

The moduli space of 2-step nilpotent Lie algebras of type (p, q)

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Introduction Simply connected homogeneous Riemannian manifolds with negative sectional curvature are closely connected with nilpotent Lie algebras, and if the sectional curvature is normalized to satisfy the inequalities $-4 \leq K \leq -1$ (quarter pinched), then the connection is with 2-step nilpotent Lie algebras. In particular, any 2-step nilpotent Lie algebra \mathfrak{N} gives rise to a homogeneous Riemannian manifold whose sectional curvature is negative and quarter pinched. See section 0 for details.

Let $p \geq 1$ and $q \geq 2$ be integers and let $\mathfrak{N}(p, q)$ denote the space of all 2-step nilpotent Lie algebras \mathfrak{N} such that the commutator ideal $[\mathfrak{N}, \mathfrak{N}]$ has dimension p and codimension q . The space $\mathfrak{N}(p, q)$ admits a C^∞ manifold structure and is a fiber bundle over the Grassmann manifold $G(p, p+q)$ of p -dimensional subspaces of \mathbb{R}^{p+q} . The group $GL(p+q, \mathbb{R})$ acts in a natural way on $\mathfrak{N}(p, q)$, and each orbit is an isomorphism class in $\mathfrak{N}(p, q)$.

In this article we study $X(p, q) = \mathfrak{N}(p, q) / GL(p+q, \mathbb{R})$, the space of isomorphism classes in $\mathfrak{N}(p, q)$. We show that $X(p, q)$ with the quotient topology is homeomorphic to the compact, connected quotient space $G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R})$, where $SL(q, \mathbb{R})$ acts on $\mathcal{SO}(q, \mathbb{R})$ and the associated Grassmann manifold $G(p, \mathcal{SO}(q, \mathbb{R}))$ by $g(Z) = gZg^t$ for $Z \in \mathcal{SO}(q, \mathbb{R})$ and $g \in SL(q, \mathbb{R})$. It follows immediately that $X(p, q)$ is homeomorphic to $X(D-p, q)$, where $D = \dim \mathcal{SO}(q, \mathbb{R}) = (1/2)q(q-1)$ (duality).

For each (p, q) we compute the dimension of $X(p, q) = G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R})$, which is defined to be the minimum codimension of an $SL(q, \mathbb{R})$ orbit in $G(p, \mathcal{SO}(q, \mathbb{R}))$. We reduce the problem to computing the integer $cd(p, q)$, which is defined to be the minimum codimension of $\mathcal{S}\ell(q, \mathbb{C}) \oplus \mathcal{S}\ell(p, \mathbb{C})$ acting on $V(p, q) = \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^p) \approx \mathcal{SO}(q, \mathbb{C}) \otimes \mathbb{C}^p$. The action is that induced by the standard action of $\mathcal{S}\ell(p, \mathbb{C})$ on \mathbb{C}^p and the action of $\mathcal{S}\ell(q, \mathbb{C})$ on $\mathcal{SO}(q, \mathbb{C})$ given by $X(Z) = XZ + ZX^t$ for all $X \in \mathcal{S}\ell(q, \mathbb{C})$ and all $Z \in \mathcal{SO}(q, \mathbb{C})$.

For $\varphi \in V(p, q)$ let $\mathcal{S}\ell(q, \mathbb{C}) \oplus \mathcal{S}\ell(p, \mathbb{C})_\varphi = \{(X, Y) \in \mathcal{S}\ell(q, \mathbb{C}) \oplus \mathcal{S}\ell(p, \mathbb{C}) : (X, Y)(\varphi) = 0\}$, and define $d'_{p, q}$ to be the minimum dimension of a stabilizer $\mathcal{S}\ell(q, \mathbb{C}) \oplus \mathcal{S}\ell(p, \mathbb{C})_\varphi$, $\varphi \in V(p, q)$. Clearly

$$cd(p, q) = p(D-p) + 2 - q^2 + d'_{p, q} = \binom{p}{2} - 1 q^2 - \binom{p}{2} q + (2-p^2) + d'_{p, q}.$$

For $p \geq 3$ very few of the integers $d'_{p, q}$ are nonzero, and work of Elashvili [El] allows one to compute the nonzero integers $d'_{p, q}$, where $p \geq 1$ and $q \geq 2$. See also Tables 2a, 2b of [KL], which add the case $d'_{3, 6} = 1$ to the list of [El].

We show that $\dim X(p,q) = 0$ if $cd(p,q) = 0$ or 1. More generally, if $cd(p,q) \geq 1$ then $\dim X(p,q) = cd(p,q) - 1$.

In particular we find that $X(p,q)$ has dimension zero precisely for the following pairs (p,q) and their dual pairs $(D-p,q)$:

- | | |
|--|---|
| 1) $(1, q)$ and $(D-1, q)$, $q \geq 2$ | 2) $(2, 2k+1)$ and $(D-2, 2k+1)$ $k \geq 1$ |
| 3) (D,q) , $q \geq 2$ (free on q generators) | 4) $(2, 4)$ and $(4, 4)$ |
| 5) $(2, 6)$ and $(13, 6)$ | 6) $(3, 4)$ (self dual) |
| 7) $(3, 5)$ and $(7, 5)$ | 8) $(4, 5)$ and $(6, 5)$. |

The first four examples are well known.

To complete the description, $\dim X(3,6) = \dim X(12,6) = 2$, $\dim X(2, 2k) = \dim X(2k^2-k-2, 2k) = k-3$ for $k \geq 4$ and $\dim X(p,q) = p(D-p)+1-q^2 > 0$ for all remaining cases.

Remark (Scarcity of lattices) Let \mathfrak{N} be a nilpotent Lie algebra with $k \geq 2$ steps, and let N denote the unique simply connected nilpotent Lie group with Lie algebra \mathfrak{N} . A discrete subgroup Γ of N is said to be a lattice if the quotient space $\Gamma \backslash N$ is compact. A well known criterion of Mal'cev says that N admits a lattice $\Gamma \Leftrightarrow \mathfrak{N}$ admits a basis \mathfrak{B} with rational structure constants. Specializing to the case that \mathfrak{N} is 2-step nilpotent, it follows that for each pair of integers (p,q) with $p \geq 1$ and $q \geq 2$ there are at most countably many isomorphism classes in $X(p,q)$ that admit rational structures. In particular, if $\dim X(p,q) > 0$, then for all but countably many elements of $X(p,q)$ the corresponding simply connected Lie groups N will fail to admit lattices.

Note that if $p \geq 3$, then $\binom{p}{2} - 1 > 0$ and the formula above for $cd(p,q)$ shows that there exists a positive integer q_0 , depending on p , such that $\dim X(p,q) > 0$ for all $q \geq q_0$.

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Section 0 Connection between negative curvature and nilpotent Lie algebras

The horospheres of a Riemannian symmetric space X with variable quarter pinched negative curvature are isometric to certain 2-step nilpotent Lie groups N with a left invariant metric. The space X itself is isometric to a solvable subgroup S of isometries equipped with a left invariant metric, and N is the commutator subgroup $[S,S]$ of codimension 1 in S . The Lie algebra \mathfrak{S} of S with the inherited inner product \langle, \rangle admits a unit element A orthogonal to the Lie algebra \mathfrak{N} of N such that $\text{ad } A = 2\text{Id}$ on the center \mathfrak{Z} of \mathfrak{N} and $\text{ad } A = \text{Id}$ on \mathfrak{Z}^\perp , the orthogonal complement of \mathfrak{Z} in \mathfrak{N} .

If KAN is an Iwasawa decomposition of $G = I_o(X)$, then the solvable group $S = AN$ has the properties listed above, but the isomorphism type of S is not uniquely determined by these properties ([AW], Lemma 4.2).

The metric Lie algebra $\{\mathfrak{N}, \langle, \rangle\}$ has further strong geometric properties that make it a Lie algebra of Heisenberg type. See [K1, 2] or [BTV] for further details.

More generally, Heintze showed in [H] that a homogeneous Riemannian manifold with strictly negative sectional curvature is isometric to a solvable Lie group S with a left invariant metric such that $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$ is a codimension 1 nilpotent ideal in \mathfrak{S} . In addition \mathfrak{S} admits a unit element A orthogonal to \mathfrak{N} such that $\text{ad } A$ has eigenvalues in \mathfrak{N} with positive real parts. Conversely, Heintze also showed in [H] that if \mathfrak{N} is a nilpotent Lie algebra that admits a derivation D whose eigenvalues have positive real parts, then $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}$ admits a solvable Lie algebra structure such that $\text{ad } 1 = D$ and an inner product \langle, \rangle such that the corresponding simply connected solvable Lie group S with left invariant metric \langle, \rangle has strictly negative sectional curvature.

In [EH] the method of Heintze was adapted to study homogeneous Riemannian manifolds with strictly negative and quarter pinched sectional curvature. A solvable Lie algebra \mathfrak{S} is said to be a QP-algebra if 1) The commutator ideal $\mathfrak{N} = [\mathfrak{S}, \mathfrak{S}]$ has codimension 1 in S and 2) There exists a nonzero vector A in \mathfrak{S} such that a) $\mathfrak{S} = \langle A \rangle \oplus \mathfrak{N}$ b) The derivation $\text{ad } A$ of \mathfrak{N} has eigenvalues whose real parts lie in $[1, 2]$. c) If $\text{Re } \lambda = 1$ or 2 for an eigenvalue λ of $\text{ad } A$, then $\text{ad } A = \lambda \text{Id}$ on $\mathfrak{N}_\lambda^{\mathbb{C}} = \{\xi \in \mathfrak{N}^{\mathbb{C}} : (\text{ad } A - \lambda \text{Id})^n(\xi) = 0 \text{ for some positive integer } n\}$.

In [EH] it was shown that a simply connected solvable Lie group S admits a left invariant Riemannian metric with sectional curvature satisfying $-4 \leq K \leq -1 \Leftrightarrow$ the Lie algebra \mathfrak{S} is a QP-algebra.

The simplest QP-algebras arise from 2-step nilpotent Lie algebras \mathfrak{N} . Let \mathfrak{N} denote a 2-step nilpotent Lie algebra with center \mathfrak{Z} and let \mathfrak{V} be a subspace of \mathfrak{N} such that $\mathfrak{N} = \mathfrak{V} \oplus \mathfrak{Z}$. Let D be the derivation of \mathfrak{N} such that $D = \text{Id}$ on \mathfrak{V} and $D = 2 \text{Id}$ on \mathfrak{Z} . Let $\mathfrak{S} = \mathbb{R} \oplus \mathfrak{N}$ be given the solvable Lie algebra structure such that $\text{ad } 1 = D$ on \mathfrak{N} . Then \mathfrak{S} is clearly a QP-algebra.

Section 1 Basic framework for studying 2-step nilpotent structures on \mathbb{R}^n

1.1 Definitions and notation

Let V, W be finite dimensional real vector spaces, and let $T : V \times V \rightarrow W$ be an alternating bilinear map. Recall that there exists a unique linear map $\hat{T} : \Lambda^2(V) \rightarrow W$ such that $\hat{T}(v \wedge v^*) = T(v, v^*)$ for all v, v^* in V .

A 2-step nilpotent structure on \mathbb{R}^n is a nonzero alternating, bilinear map $[\cdot, \cdot] : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $[x, [y, z]] = 0$ for all x, y, z in \mathbb{R}^n . Let $\rho : \Lambda^2(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ be the nonzero linear map such that $[x, y] = \rho(x \wedge y)$ for all x, y in \mathbb{R}^n . If $Z(\rho) = \{x \in \mathbb{R}^n : \rho(x \wedge y) = 0 \text{ for all } y \text{ in } \mathbb{R}^n\}$, then the condition that $[x, [y, z]] = 0$ for all x, y, z in \mathbb{R}^n is equivalent to the condition that $\rho\{\Lambda^2(\mathbb{R}^n)\} \subseteq Z(\rho)$. Hence we arrive at the following :

Definition

A 2-step nilpotent structure on \mathbb{R}^n is a linear map $\rho : \Lambda^2(\mathbb{R}^n) \rightarrow \mathbb{R}^n$ such that $\rho\{\Lambda^2(\mathbb{R}^n)\} \subseteq Z(\rho)$, where $Z(\rho) = \{x \in \mathbb{R}^n : \rho(x \wedge y) = 0 \text{ for all } y \text{ in } \mathbb{R}^n\}$. The center of ρ is defined to be the subspace $Z(\rho)$.

Notation

Let $\mathfrak{N}_2(n)$ denote the set of all 2-step nilpotent structures on \mathbb{R}^n .

1.2 Action of $GL(n, \mathbb{R})$ on $\mathfrak{N}_2(n)$

For every $g \in G = GL(n, \mathbb{R})$ and every $\rho \in \mathfrak{N}_2(n)$ let $g\rho \in \mathfrak{N}_2(n)$ be defined by $(g\rho)(\xi) = g(\rho(g^{-1}\xi))$ for all $\xi \in \Lambda^2(\mathbb{R}^n)$. Equivalently, if $[x, y] = \rho(x \wedge y)$ for x, y in \mathbb{R}^n , and $[\cdot, \cdot]_g = g([\cdot, \cdot])$, then $[x, y]_g = g([g^{-1}x, g^{-1}y])$ for all x, y in \mathbb{R}^n .

