\mathfrak{H} = W \oplus [W, W]$. In the second case $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K} \supseteq \mathfrak{K} \cap \mathfrak{P} = \mathfrak{P}$, which implies that $[\mathfrak{P}, \mathfrak{P}] \subseteq [[\mathfrak{P}, \mathfrak{P}], \mathfrak{P}] \subseteq \mathfrak{P}$. We conclude that $W = \mathfrak{P} = [\mathfrak{P}, \mathfrak{P}]$ and $\mathfrak{H} = W + [W, W] = W = [W, W]$.

To show that $\mathfrak{H}$ is semisimple in either case it suffices by Corollary 2 to show that $\mathfrak{H}$ has trivial center $\mathfrak{Z}$. If $\mathfrak{Z} \neq \{0\}$, then $W_o \neq \{0\}$ by b) of Lemma 3, and W is a reducible Lie triple system by a) of Lemma 4. This contradicts the irreducibility hypothesis and completes the proof of a)

b) For any elements $K \in \mathfrak{K}$ and $P \in \mathfrak{P}$, we obtain $B_{\mathfrak{H}}(K, P) = \text{trace}_{\mathfrak{H}}$ (ad K o ad P) = {0} since (ad K o ad P)$(\mathfrak{K}) \subseteq \mathfrak{P}$, (ad K o ad P)$(\mathfrak{P}) \subseteq \mathfrak{K}$ and $\mathfrak{H} = \mathfrak{K} \oplus \mathfrak{P}$. Hence $B_{\mathfrak{H}}(\mathfrak{K}, \mathfrak{P}) = \{0\}$.

Proof of the Proposition

We show first that W_o is the center of $\mathfrak{H} = W + [W, W]$. By b) of Lemma 4 we may write $W = W_o \oplus \{\bigoplus_{i=1}^{N} W_i\}$, where for $1 \leq i \leq N$ each W_i is an irreducible Lie triple system with $[W_i, W_j] = 0$ for $i \neq j$. If $\mathfrak{H}_i = W_i + [W_i, W_i]$, then $\mathfrak{H}_i$ is semisimple for $1 \leq i \leq N$ by a) of Lemma 5. From the definitions it follows that $\mathfrak{H} = W + [W, W] = W_o \oplus \{\bigoplus_{i=1}^{N} \mathfrak{H}_i\}$. Note that $[\mathfrak{H}_i, \mathfrak{H}_j] = \{0\}$ for $i \neq j$ since $[W_i, \mathfrak{H}_j] = \{0\}$ by the Jacobi identity and W_i generates $\mathfrak{H}_i$. The subalgebra $\mathfrak{H}_o = \bigoplus_{i=1}^{N} \mathfrak{H}_i$ is semisimple since each $\mathfrak{H}_i$ is semisimple, and hence $\mathfrak{H}_o$ has trivial center by Lemma 2. Therefore $W_o = \mathfrak{Z}$, the center of $\mathfrak{H}$, since $W_o \subseteq \mathfrak{Z}$ by a) of Lemma 3.

Since $\mathfrak{H} = \mathfrak{Z} \oplus \mathfrak{H}_o$ it follows that $[\mathfrak{H}, \mathfrak{H}] = [\mathfrak{H}_o, \mathfrak{H}_o] = \mathfrak{H}_o$ since $\mathfrak{H}_o$ is semisimple. Clearly $\mathfrak{H}_o = [\mathfrak{H}, \mathfrak{H}]$ is an ideal of $\mathfrak{H}$. If $< , >$ is any $\text{ad}_{\mathfrak{H}}$-invariant inner product on $\mathfrak{H}$, then $\mathfrak{Z}$ and $\mathfrak{H}_o$ are orthogonal by 2) of Lemma 2.

Warning

From b) of Lemma 5 it is tempting to believe that if W is an irreducible nonabelian Lie triple system in a Lie algebra $\mathfrak{G}$ whose Killing form is negative definite and if $W \cap [W, W] = \{0\}$, then 1) $\mathfrak{H} = W \oplus [W, W]$ must be a simple Lie algebra and 2) W and

[W, W] are orthogonal with respect to any $ad_{\mathfrak{H}}$-invariant inner product < , > on $\mathfrak{H}$. Indeed, if 1) holds then so does 2) by b) of Lemma 5 since if $\mathfrak{H}$ is a simple Lie algebra, then any $ad_{\mathfrak{H}}$-invariant inner product < , > on $\mathfrak{H}$ must be a negative multiple of the Killing form $B_{\mathfrak{H}}$. However, both 1) and 2) may fail, even in the case that $\mathfrak{G} = \mathcal{O}(m)$ and < , > is the inner product on $\mathcal{O}(m)$ given by < X, Y> = – trace XY for all X,Y in $\mathcal{O}(m)$.

Example

Let $\mathfrak{H}$ be the Lie algebra of a compact, simple Lie group H, and let $\rho_i : \mathfrak{H} \rightarrow \mathrm{End}(U_i)$ be an irreducible representation of $\mathfrak{H}$ on a finite dimensional real vector space U_i for i = 1,2. Assume that $\dim U_1 \neq \dim U_2$. Let $U = U_1 \oplus U_2$ and let $\rho : \mathfrak{H} \oplus \mathfrak{H} \rightarrow \mathrm{End}(U)$ be the representation such that $\rho(\mathfrak{H} \times \{0\}) = \rho_1(\mathfrak{H}) \subseteq \mathrm{End}(U_1)$ and $\rho(\{0\} \times \mathfrak{H}) = \rho_2(\mathfrak{H}) \subseteq \mathrm{End}(U_2)$.

Let < , > denote an $ad_{\mathfrak{H} \oplus \mathfrak{H}}$-invariant inner product on U. For example, let < , > be an inner product on U invariant under σ(H x H), where H is a compact simply connected Lie group with Lie algebra $\mathfrak{H}$ and σ is the product representation $\sigma_1 \times \sigma_2$: H x H → GL(U), where $\sigma_i : H \rightarrow GL(U_i)$ is the representation such that $d\sigma_i = \rho_i$ for i = 1, 2. Then $\rho(\mathfrak{H} \oplus \mathfrak{H}) \subseteq \mathfrak{G} = \mathcal{O}(U, < , >)$, the Lie algebra of skew symmetric linear transformations of {U, < , >}, and we may define an ad-invariant inner product << , >> on $\mathcal{O}(U, < , >)$ by << S, T >> = – trace (S o T). The ad-invariant inner products on a compact simple Lie algebra are unique up to constant multiples, and hence there exist positive constants λ_1 and λ_2 such that << , >> = $-\lambda_1 B$ on $\mathfrak{H} \times \{0\}$ and << , >> = $-\lambda_2 B$ on $\{0\} \times \mathfrak{H}$, where B denotes the Killing form for $\mathfrak{H}$. The constants λ_1, λ_2 depend on $\dim U_1$, $\dim U_2$ and in general $\lambda_1 \neq \lambda_2$ if $\dim U_1 \neq \dim U_2$. This can be readily computed, for example, in the simplest case where G = SU(2).

Let $\mathfrak{P} = \{(\rho_1(X), -\rho_2(X)) = \rho(X, -X) : X \in \mathfrak{H}\}$ and let $\mathfrak{K} = \{(\rho_1(X), \rho_2(X)) = \rho(X, X) : X \in \mathfrak{H}\}$. Then $[\mathfrak{P}, \mathfrak{P}] = \mathfrak{K}$, $[\mathfrak{P}, [\mathfrak{P}, \mathfrak{P}]] \subseteq [\mathfrak{P}, \mathfrak{K}] \subseteq \mathfrak{P}$ and $\mathfrak{K} \oplus \mathfrak{P} = \mathfrak{H} \oplus \mathfrak{H}$. Hence $\mathfrak{P}$ is an irreducible Lie triple system in $\mathfrak{G} = \mathcal{O}(U, < , >)$, but clearly $\mathfrak{P} \oplus [\mathfrak{P}, \mathfrak{P}] = \mathfrak{H} \oplus \mathfrak{H}$ is not a simple Lie algebra.

Let < , > be any ad-invariant inner product on $\mathfrak{G}^* = \mathfrak{G}_1^* \oplus \mathfrak{G}_2^*$, where $\mathfrak{G}^*$ is a semisimple Lie algebra and $\mathfrak{G}_1^*$, $\mathfrak{G}_2^*$ are (semisimple) ideals of $\mathfrak{G}^*$. Then $[\mathfrak{G}_1^*, \mathfrak{G}_2^*] \subseteq \mathfrak{G}_1^* \cap \mathfrak{G}_2^* = \{0\}$ and $< X_1, [X_2, Y_2] > = < [X_1, X_2], Y_2> = 0$ for any elements $X_1 \in \mathfrak{G}_1^*$ and $X_2, Y_2 \in \mathfrak{G}_2^*$. Hence $< \mathfrak{G}_1^*, \mathfrak{G}_2^* > = \{0\}$ since $[\mathfrak{G}_2^*, \mathfrak{G}_2^*] = \mathfrak{G}_2^*$. Applying this observation in our situation we conclude that $<< \mathfrak{H} \times \{0\}, \{0\} \times \mathfrak{H} >> = \{0\}$. It follows immediately that $\mathfrak{K}$ and $\mathfrak{P}$ are not orthogonal relative to << , >> if $\lambda_1 \neq \lambda_2$. □