The action of $G = GL(n, \mathbb{R})$ on $\mathfrak{N}_2(n)$ is a left action, and the orbit $G(\rho)$ is the set of 2-step nilpotent structures in $\mathfrak{N}_2(n)$ that are Lie algebra isomorphic to ρ . The group $G_\rho = \{g \in G : g\rho = \rho\}$ is the automorphism group of the the Lie algebra ρ .

1.3 Abelian and nonabelian factors

It is useful to observe that one may always split off an abelian Lie algebra \mathfrak{E} from a 2-step nilpotent Lie algebra \mathfrak{N} and thereby reduce consideration to the case that $[\mathfrak{N}, \mathfrak{N}] = \mathfrak{Z}$, the center of \mathfrak{N} . We call the ideals \mathfrak{E} and \mathfrak{N}^* in the proposition below the abelian and nonabelian factors of the 2-step nilpotent Lie algebra \mathfrak{N} .

Proposition

Let \mathfrak{N} be a 2-step nilpotent Lie algebra with center \mathfrak{Z} . Then there exist ideals \mathfrak{N}^* and \mathfrak{E} of \mathfrak{N} with $\mathfrak{E} \subseteq \mathfrak{Z}$ such that

- 1) $\mathfrak{N} = \mathfrak{N}^* \oplus \mathfrak{E}$ and $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}] \oplus \mathfrak{E}$.
- 2) \mathfrak{N}^* is a 2-step nilpotent Lie algebra such that $[\mathfrak{N}, \mathfrak{N}] = [\mathfrak{N}^*, \mathfrak{N}^*] = \mathfrak{Z}^*$, the center of \mathfrak{N}^* .
- 3) The ideals \mathfrak{N}^* and \mathfrak{E} are uniquely determined up to isomorphism by 1).
- 4) $\mathfrak{E} = \{0\} \Leftrightarrow [\mathfrak{N}, \mathfrak{N}] = \mathfrak{Z}$.

Proof

We begin by proving 3). Suppose that we can write $\mathfrak{N} = \mathfrak{N}_1^* \oplus \mathfrak{E}_1 = \mathfrak{N}_2^* \oplus \mathfrak{E}_2$, where $\{\mathfrak{N}_1^*, \mathfrak{E}_1\}$ and $\{\mathfrak{N}_2^*, \mathfrak{E}_2\}$ satisfy hypothesis 1) of the Proposition. If \mathcal{V} is a subspace of \mathfrak{N} such that $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$, then $\mathfrak{N} = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}] \oplus \mathfrak{E}_i$ for $i = 1, 2$. Let $T : \mathfrak{N} \rightarrow \mathfrak{N}$ be a linear isomorphism such that $T = \text{Id}$ on $\mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$ and $T(\mathfrak{E}_1) = \mathfrak{E}_2$. It is easy to check that T is a Lie algebra isomorphism, and hence T induces a Lie algebra isomorphism $\bar{T} : \mathfrak{N} / \mathfrak{E}_1 \rightarrow \mathfrak{N} / \mathfrak{E}_2$. However, $\mathfrak{N} / \mathfrak{E}_1 \cong \mathfrak{N}_1^*$ and $\mathfrak{N} / \mathfrak{E}_2 \cong \mathfrak{N}_2^*$ by 1). This proves that $\mathfrak{N}_1^* \cong \mathfrak{N}_2^*$, and $\mathfrak{E}_1^* \cong \mathfrak{E}_2^*$ since by 1) \mathfrak{E}_1^* and \mathfrak{E}_2^* are abelian Lie algebras of the same dimension.

To prove the existence of \mathfrak{N}^* and \mathfrak{E} we choose \mathfrak{E} to be any subspace of \mathfrak{Z} such that $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}] \oplus \mathfrak{E}$. Let \mathcal{V} be a subspace of \mathfrak{N} such that $\mathfrak{N} = \mathcal{V} \oplus \mathfrak{Z}$. If $\mathfrak{N}^* = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$, then it is easy to check that \mathfrak{N}^* and \mathfrak{E} satisfy 1) and 2). Finally, 4) is an immediate consequence of 1).

Section 2 Free 2-step nilpotent Lie algebras

We define and discuss some important examples that can be regarded as universal covers in the collection of all 2-step nilpotent Lie algebras. See assertion 2) of proposition 2.2a.

2.1 Definition

A 2-step nilpotent Lie algebra \mathfrak{N} is said to be free on $q \geq 2$ generators if there exists a generating set $\{x_1, \dots, x_q\}$ for \mathfrak{N} with the following property :

(#) Let \mathfrak{N}^* be any 2-step nilpotent Lie algebra, and let $\{x_1^*, \dots, x_q^*\}$ be any subset of q elements in \mathfrak{N}^* . Then there exists a unique Lie algebra homomorphism $T : \mathfrak{N} \rightarrow \mathfrak{N}^*$ such that $T(x_i) = x_i^*$ for $1 \leq i \leq q$.

A generating set $\{x_1, \dots, x_q\}$ of the type above will be called an admissible generating set for the free 2-step nilpotent Lie algebra \mathfrak{N} .

2.2 Basic results

We list some basic results about free 2-step nilpotent Lie algebras on q generators and for each integer $q \geq 2$ we construct two models of such Lie algebras. Proofs of these results are given in Appendix 2 if they are not included in this section.

Let $\mathfrak{N}_2(q)$ denote the collection of all 2-step nilpotent Lie algebras on q generators. We show next that the elements of $\mathfrak{N}_2(q)$ are all isomorphic and have dimension $(1/2)q(q+1)$. Equivalently, the group $GL(N, \mathbb{R})$ acts transitively on $\mathfrak{N}_2(q)$, where $N = (1/2)q(q+1)$. In section 4 (Proposition 4.2b) we show that $\mathfrak{N}_2(q)$ has the

structure of a smooth manifold that is also a fiber bundle over a Grassmann manifold. The manifold $\mathfrak{N}_2(q)$ has two connected components, which are the orbits of the index two subgroup $GL(N, \mathbb{R})^+$.

2.2a Proposition

1) For every integer $q \geq 2$ there exists a free 2-step nilpotent Lie algebra \mathfrak{N} on q generators, and \mathfrak{N} is unique up to Lie algebra isomorphism. Moreover, $\dim \mathfrak{N} = (1/2)q(q+1)$.

2) Let \mathfrak{N}^* be a 2-step nilpotent Lie algebra such that $\dim \mathfrak{N}^* / [\mathfrak{N}^*, \mathfrak{N}^*] \leq q$. Then \mathfrak{N}^* is the homomorphic image of a free 2-step nilpotent Lie algebra on q generators.

Remark The proof of this result will show that a 2-step nilpotent Lie algebra \mathfrak{N}^* has a generating set with $r = \dim \mathfrak{N}^* / [\mathfrak{N}^*, \mathfrak{N}^*]$ elements, and every generating set for \mathfrak{N}^* has at least r elements.

2.2b Proposition Let $q \geq 2$ be an integer, and let \mathfrak{N} be a free 2-step nilpotent Lie algebra on q generators.

1) Any generating set for \mathfrak{N} contains at least q elements. Any generating set for \mathfrak{N} with exactly q elements is admissible.

2) If $\{x_1, \dots, x_q\}$ and $\{x_1^*, \dots, x_q^*\}$ are any two generating sets for \mathfrak{N} with q elements, then there exists a unique Lie algebra isomorphism $T : \mathfrak{N} \rightarrow \mathfrak{N}$ such that $T(x_i) = x_i^*$ for $1 \leq i \leq q$.

2.2c Proposition Let $q \geq 2$ be an integer, and let \mathfrak{N} be a free 2-step nilpotent Lie algebra on q generators. Let \mathfrak{Z} denote the center of \mathfrak{N} . Then

1) $[\mathfrak{N}, \mathfrak{N}] = \mathfrak{Z}$ has dimension $(1/2)q(q-1)$.

2) If $\{x_1, \dots, x_q\}$ is any generating set for \mathfrak{N} with q elements, then $\{[x_i, x_j] : 1 \leq i < j \leq q\}$ is a basis for \mathfrak{Z} .

2.2d Proposition (Homomorphism lifting property) Let \mathfrak{N} and \mathfrak{N}' be 2-step nilpotent Lie algebras with generating sets of at most q elements. Let $F_2(q)$ be a free 2-step nilpotent Lie algebra with q generators, and choose surjective homomorphisms $\rho : F_2(q) \rightarrow \mathfrak{N}$ and $\rho' : F_2(q) \rightarrow \mathfrak{N}'$. If $T : \mathfrak{N} \rightarrow \mathfrak{N}'$ is a Lie algebra homomorphism, then

1) There exists a Lie algebra homomorphism $\tilde{T} : F_2(q) \rightarrow F_2(q)$ such that $\rho' \circ \tilde{T} = T \circ \rho$.

2) If T is an isomorphism and $q = \dim \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}] = \dim \mathfrak{N}' / [\mathfrak{N}', \mathfrak{N}']$, then the Lie algebra homomorphism \tilde{T} from 1) is an isomorphism.

Other characterizations of free 2-step nilpotent Lie algebras

If $[\cdot, \cdot]$ is a 2-step nilpotent structure on \mathbb{R}^n associated to $\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$, then since $\text{Im } \rho \subseteq \mathfrak{Z}$, the center of \mathbb{R}^n , we may define an alternating bilinear map $[\cdot, \cdot]': (\mathbb{R}^n / \text{Im } \rho) \times (\mathbb{R}^n / \text{Im } \rho) \rightarrow \text{Im } \rho = [\mathbb{R}^n, \mathbb{R}^n] \subseteq \mathbb{R}^n$ by $[x + \text{Im } \rho, y + \text{Im } \rho]' = [x, y]$ for all $x, y \in \mathbb{R}^n$. Let $\rho': \Lambda^2(\mathbb{R}^n / \text{Im } \rho) \rightarrow \text{Im } \rho$ be the unique linear map such that $\rho'((x + \text{Im } \rho) \wedge (y + \text{Im } \rho)) = [x + \text{Im } \rho, y + \text{Im } \rho]' = [x, y] = \rho(x \wedge y)$ for all $x, y \in \mathbb{R}^n$. The map ρ' is obviously surjective, but ρ' is injective only under special circumstances.

2.2e Proposition

Let $\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$ be a 2-step nilpotent Lie algebra structure on \mathbb{R}^n , and let $\rho': \Lambda^2(\mathbb{R}^n / \text{Im } \rho) \rightarrow \text{Im } \rho$ be the associated surjective linear map. Then ρ' is an isomorphism $\Leftrightarrow \{\mathbb{R}^n, \rho\}$ is a free 2-step nilpotent Lie algebra on $q = n - \dim(\text{Im } \rho)$ generators. In this case $n = (1/2)q(q+1)$.

2.2f Corollary

Let $\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$ be a 2-step nilpotent Lie algebra structure on \mathbb{R}^n and let $p = \dim \text{Im } \rho$. If $q = n - p$ and $D = (1/2)q(q-1)$, then $p \leq D$ with equality $\Leftrightarrow \{\mathbb{R}^n, \rho\}$ is a free 2-step nilpotent Lie algebra on q generators.

Proof If $p = \dim \text{Im } \rho$, then $q = \dim \mathbb{R}^n / \text{Im } \rho$. By the work above $\text{Im } \rho = \rho(\Lambda^2(\mathbb{R}^n)) = \rho'(\Lambda^2(\mathbb{R}^n / \text{Im } \rho))$, and hence $p = \dim \text{Im } \rho \leq \dim \Lambda^2(\mathbb{R}^n / \text{Im } \rho) = D$. Equality holds $\Leftrightarrow \rho'$ is injective, but by Proposition 2.2e ρ' is injective $\Leftrightarrow \{\mathbb{R}^n, \rho\}$ is a free 2-step nilpotent Lie algebra on q generators. \square

The next result is a restatement of part of Proposition 4.2b.

2.2g Proposition

For any integer $q \geq 2$ the set $\mathfrak{N}_2(q)$ has the structure of a smooth manifold of dimension $(1/4)q^2(q^2-1)$ with two connected components. Moreover, $\mathfrak{N}_2(q)$ is a fiber bundle over $G(D, n)$, where $D = (1/2)q(q-1)$ and $n = (1/2)q(q+1)$.

2.3 Existence of a free 2-step nilpotent Lie algebra on q generators

Let $q \geq 2$ be given and let $D = (1/2)q(q-1)$. We construct two models of a free 2-step nilpotent Lie algebra on q generators, and we define an explicit isomorphism between them. In Appendix 2, in the proof of Proposition 2.2a, we show that the first model is a free 2-step nilpotent Lie algebra on q generators.

Model 1 Let $F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$ and let $[\cdot, \cdot]$ denote the 2-step nilpotent Lie algebra structure on $F_2(q)$ such that $\Lambda^2(\mathbb{R}^q)$ lies in the center of \mathbb{R}^q and $[v, w] = v \wedge w$ for all $v, w \in \mathbb{R}^q$. It is easy to check that $[F_2(q), F_2(q)] = \Lambda^2(\mathbb{R}^q)$ is the center of $F_2(q)$.

Model 2 Let $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$, where $\mathcal{SO}(q, \mathbb{R})$ denotes the set of $q \times q$ real skew symmetric matrices. Let $[\cdot, \cdot]^*$ denote the 2-step nilpotent Lie algebra structure on $F_2(q)^*$ such that $\mathcal{SO}(q, \mathbb{R})$ lies in the center of \mathbb{R}^q and $[v, w]^* = -(1/2) \{vw^t - wv^t\}$ for all $v, w \in \mathbb{R}^q$, where we regard the elements of \mathbb{R}^q as $q \times 1$ column vectors. It is again easy to check that $[F_2(q)^*, F_2(q)^*] = \mathcal{SO}(q, \mathbb{R})$ is the center of $F_2(q)^*$ since $[e_i, e_j]^* = (1/2)\{E_{ji} - E_{ij}\}$ for $1 \leq i, j \leq q$, where $\{e_1, \dots, e_q\}$ is the natural basis of \mathbb{R}^q and $E_{\alpha\beta}$ is the $q \times q$ matrix with 1 in position (α, β) and zeros elsewhere.

Isomorphism between the models

The map $\varphi: \mathbb{R}^q \times \mathbb{R}^q \rightarrow \mathcal{SO}(q, \mathbb{R})$ defined by $\varphi(v, w) = -(1/2) \{vw^t - wv^t\}$ is alternating and bilinear, and hence there is a unique linear map $T: \Lambda^2(\mathbb{R}^q) \rightarrow \mathcal{SO}(q, \mathbb{R})$ such that $T(v \wedge w) = -(1/2) \{vw^t - wv^t\} = [v, w]^*$ for all $v, w \in \mathbb{R}^q$. The map T is surjective by the remarks above, and hence T is an isomorphism since $\Lambda^2(\mathbb{R}^q)$ and $\mathcal{SO}(q, \mathbb{R})$ both have dimension $D = (1/2)q(q-1)$. If $F: F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q) \rightarrow \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R}) = F_2(q)^*$ is given by $F(v, \xi) = (v, T(\xi))$, then it is easy to check that F is a Lie algebra isomorphism.

2.4 A metric description of $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$

Proposition Let $\langle \cdot, \cdot \rangle^*$ denote the standard inner product on \mathbb{R}^q that makes the standard basis $\{e_1, \dots, e_q\}$ orthonormal. Let $\langle \cdot, \cdot \rangle$ denote the canonical inner product on $\mathcal{SO}(q, \mathbb{R})$ such that $\langle Z, Z' \rangle = -\text{trace}(ZZ')$ for all $Z, Z' \in \mathcal{SO}(q, \mathbb{R})$. Then $\langle [v, w]^*, Z \rangle = \langle Zv, w \rangle^*$ for all $v, w \in \mathbb{R}^q$ and all $Z \in \mathcal{SO}(q, \mathbb{R})$.

Proof Regarding the elements of \mathbb{R}^q as column vectors we note that $\langle u, v \rangle^* = \text{trace}(uv^t)$ for all $u, v \in \mathbb{R}^q$. Hence for $v, w \in \mathbb{R}^q$ and $Z \in \mathcal{SO}(q, \mathbb{R})$ we have

$$(*) \quad \langle v, Zw \rangle^* = \text{trace}(v(Zw)^t) = -\text{trace}(vw^tZ)$$

From (*) and the definition of $[\cdot, \cdot]^*$ we obtain $\langle [v, w]^*, Z \rangle = -\text{trace}\{[v, w]^*Z\} = (1/2)\text{trace}(vw^tZ) - (1/2)\text{trace}(wv^tZ) = -(1/2)\langle v, Zw \rangle^* + (1/2)\langle w, Zv \rangle^* = \langle Zv, w \rangle^*$ since Z is skew symmetric. \square

2.5 The automorphisms of $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$

As in (2.3) we regard \mathbb{R}^q as the set of $q \times 1$ column vectors, and we regard $GL(q, \mathbb{R})$ as the group of invertible real $q \times q$ matrices acting on \mathbb{R}^q by left multiplication.

Proposition For every automorphism φ of $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ there exist unique elements g of $GL(q, \mathbb{R})$ and S of $\text{Hom}(\mathbb{R}^q, \mathcal{SO}(q, \mathbb{R}))$ such that

- a) $\varphi(v) = gv + S(v)$ for all $v \in \mathbb{R}^q$.
 b) $\varphi(Z) = gZg^t$ for all $Z \in \mathcal{SO}(q, \mathbb{R})$.

Conversely, given $(g, S) \in GL(q, \mathbb{R}) \times \text{Hom}(\mathbb{R}^q, \mathcal{SO}(q, \mathbb{R}))$ there is a unique automorphism φ of $F_2(q)^*$ that satisfies a) and b).

In particular $\text{Aut}(F_2(q)^*)$ is a closed Lie subgroup of $GL((F_2(q)^*))$ of dimension $q^2 + qD$, where $D = (1/2)q(q-1)$.

Remark If we define $g(Z) = gZg^t$ for $Z \in \mathcal{SO}(q, \mathbb{R})$, then we obtain a left action of $GL(q, \mathbb{R})$ on $\mathcal{SO}(q, \mathbb{R})$. The set $GL(q, \mathbb{R}) \times \text{Hom}(\mathbb{R}^q, \mathcal{SO}(q, \mathbb{R}))$ becomes a group with unit $(\text{Id}, 0)$ under the multiplication $(g, S) \cdot (g^*, S^*) = (gg^*, S \circ g^* + g \circ S^*)$. The bijection $\varphi \rightarrow (g, S)$ defined above between $\text{Aut}(F_2(q)^*)$ and $GL(q, \mathbb{R}) \times \text{Hom}(\mathbb{R}^q, \mathcal{SO}(q, \mathbb{R}))$ now becomes an isomorphism of groups.

Proof of the proposition Given an automorphism φ of $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ there are unique elements g, h of $\text{End}(\mathbb{R}^q)$ and S, T of $\text{Hom}(\mathbb{R}^q, \mathcal{SO}(q, \mathbb{R}))$ such that a) holds for φ and φ^{-1} respectively. It is easy to verify that g and h are inverses and hence are elements of $GL(n, \mathbb{R})$.

We now show that b) holds for φ and g as well. Given elements v, w of \mathbb{R}^q we know that $[v, w]^* = - (1/2) \{vw^t - wv^t\}$ by the definition of the bracket $[,]^*$ in $F_2(q)^*$. Since φ is a Lie algebra homomorphism we obtain $\varphi([v, w]^*) = [\varphi(v), \varphi(w)]^* = [gv + S(v), gw + S(w)]^* = [gv, gw]^* = - (1/2) \{gv(gw)^t - gw(gv)^t\} = - (1/2) \{g(vw^t - wv^t)g^t\} = g \{ - (1/2) \{vw^t - wv^t\} \} g^t = g [v, w]^* g^t$. Hence b) holds for all linear combinations of elements Z in $\mathcal{SO}(q, \mathbb{R})$ of the form $Z = [v, w]^*$, where v, w are arbitrary elements of \mathbb{R}^q . Since $[F_2(q)^*, F_2(q)^*] = \mathcal{SO}(q, \mathbb{R})$ it follows that b) holds for all Z in $\mathcal{SO}(q, \mathbb{R})$.

Conversely, if we are given (g, S) in $GL(q, \mathbb{R}) \times \text{Hom}(\mathbb{R}^q, \mathcal{SO}(q, \mathbb{R}))$, then we may define a linear isomorphism $\varphi : F_2(q)^* \rightarrow F_2(q)^*$ uniquely by conditions a) and b) above. The argument of the previous paragraph shows that $\varphi([v, w]^*) = [\varphi(v), \varphi(w)]^*$ for all $v, w \in \mathbb{R}^q$, and this proves that φ is an automorphism of $F_2(q)^*$. \square

Section 3

Models for 2-step nilpotent Lie algebras

3.1 The structure constants and the structure space

Let \mathfrak{N} be a 2-step nilpotent Lie algebra with dimension $p+q$, where $p = \dim [\mathfrak{N}, \mathfrak{N}]$. Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ be a basis of \mathfrak{N} such that $\{Z_1, \dots, Z_p\}$ is a basis of $[\mathfrak{N}, \mathfrak{N}]$. We call \mathfrak{B} an adapted basis of \mathfrak{N} . Let $\{C^1, \dots, C^p\}$ be the skew symmetric real $q \times q$ matrices such that $[v_i, v_j] = \sum_{k=1}^p C_{ij}^k Z_k$ for $1 \leq i, j \leq q$. We call the matrices

$\{C^1, \dots, C^p\}$ the structure matrices determined by \mathfrak{B} and the space $W = \text{span} \{C^1, \dots, C^p\}$ the structure space determined by \mathfrak{B} .

We show first that the structure space W has dimension p by showing that the matrices $\{C^1, C^2, \dots, C^p\}$ are linearly independent in $so(q, \mathbb{R})$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_p\}$ be real numbers such that $0 = \sum_{k=1}^p \alpha_k C^k$. Let $Z^* = \sum_{k=1}^p \alpha_k Z_k$. Define $\langle \cdot, \cdot \rangle$ to be the inner product on $[\mathfrak{N}, \mathfrak{N}]$ that makes $\{Z_1, Z_2, \dots, Z_p\}$ an orthonormal basis. Then for $1 \leq r, s \leq p$ we have $\langle [v_r, v_s], Z^* \rangle = \sum_{k=1}^p \alpha_k C_{rs}^k = \left(\sum_{k=1}^p \alpha_k C^k \right)_{rs} = 0$. Hence Z^* is orthogonal to $[\mathfrak{N}, \mathfrak{N}] = \text{span}\{Z_1, Z_2, \dots, Z_p\}$. It follows that $Z^* = 0$, which implies that $\alpha_k = 0$ for all k . \square

Next we show that the collection of structure spaces W , taken over all adapted bases \mathfrak{B} of a 2-step nilpotent Lie algebra \mathfrak{N} , is a single $GL(q, \mathbb{R})$ orbit in the Grassmann manifold $G(p, so(q, \mathbb{R}))$ of p -dimensional subspaces of $so(q, \mathbb{R})$,

Proposition Let \mathfrak{N} be a 2-step nilpotent Lie algebra with dimension $p+q$, where $p = \dim [\mathfrak{N}, \mathfrak{N}]$. Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ and $\mathfrak{B}' = \{v'_1, \dots, v'_p, Z_1, \dots, Z_p\}$ be adapted bases of \mathfrak{N} , and let $W = \text{span} \{C^1, \dots, C^p\}$ and $W' = \text{span} \{C'^1, \dots, C'^p\}$ be the structure spaces defined by \mathfrak{B} and \mathfrak{B}' . Let $g \in GL(q, \mathbb{R})$ be the nonsingular $q \times q$ matrix such that $v'_i = \sum_{k=1}^q g_{ij} v_k$. Then $gWg^t = W'$.

Conversely, given an adapted basis $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ of \mathfrak{N} and an element g of $GL(q, \mathbb{R})$ let $\mathfrak{B}' = \{v'_1, \dots, v'_p, Z_1, \dots, Z_p\}$, where $v'_i = \sum_{k=1}^q g_{ij} v_k$. Then $gWg^t = W'$, where W' is the structure space defined by the adapted basis \mathfrak{B}' of \mathfrak{N} .

Proof Suppose first that \mathfrak{B} , \mathfrak{B}' and $g \in GL(q, \mathbb{R})$ are given as above. Define $h \in GL(p, \mathbb{R})$ by $Z'_k = \sum_{r=1}^p h_{kr} Z_r$. From the definition of h and the structure matrices $\{C^1, \dots, C^p\}$ a routine computation yields, for $1 \leq i, j \leq p$,

$$(1) \quad [v'_i, v'_j] = \sum_{r=1}^p \left(\sum_{k=1}^p h_{kr} C^k \right)_{ij} Z_r.$$

From the definition of g and the structure matrices $\{C^1, \dots, C^p\}$ another computation yields

$$(2) \quad [v'_i, v'_j] = \sum_{r=1}^p \left\{ \sum_{k, \ell=1}^q g_{ik} C_{k\ell}^r g_{j\ell} \right\} Z_r = \sum_{r=1}^p \{g C^r g^t\}_{ij} Z_r$$

Comparing (1) and (2) yields

$$(3) \quad g C^r g^t = \sum_{k=1}^p h_{kr} C^k \quad \text{for } 1 \leq r \leq p$$

Hence $gWg^t = \text{span} \{ g C^r g^t : 1 \leq r \leq p \} = \text{span} \{ \sum_{k=1}^p h_{kr} C^k : 1 \leq r \leq p \} \subseteq$

$\text{span} \{ C^1, \dots, C^p \} = W'$. Equality holds since both W and W' have dimension p in $\mathcal{SO}(q, \mathbb{R})$ and the map $Z \rightarrow gZg^t$ is an invertible linear map of $\mathcal{SO}(q, \mathbb{R})$.

Conversely, let an adapted basis $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ of \mathfrak{N} and an element g of $GL(q, \mathbb{R})$ be given. Let $\mathfrak{B}' = \{v_1', \dots, v_p', Z_1, \dots, Z_p\}$, where $v_i' = \sum_{k=1}^q g_{ik} v_k$. Then the argument above shows that $gWg^t = W'$, where W' is the structure space defined by the adapted basis \mathfrak{B}' . \square

3.2 Standard metric 2-step nilpotent Lie algebras

We describe a simple family of 2-step nilpotent Lie algebras that come equipped with an inner product. We show that every 2-step nilpotent Lie algebra is isomorphic to one of the elements in this family. We have seen already in (2.4) that $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ belongs to this family.

3.2a Definition

Let $\mathcal{SO}(q, \mathbb{R})$ denote the real $q \times q$ skew symmetric matrices. Let p be an integer with $1 \leq p \leq D = (1/2)q(q-1)$, and let W be a p -dimensional subspace of $\mathcal{SO}(q, \mathbb{R})$. Let \mathfrak{N} denote the vector space $\mathbb{R}^q \oplus W$, where the elements of \mathbb{R}^q are regarded as column vectors. Fix an inner product $\langle \cdot, \cdot \rangle^*$ on \mathbb{R}^q , and let $\langle \cdot, \cdot \rangle$ be the canonical inner product on $\mathcal{SO}(q, \mathbb{R})$ given by $\langle Z, Z^* \rangle = -\text{trace}(ZZ^*)$. Let $\langle \cdot, \cdot \rangle$ also denote the inner product on $\mathfrak{N} = \mathbb{R}^q \oplus W$ which extends the inner products on the subspaces \mathbb{R}^q and W and which makes these subspaces orthogonal. Let $[\cdot, \cdot]$ denote the 2-step nilpotent Lie algebra structure on \mathfrak{N} such that W lies in the center of \mathfrak{N} , and $\langle [v, w], Z \rangle = \langle Zv, w \rangle^*$ for all $v, w \in \mathbb{R}^q$ and all $Z \in \mathcal{SO}(q, \mathbb{R})$. We call $\{\mathbb{R}^q \oplus W, [\cdot, \cdot]\}$ a standard metric 2-step nilpotent Lie algebra.

It is not difficult to show that if $\mathfrak{N} = \mathbb{R}^q \oplus W$ is a standard metric 2-step nilpotent Lie algebra, then $[\mathfrak{N}, \mathfrak{N}] = W$. Moreover, if \mathfrak{Z} denotes the center of \mathfrak{N} , then $\mathfrak{Z} = W \oplus \{v \in \mathbb{R}^q : Zv = 0 \text{ for all } Z \in W\}$.

3.2b Prescribing structure constants

Proposition Let p be an integer with $1 \leq p \leq D = (1/2)q(q-1)$. Let $\{e_1, \dots, e_q\}$ denote the natural basis of \mathbb{R}^q . Let W be a p -dimensional subspace of $\mathcal{SO}(q, \mathbb{R})$, and let $\mathfrak{N} = \mathbb{R}^q \oplus W$ be the corresponding standard metric 2-step nilpotent Lie algebra.

Let $\{C^1, \dots, C^p\}$ be a basis of W . Then there exists a basis $\{\rho_1, \dots, \rho_p\}$ of W such that $\mathfrak{B} = \{e_1, \dots, e_q, \rho_1, \dots, \rho_p\}$ is a basis for \mathfrak{N} with structure matrices $\{C^1, \dots, C^p\}$; that is, $[e_i, e_j] = \sum_{k=1}^q C_{ij}^k \rho_k$.

Proof Let \langle, \rangle denote the canonical inner product on $\mathcal{SO}(q, \mathbb{R})$ defined above, and let $\{\rho_1, \dots, \rho_p\}$ be the basis of W such that $\langle \rho_\alpha, C^\beta \rangle = -\delta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq p$. We show that $\mathfrak{B} = \{e_1, \dots, e_q, \rho_1, \dots, \rho_p\}$ is the desired basis of W .

Let $\{D^1, \dots, D^p\}$ be the skew symmetric structure matrices defined by the bracket relations of the basis \mathfrak{B} ; that is $[e_i, e_j] = \sum_{k=1}^q D_{ij}^k \rho_k$. Then $\langle [e_i, e_j], C^\beta \rangle =$

$$\sum_{\alpha=1}^q D_{ij}^\alpha \langle \rho_\alpha, C^\beta \rangle = -D_{ij}^\beta = D_{ji}^\beta. \text{ On the other hand } \langle [e_i, e_j], C^\beta \rangle = \langle C^\beta(e_i), e_j \rangle = C_{ji}^\beta. \text{ Hence } C^\beta = D^\beta \text{ for } 1 \leq \beta \leq p.$$

3.3 Reduction to standard form

The next result is Proposition 2.6 from [Eb], but we include the proof here for completeness.

Proposition Let \mathfrak{N} be a 2-step nilpotent Lie algebra of dimension $p+q$ such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension p . Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ be an adapted basis for \mathfrak{N} with structure matrices $\{C^1, \dots, C^p\}$ and structure space $W = \text{span}\{C^1, \dots, C^p\}$. Then \mathfrak{N} is isomorphic to the standard metric 2-step nilpotent Lie algebra $\mathbb{R}^q \oplus W$.

For $\mathfrak{N} = \mathbb{R}^q \oplus W$ we observed above that $[\mathfrak{N}, \mathfrak{N}] = W$, a p -dimensional subspace of the D -dimensional vector space $\mathcal{SO}(q, \mathbb{R})$, where $D = (1/2)q(q-1)$. From this observation and the discussion in (2.3) of models for free 2-step nilpotent Lie algebras we obtain another proof of Corollary (2.2f), namely

Corollary Let \mathfrak{N} be a 2-step nilpotent Lie algebra of dimension $p+q$ such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension p . Then $p \leq D = (1/2)q(q-1)$, with equality $\Leftrightarrow \mathfrak{N}$ is a free 2-step nilpotent Lie algebra on q generators.

Proof of the Proposition The discussion in (3.1) shows that $\{C^1, \dots, C^p\}$ is a basis for W . Let $\{\rho_1, \dots, \rho_p\}$ be the basis of W such that $\langle \rho_\alpha, C^\beta \rangle = -\delta_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq p$. Let $\{e_1, \dots, e_q\}$ be the standard basis for \mathbb{R}^q , and let $[\cdot, \cdot]^*$ denote the Lie bracket in $\mathfrak{N}^* = \mathbb{R}^q \oplus W$. Let $T: \mathfrak{N} \rightarrow \mathfrak{N}^*$ be the unique linear isomorphism such that $T(v_i) = e_i$ for $1 \leq i \leq q$ and $T(Z_\alpha) = \rho_\alpha$ for $1 \leq \alpha \leq p$. It is routine to check that $T([v_i, v_j])$ and $[e_i, e_j]^*$ have the same inner product in $\mathcal{SO}(q, \mathbb{R})$ with C^α for $1 \leq i, j \leq q$ and $1 \leq \alpha \leq p$. Since $\{C^1, \dots,$

C^p spans W it follows that $T([v_i, v_j]) = [e_i, e_j]^* = [T(v_i), T(v_j)]^*$ for $1 \leq i, j \leq q$, which proves that T is a Lie algebra isomorphism. \square

(3.4) Quotients of free 2-step nilpotent Lie algebras

Proposition Let \mathfrak{N} be a 2-step nilpotent Lie algebra of dimension $p+q$ such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension $p \geq 1$. Let $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ be an adapted basis for \mathfrak{N} with structure matrices $\{C^1, \dots, C^p\}$ and structure space $W = \text{span} \{C^1, \dots, C^p\} \subseteq \mathcal{SO}(q, \mathbb{R})$. Let $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ be the free 2-step nilpotent Lie algebra on q generators, and let $\{e_1, \dots, e_q\}$ denote the standard basis of \mathbb{R}^q .

Let $\rho : F_2(q)^* \rightarrow \mathfrak{N}$ be the unique Lie algebra homomorphism such that $\rho(e_i) = v_i$ for $1 \leq i \leq q$. Then ρ is surjective and $\text{Ker } \rho = W^\perp$, the orthogonal complement of W in $\mathcal{SO}(q, \mathbb{R})$ relative to the canonical inner product \langle, \rangle .

Proof Since $\mathfrak{B} = \{v_1, \dots, v_q, Z_1, \dots, Z_p\}$ is an adapted basis for \mathfrak{N} it is clear that $\text{span} \{Z_1, \dots, Z_p\} = [\mathfrak{N}, \mathfrak{N}] = \text{span} \{[v_i, v_j] : 1 \leq i, j \leq q\}$. Hence the vectors $\{v_1, \dots, v_q\}$ generate \mathfrak{N} , and it follows immediately that ρ is surjective.

Let $[\cdot, \cdot]^*$ denote the canonical Lie bracket in $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ defined by $[v, w]^* = -(1/2) \{vw^t - wv^t\}$ for all $v, w \in \mathbb{R}^q$. Let $Z_{ij} = [e_i, e_j]^* = (1/2) \{E_{ji} - E_{ij}\}$, where $E_{\alpha\beta}$ is the $q \times q$ matrix with 1 in position (α, β) and zeros elsewhere. Then $\rho(Z_{ij}) = [v_i, v_j] = \sum_{k=1}^p C_{ij}^k Z_k$. Given an element $A = (a_{ij})$ of $\mathcal{SO}(q, \mathbb{R})$, we write $A = -\sum_{i,j=1}^q a_{ij} Z_{ij}$, and we

$$\text{compute } \rho(A) = -\sum_{k=1}^p \left\{ \sum_{i,j=1}^q a_{ij} C_{ij}^k \right\} Z_k = \sum_{k=1}^p \text{trace}(AC^k) Z_k = -\sum_{k=1}^p \langle A, C^k \rangle Z_k.$$

Since $W = \text{span} \{C^1, \dots, C^p\}$ and $\{Z_1, \dots, Z_p\}$ is a basis for $[\mathfrak{N}, \mathfrak{N}]$ it follows that $A \in \text{Ker } \rho \Leftrightarrow \langle A, C^k \rangle = 0$ for $1 \leq k \leq p \Leftrightarrow A \in W^\perp$. \square

3.5 Isomorphism classes

Let \mathfrak{N} be a 2-step nilpotent Lie algebra of dimension $p+q$ such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension $p \geq 1$. In (3.4) we saw that \mathfrak{N} is isomorphic to $F_2(q)^* / W$, where $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$, W is a $(D-p)$ -dimensional subspace of $\mathcal{SO}(q, \mathbb{R})$ and $D = (1/2)q(q-1)$. We now state a criterion for determining when two such quotients are isomorphic as Lie algebras.

3.5a Proposition Let p be an integer with $1 \leq p \leq D-1$, where $D = (1/2)q(q-1)$. Let W_1, W_2 be p -dimensional subspaces of $\mathcal{SO}(q, \mathbb{R})$. Then the Lie algebras $F_2(q)^* / W_1$ and $F_2(q)^* / W_2$ are isomorphic \Leftrightarrow there exists an element g of $GL(q, \mathbb{R})$ such that $gW_1g^t = W_2$.

Remark Since $\mathcal{SO}(q, \mathbb{R})$ is the center of $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ the subspaces W_1 and W_2 are ideals of $F_2(q)^*$.

Proof

We suppose first that there exists an element g of $GL(q, \mathbb{R})$ such that $g(W_1) = W_2$. By (2.5) there exist inverse automorphisms \tilde{T} and \tilde{S} of $F_2(q)^*$ such that $\tilde{T} = g$ and $\tilde{S} = g^{-1}$ on $\mathcal{SO}(q, \mathbb{R})$. For $i = 1, 2$ let $\mathfrak{N}_i = F_2(q)^* / W_i$, and let $\rho_i : F_2(q)^* \rightarrow \mathfrak{N}_i$ be the projection homomorphism with $\text{Ker } \rho_i = W_i$. Since $\tilde{T}(\text{Ker } \rho_1) = \text{Ker } \rho_2$ and $\tilde{S}(\text{Ker } \rho_2) = \text{Ker } \rho_1$ there exist Lie algebra homomorphisms $T : \mathfrak{N}_1 \rightarrow \mathfrak{N}_2$ and $S : \mathfrak{N}_2 \rightarrow \mathfrak{N}_1$ such that $\rho_2 \circ \tilde{T} = T \circ \rho_1$ and $\rho_1 \circ \tilde{S} = S \circ \rho_2$. Hence $S \circ T \circ \rho_1 = S \circ \rho_2 \circ \tilde{T} = \rho_1 \circ \tilde{S} \circ \tilde{T} = \rho_1$, which proves that $S \circ T$ is the identity on \mathfrak{N}_1 . A similar argument shows that $T \circ S$ is the identity on \mathfrak{N}_2 , which proves that S and T are inverse isomorphisms.

Next, we assume that $\mathfrak{N}_1 = F_2(q)^* / W_1$ and $\mathfrak{N}_2 = F_2(q)^* / W_2$ are isomorphic Lie algebras, and for $i = 1, 2$ we let $\rho_i : F_2(q)^* \rightarrow \mathfrak{N}_i$ denote the surjective Lie algebra homomorphism with $\text{Ker } \rho_i = W_i$. Let $T : \mathfrak{N}_1 \rightarrow \mathfrak{N}_2$ be a Lie algebra isomorphism. By 1) of Proposition 2.2d there exists a Lie algebra homomorphism $\tilde{T} : F_2(q)^* \rightarrow F_2(q)^*$ such that $\rho_2 \circ \tilde{T} = T \circ \rho_1$.

It suffices to prove that \tilde{T} is an automorphism of $F_2(q)^*$. From (2.5) it will then follow that there exists $g \in GL(q, \mathbb{R})$ such that $\tilde{T}(Z) = g(Z) = gZg^t$ for all $Z \in \mathcal{SO}(q, \mathbb{R})$. In particular $g(W_1) = g(\text{Ker } \rho_1) = \tilde{T}(\text{Ker } \rho_1) = \text{Ker } \rho_2 = W_2$ since W_1, W_2 are subspaces of $\mathcal{SO}(q, \mathbb{R})$ and $\rho_2 \circ \tilde{T} = T \circ \rho_1$.

To prove that \tilde{T} is an automorphism of $F_2(q)^*$ it suffices by 2) of Proposition 2.2d to prove that $q = \dim \mathfrak{N}_i / [\mathfrak{N}_i, \mathfrak{N}_i]$ for $i = 1, 2$. We show first that $\rho_i^{-1}([\mathfrak{N}_i, \mathfrak{N}_i]) = \mathcal{SO}(q, \mathbb{R})$ for $i = 1, 2$. For $i = 1, 2$ note that $\rho_i(\mathcal{SO}(q, \mathbb{R})) = \rho_i[F_2(q)^*, F_2(q)^*] = [\mathfrak{N}_i, \mathfrak{N}_i]$ since $\rho_i : F_2(q)^* \rightarrow \mathfrak{N}_i$ is surjective. This proves that $\mathcal{SO}(q, \mathbb{R}) \subseteq \rho_i^{-1}([\mathfrak{N}_i, \mathfrak{N}_i])$. Conversely, if $\xi \in \rho_i^{-1}([\mathfrak{N}_i, \mathfrak{N}_i])$ for $i = 1$ or 2 , then choose $Z \in \mathcal{SO}(q, \mathbb{R})$ such that $\rho_i(\xi) = \rho_i(Z)$. Then $\xi - Z \in \text{Ker } \rho_i = W_i \subseteq \mathcal{SO}(q, \mathbb{R})$, which proves that $\xi \in \mathcal{SO}(q, \mathbb{R})$ and $\rho_i^{-1}([\mathfrak{N}_i, \mathfrak{N}_i]) \subseteq \mathcal{SO}(q, \mathbb{R})$.

For $i = 1, 2$ let $\pi_i : \mathfrak{N}_i \rightarrow \mathfrak{N}_i / [\mathfrak{N}_i, \mathfrak{N}_i]$ denote the projection homomorphism. For $i = 1, 2$ the map $\pi_i \circ \rho_i : F_2(q)^* \rightarrow \mathfrak{N}_i / [\mathfrak{N}_i, \mathfrak{N}_i]$ is surjective, and $\text{Ker } (\pi_i \circ \rho_i) = \rho_i^{-1}([\mathfrak{N}_i, \mathfrak{N}_i]) = \mathcal{SO}(q, \mathbb{R})$ by the discussion above. Hence $\mathfrak{N}_i / [\mathfrak{N}_i, \mathfrak{N}_i]$ is isomorphic to $F_2(q)^* / \mathcal{SO}(q, \mathbb{R}) \approx \mathbb{R}^q$, and in particular $\dim \mathfrak{N}_i / [\mathfrak{N}_i, \mathfrak{N}_i] = q$ for $i = 1, 2$. \square

For a subspace W of $\mathcal{SO}(q, \mathbb{R})$ let W^\perp denote the orthogonal complement of W in $\mathcal{SO}(q, \mathbb{R})$ relative to the canonical inner product $\langle \cdot, \cdot \rangle$ defined in (2.4).

3.5b Proposition Let p be an integer with $1 \leq p \leq D-1$, where $D = (1/2)q(q-1)$. Let W_1, W_2 be p -dimensional subspaces of $\mathcal{SO}(q, \mathbb{R})$. Then the Lie algebras $F_2(q)^* / W_1$ and $F_2(q)^* / W_2$ are isomorphic \Leftrightarrow the Lie algebras $F_2(q)^* / W_1^\perp$ and $F_2(q)^* / W_2^\perp$ are isomorphic.

Proof

If $F_2(q)^* / W_1$ is Lie algebra isomorphic to $F_2(q)^* / W_2$, then $g(W_1) = W_2$ for some element $g \in GL(q, \mathbb{R})$ by Proposition 3.5a. It follows that $h(W_1^\perp) = W_2^\perp$, where $h = (g^t)^{-1} = (g^{-1})^t$, and Proposition 3.5a now says that $F_2(q)^* / W_1^\perp$ is Lie algebra isomorphic to $F_2(q)^* / W_2^\perp$. Conversely, since $(W_i^\perp)^\perp = W_i$ for $i = 1, 2$ this argument also shows that if $F_2(q)^* / W_1^\perp$ is Lie algebra isomorphic to $F_2(q)^* / W_2^\perp$, then $F_2(q)^* / W_1$ is Lie algebra isomorphic to $F_2(q)^* / W_2$. \square

3.5c Corollary Let W_1 and W_2 be p -dimensional subspaces of $\mathcal{SO}(q, \mathbb{R})$, and let $\mathfrak{N}_1 = \mathbb{R}^q \oplus W_1$ and $\mathfrak{N}_2 = \mathbb{R}^q \oplus W_2$ be the corresponding standard metric 2-step nilpotent Lie algebras. Then \mathfrak{N}_1 is isomorphic to $\mathfrak{N}_2 \Leftrightarrow$ there exists $g \in GL(q, \mathbb{R})$ such that $gW_1g^t = W_2$.

Proof By the proposition in (3.4) \mathfrak{N}_i is isomorphic to $F_2(q)^* / W_i^\perp$ for $i = 1, 2$. The assertion now follows immediately from Propositions 3.5a and 3.5b. \square

(3.6) The case of a 1-dimensional commutator ideal

Let \mathfrak{N} be a 2-step nilpotent Lie algebra with a 1-dimensional commutator ideal $[\mathfrak{N}, \mathfrak{N}]$. By the proposition in (3.3) \mathfrak{N} is isomorphic to $\mathbb{R}^q \oplus W$ for some integer $q \geq 2$, where W is a 1-dimensional subspace of $\mathcal{SO}(q, \mathbb{R})$. If $O = \{Z \in \mathcal{SO}(q, \mathbb{R}) : \text{rank } Z \text{ is maximal}\}$, then O is Zariski open in $\mathcal{SO}(q, \mathbb{R})$ and standard linear algebra shows that $O(q, \mathbb{R})$ acts transitively by conjugation on O .

Let $\mathfrak{N}_O(1, q) = \{\mathbb{R}^q \oplus \mathbb{R}Z : Z \in O\}$. Since $gZg^t = gZg^{-1}$ for $g \in O(q, \mathbb{R})$ it follows from Proposition 3.5a that the elements of $\mathfrak{N}_O(1, q)$ are all isomorphic.

If $q = 2k$, then the elements of O are nonsingular. The elements of $\mathfrak{N}_O(1, q)$ are isomorphic to the $(2k+1)$ -dimensional Heisenberg algebra that is typically described by the basis $\{X_1, Y_1, \dots, X_k, Y_k, Z\}$ and the nonzero bracket relations $[X_i, Y_i,] = -[Y_i, X_i,] = Z$ for $1 \leq i \leq k$. In this case \mathfrak{N} has a 1-dimensional center $\mathfrak{Z} = [\mathfrak{N}, \mathfrak{N}]$ spanned by Z .

If $q = 2k+1$, then the elements Z of O have a 1-dimensional kernel in \mathbb{R}^q . If A spans the kernel of Z and Z spans $W = [\mathfrak{N}, \mathfrak{N}]$, then $\mathfrak{N} = \mathbb{R}^q \oplus W$ has a basis $\{X_1, Y_1, \dots,$

X_k, Y_k, A, Z }, where $\{X_1, Y_1, \dots, X_k, Y_k, A\}$ spans \mathbb{R}^q , $\{A, Z\}$ spans the center \mathfrak{Z} and the nonzero bracket relations of the basis are given by $[X_i, Y_i,] = -[Y_i, X_i,] = Z$ for $1 \leq i \leq k$. Alternatively, we may say that if $Z \in \mathfrak{O}$ and $W = \mathbb{R}Z$, then $\mathfrak{N} = \mathbb{R}^q \oplus W = \mathfrak{N}^* \oplus \mathbb{R}$, where \mathfrak{N}^* is a Heisenberg algebra of dimension $2k+1$ and \mathbb{R} is a 1-dimensional abelian factor.

Section 4 Stratification of the space of 2-step nilpotent structures on \mathbb{R}^n

We recall from (1.1) that $\mathfrak{N}_2(n)$ denotes the set of all 2-step nilpotent structures on \mathbb{R}^n . There are two basic ways to stratify the space $\mathfrak{N}_2(n)$ into sets that are invariant under $GL(n, \mathbb{R})$. One way is defined in terms of the dimension of $[\mathbb{R}^n, \mathbb{R}^n]$ and the other in terms of the dimension of the center of \mathbb{R}^n . The results of section 3 suggest that the first stratification is the most natural. One may further choose to define strata of $\mathfrak{N}_2(n)$ that have the structure of a real algebraic variety or the structure of a smooth manifold that fibers over a Grassmann manifold.

We make the statements above more precise by defining the strata and some of their qualitative properties. We postpone the proofs of these statements until section 6.

4.1 Varieties of 2-step nilpotent structures

In this section p, q, n will always denote positive integers.

4.1a Proposition

Let $\mathfrak{N}_2(n)$ denote the space of 2-step nilpotent structures on \mathbb{R}^n . Then $\mathfrak{N}_2(n)$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$ that is invariant under $GL(n, \mathbb{R})$.

4.1b Proposition

Let $Z_2(p, q) = \{\rho \in \mathfrak{N}_2(p+q) : \dim Z(\rho) \geq p\}$. Then $Z_2(p, q)$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^{p+q}), \mathbb{R}^{p+q})$ that is invariant under $GL(p+q, \mathbb{R})$.

4.1c Proposition

Let $R_2(p, q) = \{\rho \in \mathfrak{N}_2(p+q) : \text{rank}(\rho) \leq p\}$. Then $R_2(p, q)$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^{p+q}), \mathbb{R}^{p+q})$ that is invariant under $GL(p+q, \mathbb{R})$.

4.2 Manifolds of 2-step nilpotent structures

Notation

We standardize some notation that will be used in the discussion of the next three results. Let $G(k, n)$ denote the Grassmann manifold of k -dimensional subspaces of \mathbb{R}^n . Let $p \geq 1$ and $q \geq 2$ be integers and let $D = (1/2)q(q-1)$. Let U be the vector space $\mathfrak{so}(q, \mathbb{R}) \times \dots$

$\times \mathfrak{so}(q, \mathbb{R})$ (p times). Recall that $\mathfrak{N}_2(q)$ denotes the set of 2-step nilpotent Lie algebra structures on $\mathbb{R}^{D+q} = \mathbb{R}^{(1/2)q(q+1)}$ that are free on q generators.

The next three results describe a stratification of $\mathfrak{N}_2(p+q)$ into strata that are smooth manifolds that fiber over a Grassmann manifold. These strata are Zariski open subsets of the varieties above and are also invariant under the action of $GL(p+q, \mathbb{R})$.

4.2a Proposition

Let $p \geq 2$ and $q \geq 2$. Let $Z_2^0(p, q) = \{\rho \in \mathfrak{N}_2(p+q) : \dim Z(\rho) = p\}$. Then $Z_2^0(p, q)$ is a connected smooth manifold of dimension $pq + pD$ and a fiber bundle over $G(p, p+q)$. Moreover, $Z_2^0(p, q)$ is a Zariski open subset of $Z_2(p, q)$.

4.2b Proposition

Let $2 \leq p \leq D$ and $q \geq 3$. Let $\mathfrak{N}(p, q) = \{\rho \in \mathfrak{N}_2(p+q) : \dim \text{rank}(\rho) = p\}$. Then $\mathfrak{N}(p, q)$ is a smooth manifold of dimension $pq + pD$ and a fiber bundle over $G(p, p+q)$. Moreover, $\mathfrak{N}(p, q)$ is a Zariski open subset of $R_2(p, q)$, and $\mathfrak{N}(p, q)$ is connected if $p \neq D$. If $p = D$, then $\mathfrak{N}(p, q) = \mathfrak{N}_2(q)$, and $\mathfrak{N}(p, q)$ has two connected components.

4.2c Proposition

Let $2 \leq p \leq D$ and $q \geq 3$. Let $\mathfrak{N}_0(p, q) = \mathfrak{N}(p, q) \cap Z_2^0(p, q) = \{\rho \in \mathfrak{N}_2(p+q) : \text{Im}(\rho) = Z(\rho) \in G(p, p+q)\}$. Then $\mathfrak{N}_0(p, q)$ is a smooth manifold of dimension $pq + pD$ and a fiber bundle over $G(p, p+q)$. Moreover, $\mathfrak{N}_0(p, q)$ is a Zariski open subset of both $R_2(p, q)$ and $Z_2(p, q)$, and $\mathfrak{N}_0(p, q)$ is connected if $p \neq D$. If $p = D$, then $\mathfrak{N}_0(p, q) = \mathfrak{N}_2(q)$.

Remark

The restriction $2 \leq p \leq D$ and $q \geq 3$ is inessential. We observed already in Proposition 2.2f or the corollary in (3.3) that $p \leq D$ always holds. If $q = 2$, then $D=1$ and $p = 1$, which is the 3-dimensional Heisenberg Lie algebra and also the free 2-step nilpotent Lie algebra on two generators.

Section 5 Duality and isomorphism classes

5.1 Duality

If $\langle \cdot, \cdot \rangle$ is the natural inner product on $\mathcal{SO}(q, \mathbb{R})$ given by $\langle Z, Z^* \rangle = -\text{trace}(ZZ^*)$, then the map ${}^\perp : W \rightarrow W^\perp$ is a diffeomorphism of $G(p, \mathcal{SO}(q, \mathbb{R}))$ onto $G(D-p, \mathcal{SO}(q, \mathbb{R}))$. Given $W \in G(p, \mathcal{SO}(q, \mathbb{R}))$ and $g \in SL(q, \mathbb{R})$ it is routine to check that if $g(W) = W^*$, then $h(W^\perp) = W^{*\perp}$, where $h = (g^{-1})^t = (g^t)^{-1}$. Hence the map ${}^\perp$ carries $SL(q, \mathbb{R})$ orbits in $G(p, \mathcal{SO}(q, \mathbb{R}))$ onto $SL(q, \mathbb{R})$ orbits in $G(D-p, \mathcal{SO}(q, \mathbb{R}))$. It follows that ${}^\perp$ induces a

homeomorphism of $G(p, \mathcal{SO}(q, \mathbb{R})) / \mathrm{SL}(q, \mathbb{R})$ onto $G(D-p, \mathcal{SO}(q, \mathbb{R})) / \mathrm{SL}(q, \mathbb{R})$. We call this involutive process duality. For an application of duality see b) of the next result.

5.2 The space of isomorphism classes

If X denotes any of the spaces considered in section 4, then the quotient space X / G is the space of isomorphism classes in X , where $G = \mathrm{GL}(n, \mathbb{R})$ if $X = \mathfrak{N}_2(n)$ and $G = \mathrm{GL}(p+q, \mathbb{R})$ if X is any of the spaces described by pairs of positive integers p, q . Each of the spaces X has a metric space topology since X is a subset of the Euclidean space $\mathrm{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$, where $n = p+q$ in the second case.

It would be interesting to determine the topology of these isomorphism classes X / G . In this article we consider only the questions of compactness and dimension. For compactness it suffices to consider sequential compactness, which is an equivalent property in a metric space. The main result is the following

Proposition The spaces $\mathfrak{N}_2(n) / \mathrm{GL}(n, \mathbb{R})$, $Z_2(p, q) / \mathrm{GL}(p+q, \mathbb{R})$, $R_2(p, q) / \mathrm{GL}(p+q, \mathbb{R})$ and $\mathfrak{N}(p, q) / \mathrm{GL}(p+q, \mathbb{R})$ are compact in the quotient topology. Moreover,

- The space $X(p, q) = \mathfrak{N}(p, q) / \mathrm{GL}(p+q, \mathbb{R})$ is homeomorphic to $G(p, \mathcal{SO}(q, \mathbb{R})) / \mathrm{SL}(q, \mathbb{R})$. In particular $X(p, q)$ is compact.
- $X(p, q)$ is homeomorphic to $X(D-p, q)$, where $D = (1/2)q(q-1)$.
- The space $\mathfrak{N}_0(p, q) / \mathrm{GL}(p+q, \mathbb{R})$ is a dense open subset of $X(p, q)$.
- The space $Z_2^0(p, q) / \mathrm{GL}(p+q, \mathbb{R})$ is a dense open subset of $Z_2(p, q) / \mathrm{GL}(p+q, \mathbb{R})$.

Proof We first prove a). Let $D = (1/2)q(q-1)$. An element of $\mathfrak{N}(p, q)$ is by definition a 2-step nilpotent Lie algebra \mathfrak{N} of dimension $p+q$ such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension p . The discussion in section (3.4) shows that \mathfrak{N} is isomorphic to $F_2(q)^* / W$, where $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(q, \mathbb{R})$ is a free 2-step nilpotent Lie algebra with q generators and W is a subspace of $\mathcal{SO}(q, \mathbb{R})$ of dimension $D-p$. The subspace W is not unique, but Proposition 3.5a shows that the other possibilities for W are precisely those elements in the $\mathrm{GL}(q, \mathbb{R})$ orbit of W in $G(D-p, \mathcal{SO}(q, \mathbb{R}))$, the Grassmann manifold of $(D-p)$ -dimensional subspaces of $\mathcal{SO}(q, \mathbb{R})$. By the discussion of duality in (5.1) the map ${}^\perp: W \rightarrow W^\perp$ maps $\mathrm{GL}(q, \mathbb{R})$ orbits in $G(p, \mathcal{SO}(q, \mathbb{R}))$ onto $\mathrm{GL}(q, \mathbb{R})$ orbits in $G(D-p, \mathcal{SO}(q, \mathbb{R}))$. Note that the $\mathrm{GL}(q, \mathbb{R})$ orbits and the $\mathrm{SL}(q, \mathbb{R})$ orbits in $G(D-p, \mathcal{SO}(q, \mathbb{R}))$ or $G(p, \mathcal{SO}(q, \mathbb{R}))$ are the same.

Hence we obtain a map $f: \mathfrak{N}(p, q) / \mathrm{GL}(p+q, \mathbb{R}) \rightarrow G(p, \mathcal{SO}(q, \mathbb{R})) / \mathrm{SL}(q, \mathbb{R})$ in which the isomorphism class of an element \mathfrak{N} in $\mathfrak{N}(p, q)$ is mapped to the orbit $\mathrm{SL}(q, \mathbb{R})(W^\perp)$ in $G(p, \mathcal{SO}(q, \mathbb{R}))$, where W is any subspace of dimension $D-p$ in $\mathcal{SO}(q, \mathbb{R})$ such that \mathfrak{N} is isomorphic to $F_2(q)^* / W$. Proposition 3.5a and duality say that the map f is injective. To see that f is surjective let W be any p -dimensional subspace of $\mathcal{SO}(q, \mathbb{R})$, and let $\mathfrak{N} = \mathbb{R}^q \oplus W$, the standard metric 2-step nilpotent Lie algebra determined by W . The

propositions in (3.2b) and (3.4) show that \mathfrak{N} is isomorphic to $F_2(q)^* / W^\perp$. Hence f maps the isomorphism class of \mathfrak{N} onto the orbit $SL(q, \mathbb{R})(W) \in G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R})$. The inverse function $g = f^{-1} : G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R}) \rightarrow \mathfrak{N}(p, q) / GL(p+q, \mathbb{R})$ maps the orbit $SL(q, \mathbb{R})(W)$ in $G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R})$ onto the isomorphism class of $F_2(q)^* / W^\perp$.

It is routine but somewhat tedious to show that both f and g are continuous with respect to the quotient topologies. The space $X(p, q) \approx G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R})$ is compact since $G(p, \mathcal{SO}(q, \mathbb{R}))$ is compact. This completes the proof of a). The proof of b) follows immediately from a) and the discussion of duality in (5.1).

Next we prove that $\mathfrak{N}_2(n) / GL(n, \mathbb{R})$, $Z_2(p, q) / GL(p+q, \mathbb{R})$ and $R_2(p, q) / GL(p+q, \mathbb{R})$ are compact in the quotient topologies, and we conclude with the proofs of b) and c).

Let $\{\rho_k\}$ denote a sequence in $\mathfrak{N}_2(n)$ and let $\{[\rho_k]\}$ denote the image sequence in $\mathfrak{N}_2(n) / GL(n, \mathbb{R})$. Every sequence $\{\rho_k\}$ in $\mathfrak{N}_2(n)$ has a subsequence $\{\rho'_k\}$ whose commutator ideals have the same dimension p , or equivalently, every sequence $\{\rho_k\}$ in $\mathfrak{N}_2(n)$ has a subsequence $\{\rho'_k\}$ in $\mathfrak{N}(p, q)$ for some positive integers p, q with $p+q = n$. By a) there exists an element ρ of $\mathfrak{N}(p, q)$ and a further subsequence $\{\rho''_k\}$ of $\{\rho'_k\}$ such that the isomorphism classes of $\{\rho''_k\}$ converge to the isomorphism class of $\rho \in \mathfrak{N}(p, q) \subseteq \mathfrak{N}_2(n)$. This proves that $\mathfrak{N}_2(n) / GL(n, \mathbb{R})$ is sequentially compact.

Now consider $Z_2(p, q) \subseteq \mathfrak{N}_2(n)$, where $n = p+q$. As above, any sequence $\{\rho_k\}$ in $Z_2(p, q)$ has a subsequence $\{\rho''_k\}$ whose isomorphism classes converge to the isomorphism class of an element $\rho \in \mathfrak{N}_2(n)$. Since $Z_2(p, q)$ is a Zariski closed subset of $\mathfrak{N}_2(n)$ by Proposition 4.1b, where $n = p+q$, it follows that $Z_2(p, q) / GL(p+q, \mathbb{R})$ is also sequentially compact. A similar argument shows that $R_2(p, q) / GL(p+q, \mathbb{R})$ is sequentially compact since $R_2(p, q)$ is a Zariski closed subset of $\mathfrak{N}_2(n)$ by Proposition 4.1c.

We prove c) and d) of the proposition. By Propositions 4.2a and 4.2c the sets $Z_2^0(p, q)$ and $\mathfrak{N}_0(p, q)$ are both Zariski open subsets of $Z_2(p, q)$. Hence $Z_2^0(p, q) / GL(p+q, \mathbb{R})$ and $\mathfrak{N}_0(p, q) / GL(p+q, \mathbb{R})$ are dense open subsets of $Z_2(p, q) / GL(p+q, \mathbb{R})$. Since $\mathfrak{N}_0(p, q) \subseteq \mathfrak{N}(p, q) \subseteq Z_2(p, q)$ it follows that $\mathfrak{N}_0(p, q) / GL(p+q, \mathbb{R})$ is also a dense open subset of $X(p, q) = \mathfrak{N}(p, q) / GL(p+q, \mathbb{R})$. This completes the proof of c) and d), which completes the proof of the Proposition. \square

5.3 Basic facts about real Lie group actions

If G is a Lie group acting on a differentiable manifold X , then the dimension of the quotient space X / G with the quotient topology is by definition the minimal codimension of a G orbit in X . Hence $\dim X / G = \dim X - \dim G + d$, where d is smallest dimension of a stabilizer $G_x : x \in X$.

We collect some useful facts about the stabilizers of G acting on finite dimensional real vector spaces U and the associated actions of G on the Grassmann manifolds $G(p,U)$, $1 \leq p \leq \dim U - 1$. Recall that $G(p,U)$ denotes the set of p -dimensional subspaces of U . We state these results only in the real case, but the proofs work equally well in the complex case. In some cases the results are better in the complex case. For example, the Zariski open subset O discussed below is connected in the complex case but not necessarily in the real case as the simple example in (5.3b) illustrates.

(5.3a) Stabilizers of minimum dimension

Let G be a Lie group. Let $\rho : G \rightarrow GL(U)$ be a representation of G on a real vector space U of dimension N , and let $d\rho : \mathfrak{G} \rightarrow \text{End}(U)$ denote the induced representation, where \mathfrak{G} is the Lie algebra of G . In what follows we suppress the notations ρ and $d\rho$.

We may regard U as a subspace of $\text{Hom}(\mathfrak{G}, U)$ by defining $u(X) = X(u)$ for all $u \in U$ and $X \in \mathfrak{G}$. For $u \in U$ let $G_u = \{g \in G : g(u) = u\}$, and let $\mathfrak{G}_u = L(G_u) = \{X \in \mathfrak{G} : X(u) = 0\} = \text{Ker } u$, where $u \in \text{Hom}(\mathfrak{G}, U)$. For each integer $k \geq 0$ let $\Sigma_k = \{u \in U : \dim \mathfrak{G}_u \geq k\} = \{u \in U : \text{nullity}(u) \geq k\}$. Then Σ_k is Zariski closed in U for every $k \geq 0$ by the corollary to Proposition 1 in Appendix 1. Let $r = \min \{\dim \mathfrak{G}_u : u \in U\}$, and let $O = \{u \in U : \dim \mathfrak{G}_u = r\}$. Then $\Sigma_r = U$ by the definition of r , and $O = \Sigma_r - \Sigma_{r+1} = U - \Sigma_{r+1}$ is a Zariski open subset of U .

For $u \in U$ the orbit $G(u)$ is diffeomorphic to the coset space G/G_u , and hence $\dim G(u) = \dim G - \dim G_u = \dim G - \dim \mathfrak{G}_u \leq \dim G - r$, with equality $\Leftrightarrow u \in O$.

We have proved

Proposition Let G be a Lie group, and let $\rho : G \rightarrow GL(U)$ be a representation of G on a real vector space U of dimension N . Let $r = \min \{\dim G_u : u \in U\}$, and let $O = \{u \in U : \dim G_u = r\}$. Then O is Zariski open in U . Moreover, for every $u \in U$ $\dim G(u) \leq \dim G - r$ with equality $\Leftrightarrow u \in O$.

(5.3b) Connectivity of the set O

The Zariski open set O in the proposition above may not be connected in the Euclidean topology of U . However, in each connected component O_α of O the the orbits $\{G(v) : v \in O_\alpha\}$ are all diffeomorphic. In different connected components of O the diffeomorphism types of the G orbits may be different.

Example Let U be the 3-dimensional real vector space of 3×3 real symmetric matrices, and introduce coordinates x, y, z so that (x, y, z) denotes the matrix $\begin{bmatrix} x & z \\ z & y \end{bmatrix}$. Let $G =$

$GL(3, \mathbb{R})$ act on U by $g(A) = gAg^t$. Set $\Sigma = \{A \in U : \det A = 0\} = \{(x, y, z) \in \mathbb{R}^3 : xy - z^2 = 0\}$. Note that G leaves invariant the cone Σ so that $\dim G_u = 7$ if $u \in \Sigma, u \neq 0$. If $O = U - \Sigma$, then O is Zariski open in U and $\dim G_u = 6$ for all $u \in O$. The set O has three G -invariant connected components containing the elements $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ respectively, and G acts transitively on each component. The first two orbits are convex sets and have the homotopy type of a point. The third orbit has the circle as a deformation retract.

5.3c Reduction from the real to the complex case

In computing the minimum dimension of a stabilizer of a real Lie algebra \mathfrak{G} acting on a real vector space U it is convenient to reduce consideration to the associated complex $\mathfrak{G}^{\mathbb{C}}$ -module $V = U^{\mathbb{C}}$. The next result accomplishes this.

Proposition Let \mathfrak{G} be a finite dimensional real Lie algebra, and let U be a finite dimensional real \mathfrak{G} -module. Let $V = U^{\mathbb{C}}$ be the associated complex $\mathfrak{G}^{\mathbb{C}}$ -module.

- Let $r = \min \{\dim \mathfrak{G}_u : u \in U\}$ and let $s = \min \{\dim \mathfrak{G}_v^{\mathbb{C}} : v \in V\}$. Then $r = s$.
- Let $r' = \min \{\text{codim} \mathfrak{G}(u) : u \in U\}$ and let $s' = \min \{\text{codim} \mathfrak{G}^{\mathbb{C}}(v) : v \in V\}$. Then $r' = s'$.

Proof It suffices to prove a) since b) is an immediate consequence of a). Let $O = \{u \in U : \dim \mathfrak{G}_u = r\}$, and let $W = \{v \in V : \dim \mathfrak{G}_v^{\mathbb{C}} = s\}$. Then O is \mathbb{R} -Zariski open in U and W is \mathbb{C} -Zariski open in V by the discussion in (5.3a). Regarding U as a subset of V it is easy to check that $(\mathfrak{G}_u)^{\mathbb{C}} = (\mathfrak{G}^{\mathbb{C}})_u$. To show that $r = s$ it suffices to show that O is \mathbb{C} -Zariski dense in V , which will prove that $O \cap W$ is nonempty.

Without loss of generality we may assume that $U = \mathbb{R}^n$ and $V = \mathbb{C}^n$. If Z is the \mathbb{C} -Zariski closure of O in V , then Z is closed in the Euclidean topology of V . Hence Z contains \mathbb{R}^n , the closure of O in the Euclidean topology of \mathbb{R}^n . Any complex polynomial f that vanishes on Z must vanish on \mathbb{R}^n , and it is now easy to see that f must also vanish on \mathbb{C}^n . It follows that $Z = \mathbb{C}^n$. \square

5.3d Group actions on Grassmann manifolds

Let G be a Lie group, and let $\rho : G \rightarrow GL(U)$ be a representation of G on a real vector space U of dimension N . For each integer p with $1 \leq p \leq \dim U - 1$ there is an obvious extension of the action of G on U to an action of G on $G(p, U)$. The space $G(p, U)$ is a compact, connected differentiable manifold and not a linear space. However, the group G still has stabilizers of minimal dimension r on a dense open subset O_p of $G(p, U)$ for each

integer p . To show this and to compute the integer r we may reduce to the linear case of (5.3a) by a well known method that we describe now, following [E1].

Fix an integer p with $1 \leq p \leq \dim U - 1$. Note first that G acts on U^* by $g(u^*)(u) = u^*(g^{-1}u)$ for all $g \in G$, $u \in U$ and $u^* \in U^*$. Hence $H = G \times GL(p, \mathbb{R})$ acts on $\text{Hom}(U^*, \mathbb{R}^p) \approx U \otimes \mathbb{R}^p$ by $(g,t)(\varphi) = t \circ \varphi \circ g^{-1}$. Let $A = \{\varphi \in \text{Hom}(U^*, \mathbb{R}^p) : \varphi \text{ is surjective}\}$. Then $A = \{\varphi \in \text{Hom}(U^*, \mathbb{R}^p) : \dim \text{Ker}(\varphi) = N-p\} = \{\varphi \in \text{Hom}(U^*, \mathbb{R}^p) : \dim \text{Ker}(\varphi)^\perp = p\}$, where $\text{Ker}(\varphi)^\perp = \{u \in U : u^*(u) = 0 \text{ for all } u^* \in \text{Ker}(\varphi)\}$. If we define a map $\psi : A \rightarrow G(p,U)$ by $\psi(\varphi) = \text{Ker}(\varphi)^\perp$, then it is easy to see that ψ is surjective.

Let $B = \{\varphi \in \text{Hom}(U^*, \mathbb{R}^p) : \dim H_\varphi = \dim \mathfrak{H}_\varphi \text{ is minimal}\}$, where \mathfrak{H}_φ denotes the Lie algebra of H . The sets A and B are nonempty Zariski open subsets of $\text{Hom}(U^*, \mathbb{R}^p)$ by (5.3a) and the corollary to Proposition 1 of Appendix 1. Hence $C = A \cap B$ is also a nonempty Zariski open subset of $\text{Hom}(U^*, \mathbb{R}^p)$.

Lemma A Fix an integer p with $1 \leq p \leq \dim U - 1$. Let $\psi : C \rightarrow G(p,U)$ be the map given by $\psi(\varphi) = \text{Ker}(\varphi)^\perp$. Let $\varphi \in C$. Then

- 1) H_φ is isomorphic to $G_{\psi(\varphi)}$, where $H = G \times GL(p, \mathbb{R})$.
- 2) The codimension of the orbit $H(\varphi)$ in $\text{Hom}(U^*, \mathbb{R}^p)$ equals the codimension of the orbit $G(\psi(\varphi))$ in $G(p,U)$.

3) Let $r = \min \{\dim G_W : W \in G(p,U)\} = \max \{\dim G(W) : W \in G(p,U)\}$. Let $O = \{W \in G(p,U) : \dim G_W = r\}$. Then O is a dense open subset of $G(p,U)$ that contains $\psi(C)$.

Proof 1) The condition $(g,t)(\varphi) = \varphi$ is equivalent to the condition

$$(*) \quad \varphi \circ g = t \circ \varphi$$

, which implies that $g(\text{Ker } \varphi) = \text{Ker } \varphi$ and $g(\text{Ker } \varphi^\perp) = \text{Ker } \varphi^\perp$. Hence if $(g,t) \in H_\varphi$, then $g \in G_{\psi(\varphi)}$.

If $\varphi \in C \subseteq A$, then $\varphi : U^* \rightarrow \mathbb{R}^p$ is surjective and the condition (*) uniquely determines t if g and φ are given. Hence the map $T : H_\varphi \rightarrow G_{\psi(\varphi)}$ given by $T(g,t) = g$ is an injective homomorphism. Conversely, if $g \in G_W$ is given, where $W = \psi(\varphi) = \text{Ker } \varphi^\perp$, then $g(\text{Ker } \varphi) = \text{Ker } \varphi$ and there is a unique element $t \in GL(p, \mathbb{R})$ such that (*) holds.

This proves that $T : H_\varphi \rightarrow G_{\psi(\varphi)}$ is surjective.

2) By 1) and the fact that $C \subseteq B$ there is an integer $d \geq 0$ such that $\dim H_\varphi = \dim G_{\psi(\varphi)} = d$ for all $\varphi \in C$. If $\varphi \in C$, then $\dim \text{Hom}(U^*, \mathbb{R}^p) = Np$ and $\text{codim } H(\varphi) = Np - \dim H(\varphi) = Np - \dim H + \dim H_\varphi = Np - \dim H + d$. On the other hand, since $\dim G(p,U) = p(N-p)$, it follows that $\text{codim } G(\psi(\varphi)) = p(N-p) - \dim G(\psi(\varphi)) = p(N-p) - \dim G + \dim G_{\psi(\varphi)} = p(N-p) - (\dim H - p^2) + d = Np - \dim H + d = \text{codim } H(\varphi)$ for all $\varphi \in C$.

3) Since $\psi(A) = G(\mathfrak{p}, U)$ and C is a Zariski open subset of $\text{Hom}(U^*, \mathbb{R}^P)$ with $C \subseteq A$ it follows that $\psi(C)$ is dense in $G(\mathfrak{p}, U)$.

The subset O is clearly open and nonempty in $G(\mathfrak{p}, U)$, and hence $O \cap \psi(C)$ is nonempty. Since $\dim G_{\psi(\varphi)} = d$ for all $\varphi \in C$ it follows that $d = r = \min \{\dim G_W : W \in G(\mathfrak{p}, U)\}$ and $O = \{W \in G(\mathfrak{p}, U) : \dim G_W = d\}$. We conclude that $\psi(C) \subseteq O$, which proves that O is dense in $G(\mathfrak{p}, U)$.

A necessary and sufficient condition for G to have open orbits in $G(\mathfrak{p}, U)$

Let $\rho: G \rightarrow GL(U)$ be an irreducible representation of a Lie group G on a real vector space U of dimension N such that the complexified representation $\rho: G \rightarrow GL(V)$, where $V = U^{\mathbb{C}}$, is also irreducible. Fix an integer p with $1 \leq p \leq N-1$, and define $H = G \times GL(p, \mathbb{R})$ and $H' = G \times SL(p, \mathbb{R})$. As above we obtain induced representations of $\mathfrak{H} = \mathfrak{G} \oplus \mathfrak{gl}(p, \mathbb{R})$ and $\mathfrak{H}' = \mathfrak{G} \oplus \mathfrak{sl}(p, \mathbb{R})$ on $\text{Hom}(U^*, \mathbb{R}^P) \approx U^* \otimes \mathbb{R}^P$. If $V = U^{\mathbb{C}}$, then $\text{Hom}(V^*, \mathbb{C}^P) \approx \text{Hom}(U^*, \mathbb{R}^P)^{\mathbb{C}}$ becomes a module for $\mathfrak{H}^{\mathbb{C}}$ and $(\mathfrak{H}')^{\mathbb{C}}$.

Lemma B Let U be a finite dimensional, irreducible real \mathfrak{G} -module such that $V = U^{\mathbb{C}}$ is a finite dimensional, irreducible complex $\mathfrak{G}^{\mathbb{C}}$ -module. Then G has an open orbit in $G(\mathfrak{p}, U) \Leftrightarrow (\mathfrak{H}')^{\mathbb{C}}$ has an orbit of codimension zero or one on $\text{Hom}(V^*, \mathbb{C}^P)$.

Remark If G has one open orbit in $G(\mathfrak{p}, U)$, then by 3) of Lemma A there exists a dense open subset U_o of $G(\mathfrak{p}, U)$ such that the orbit $G(W)$ is open in $G(\mathfrak{p}, U)$ for all $W \in U_o$.

Proof of Lemma B Suppose first the orbit $G(W)$ is open in $G(\mathfrak{p}, U)$ for some $W \in G(\mathfrak{p}, U)$. In the notation of Lemma A the subset $\psi(C)$ is dense in $G(\mathfrak{p}, U)$. Hence we may assume that $W = \text{Ker}(\varphi)^{\perp}$, where $\varphi \in C \subseteq \text{Hom}(U^*, \mathbb{R}^P)$. By 2) of Lemma A we know that $H(\varphi)$ is open in $\text{Hom}(U^*, \mathbb{R}^P)$. Hence $\mathfrak{H}(\varphi) = \text{Hom}(U^*, \mathbb{R}^P)$, and by the Proposition in (5.3c) there exists $\varphi' \in \text{Hom}(U^*, \mathbb{R}^P)^{\mathbb{C}} = \text{Hom}(V^*, \mathbb{C}^P)$ such that $\mathfrak{H}^{\mathbb{C}}(\varphi') = \text{Hom}(V^*, \mathbb{C}^P)$. Hence the orbit $(\mathfrak{H}')^{\mathbb{C}}(\varphi')$ has codimension zero or one in $\text{Hom}(V^*, \mathbb{C}^P)$ since $(\mathfrak{H}')^{\mathbb{C}}$ has codimension one in $\mathfrak{H}^{\mathbb{C}}$.

Conversely, suppose that the orbit $(\mathfrak{H}')^{\mathbb{C}}(\varphi')$ has codimension zero or one in $\text{Hom}(V^*, \mathbb{C}^P)$ for some $\varphi' \in \text{Hom}(V^*, \mathbb{C}^P)$. Let $k = \min \{\text{codim } (\mathfrak{H}')^{\mathbb{C}}(\varphi) : \varphi \in \text{Hom}(V^*, \mathbb{C}^P)\}$. Then by b) of the proposition in (5.3c) it follows that $k = \min \{\text{codim } (\mathfrak{H}')(\varphi) : \varphi \in \text{Hom}(U^*, \mathbb{R}^P)\}$. Moreover, there exist an \mathbb{R} -Zariski open set A in $\text{Hom}(U^*, \mathbb{R}^P)$ and a \mathbb{C} -Zariski open set B in $\text{Hom}(V^*, \mathbb{C}^P)$ such that $(\mathfrak{H}')(\varphi)$ has codimension k for all $\varphi \in A$, and $(\mathfrak{H}')^{\mathbb{C}}(\varphi)$ has codimension k for all $\varphi \in B$.

Suppose first that $k = 0$. Then $(\mathfrak{H}')(\varphi) = \text{Hom}(U^*, \mathbb{R}^P)$ for all $\varphi \in A$, and it follows that $H(\varphi)$ is an open subset of $\text{Hom}(U^*, \mathbb{R}^P)$ for all $\varphi \in A$. In particular $H(\varphi)$ is

also open in $\text{Hom}(U^*, \mathbb{R}^p)$ for all $\varphi \in A$ since $H \supseteq H'$. By 2) of Lemma A it follows that the orbit $G(W)$ is open in $G(p, U)$ for all W in some dense open subset U_o of $G(p, U)$.

Next suppose that $k = 1$. Let $d = \min \{\dim \mathfrak{H}^\mathbb{C}_\varphi : \varphi \in \text{Hom}(V^*, \mathbb{C}^p)\}$ and $d' = \min \{\dim (\mathfrak{H}')^\mathbb{C}_\varphi : \varphi \in \text{Hom}(V^*, \mathbb{C}^p)\}$. By hypothesis $V = U^\mathbb{C}$ is an irreducible $\mathfrak{G}^\mathbb{C}$ -module, and clearly \mathbb{C}^p is an irreducible $\mathcal{S}(p, \mathbb{C})$ module. Since $(\mathfrak{H}')^\mathbb{C} = \mathfrak{G}^\mathbb{C} \oplus \mathcal{S}(p, \mathbb{C})$ it follows that $\text{Hom}(V^*, \mathbb{C}^p) \approx V \otimes \mathbb{C}^p$ is an irreducible $(\mathfrak{H}')^\mathbb{C}$ -module. By Theorem 1 of [EI] we conclude that $d = d'$ since $\text{codim} (\mathfrak{H}')^\mathbb{C}(\varphi) \geq 1$ for all $\varphi \in \text{Hom}(V^*, \mathbb{C}^p)$. Moreover, there exists a Zariski open subset O in $\text{Hom}(V^*, \mathbb{C}^p)$ such that $\dim \mathfrak{H}^\mathbb{C}_\varphi = \dim (\mathfrak{H}')^\mathbb{C}_\varphi = d$ and $\text{codim} (\mathfrak{H}')^\mathbb{C}(\varphi) = 1$ for all $\varphi \in O$. It follows that $(\mathfrak{H}^\mathbb{C})(\varphi) = \text{Hom}(V^*, \mathbb{C}^p)$ for all $\varphi \in O$ since $\dim \mathfrak{H}^\mathbb{C}_\varphi = \dim (\mathfrak{H}')^\mathbb{C}_\varphi + 1$ and $\dim \mathfrak{H}^\mathbb{C}_\varphi = \dim (\mathfrak{H}')^\mathbb{C}_\varphi = d$. It follows from b) of the proposition in (5.3c) that $(\mathfrak{H})(\varphi) = \text{Hom}(U^*, \mathbb{R}^p)$ for all φ in some Zariski open subset of $\text{Hom}(U^*, \mathbb{R}^p)$, Hence by 2) and 3) of Lemma A there exists a dense open subset U_o in $G(p, U)$ such that the orbit $G(W)$ is open in $G(p, U)$ for all $W \in U_o$. \square

5.4 Dimension of the moduli space $X(p, q)$

The space $X(p, q) = \mathfrak{N}(p, q) / \text{GL}(p+q, \mathbb{R})$ is the space of isomorphism classes of 2-step nilpotent Lie algebras \mathfrak{N} with dimension $p+q$ such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension $p \geq 1$. In (5.2) we saw that $\mathfrak{N}(p, q) / \text{GL}(p+q, \mathbb{R})$ is homeomorphic to $G(p, \mathcal{SO}(q, \mathbb{R})) / \text{SL}(q, \mathbb{R})$. In this section we use the results of (5.3) and [EI] to determine the minimal codimension of an orbit of $\text{SL}(q, \mathbb{R})$ in $G(p, \mathcal{SO}(q, \mathbb{R}))$. See also Tables 2a, 2b in [KL].

For integers $p \geq 1$ and $q \geq 2$ let $d_{p,q} = \min \{\dim \text{SL}(q, \mathbb{R})_W : W \in G(p, \mathcal{SO}(q, \mathbb{R}))\}$. The maximal dimension of an $\text{SL}(q, \mathbb{R})$ orbit in $G(p, \mathcal{SO}(q, \mathbb{R}))$ is $\dim \text{SL}(q, \mathbb{R}) - d_{p,q} = q^2 - 1 - d_{p,q}$, and it follows that the minimum codimension of an $\text{SL}(q, \mathbb{R})$ orbit in $G(p, \mathcal{SO}(q, \mathbb{R}))$ is $p(D-p) - q^2 + 1 + d_{p,q}$ since $\dim G(p, \mathcal{SO}(q, \mathbb{R})) = p(D-p)$. Hence

$$\dim X(p, q) = p(D-p) - q^2 + 1 + d_{p,q}$$

It remains only to determine the dimension $d_{p,q}$ of a generic $\text{SL}(q, \mathbb{R})$ stabilizer in $G(p, \mathcal{SO}(q, \mathbb{R}))$.

Remark By the discussion of duality in (5.1) it is evident that $d_{p,q} = d_{D-p,q}$ for all positive integers p, q , where $2 \leq q$ and $1 \leq p \leq D = (1/2)q(q-1)$.

5.4a Local rigidity

An element ρ of $\mathfrak{N}(p, q)$ is said to be locally rigid if there exists a neighborhood U of ρ in $\mathfrak{N}(p, q)$ such that every 2-step nilpotent structure ρ' for \mathbb{R}^{p+q} that lies in U is isomorphic to ρ . If we express ρ isomorphically as a quotient $F_2(q)^* / W$, where $F_2(q)^* = \mathbb{R}^q \oplus \mathcal{SO}(\mathbb{R})$ is the free 2-step nilpotent Lie algebra on q generators and W is a subspace of

$\mathcal{SO}(\mathbb{R})$ of dimension $D-p$, then the condition that ρ be locally rigid is equivalent to the condition that the orbit $GL(q, \mathbb{R})(W^\perp)$ be an open subset of $G(p, \mathcal{SO}(\mathbb{R}))$. Hence the condition that $\mathfrak{N}(p, q)$ contain a locally rigid element ρ is the same as the condition that $X(p, q)$ have dimension zero.

The set of all locally rigid elements in $\mathfrak{N}(p, q)$ is clearly open in $\mathfrak{N}(p, q)$, and it corresponds to the union of all open $GL(q, \mathbb{R})$ orbits in $G(p, \mathcal{SO}(\mathbb{R}))$. Hence, if the set of all locally rigid elements in $\mathfrak{N}(p, q)$ is nonempty, then it is a dense, open subset of $\mathfrak{N}(p, q)$ by Lemma A in (5.3d).

5.4b A necessary condition for $X(p, q)$ to have dimension zero

If $G = SL(q, \mathbb{R})$ has an open orbit in $G(p, \mathcal{SO}(\mathbb{R}))$, then $q^2 - 1 = \dim G \geq \dim G(p, \mathcal{SO}(\mathbb{R})) = p(D-p)$. If $p = 1$ or 2 , then it is easy to check that $q^2 - 1 > p(D-p)$ for any integer $q \geq 2$. However, if $p \geq 3$, then it is very rare that $q^2 - 1 \geq p(D-p)$.

Recall from Proposition 2.2f or the corollary in (3.3) that $p \leq D$ with equality $\Leftrightarrow \mathfrak{N}$ is a free 2-step nilpotent Lie algebra on q generators.

Proposition Let p, q be positive integers such that $p \geq 3$, $D \geq p + 1$, where $D = (1/2)q(q-1)$.

If $q^2 - 1 > p(D-p)$, then exactly one of the following situations must occur.

- 1) $D-p = 1$.
- 2) $D-p = 2$.
- 3) $p = 3$ and $q = 4$.
- 4) $p = 3$ and $q = 5$ or $p = 7$ and $q = 5$ (dual cases).

If $q^2 - 1 = p(D-p)$, then

- 5) $p = 4$ and $q = 5$ or $p = 6$ and $q = 5$ (dual cases)

Corollary If $X(p, q)$ has dimension zero, then either $p = 1$, $q \geq 2$ or $p = 2$, $q \geq 2$ or $p \geq 3$ and (p, q) satisfies one of the five cases of the Proposition.

Remarks

1) We shall see shortly that $X(p, q)$ has dimension zero in all of the cases listed in the corollary except for the case $(p, q) = (2, 2k)$ where $k \geq 4$. In this case $\dim X(2, 2k) = k-3$.

2) The duality between p and $D-p$ expressed in the fact $d_{p, q} = d_{D-p, q}$ is seen again in cases 4) and 5) of the proposition. Case 3) is self dual.

Proof of the Proposition We show first that cases 1 and 2 of the Proposition always occur. Suppose first that $D-p = 1$ and $p \geq 3$. Then $q \geq 4$ and the condition $q^2 - 1 > p(D-p)$ is equivalent to the condition $q^2 + q > 0$, which clearly holds. Next, suppose that $D-p = 2$ and $p \geq 3$. Then $q \geq 4$ and the condition $q^2 - 1 > p(D-p)$ is equivalent to the condition $q+3 > 0$, which also clearly holds.

We assume now that $k = D-p \geq 3$.

Lemma Let $k = D-p \geq 3$, where $p \geq 3$. Then $q^2 - 1 > p(D-p)$
 $\Leftrightarrow \frac{1+\sqrt{25+8k}}{2} \leq q < \left(\frac{k}{2k-4}\right) \left(1+\sqrt{8k-15-[(8k-16)/k^2]}\right)$

Proof The condition $p = D-k \geq 3$ holds $\Leftrightarrow f_k(q) \geq 0$, where $f_k(x) = x^2 - x - (6+2k)$.

The function $f_k(x)$ is negative in the region between its roots, which are $r_1 = \frac{1-\sqrt{25+8k}}{2}$
 < 0 and $r_2 = \frac{1+\sqrt{25+8k}}{2} > 0$. Hence $p = D-k \geq 3 \Leftrightarrow \frac{1+\sqrt{25+8k}}{2} \leq q$.

Since $p = D-k$ the condition $q^2 - 1 > p(D-p)$ holds $\Leftrightarrow g_k(q) > 0$, where $g_k(x) =$
 $\left(1-\frac{k}{2}\right)x^2 + \left(\frac{k}{2}\right)x + (k^2 - 1)$. The roots of $g_k(x)$ are

$\left(\frac{k}{2k-4}\right)\left(1-\sqrt{8k-15-[(8k-16)/k^2]}\right)$ and $\left(\frac{k}{2k-4}\right)\left(1+\sqrt{8k-15-[(8k-16)/k^2]}\right)$.

Since $k \geq 3$ it follows that $\frac{k}{2k-4} > 0$ and $\sqrt{8k-15-[(8k-16)/k^2]} > 1$. Hence the first

root is negative and the second is positive. The function $g_k(x) = \left(1-\frac{k}{2}\right)x^2 + \left(\frac{k}{2}\right)x +$

$(k^2 - 1)$ is positive only in the open interval between the roots since $1-\frac{k}{2} < 0$. Hence $g_k(q)$

> 0 for $q > 0 \Leftrightarrow q < \left(\frac{k}{2k-4}\right) \left(1+\sqrt{8k-15-[(8k-16)/k^2]}\right)$, which completes the proof of
the lemma. \square

We now complete the proof of the Proposition by finding all solutions of the
inequalities of the lemma under the condition $k = D-p \geq 3$. One may show that for $k \geq 11$
 $\frac{1+\sqrt{25+8k}}{2} > \left(\frac{k}{2k-4}\right) \left(1+\sqrt{8k-15-[(8k-16)/k^2]}\right)$. It follows from the lemma and its
proof that there are no solutions for $k \geq 11$ to the inequality $q^2 - 1 \geq p(D-p)$, where $k =$
 $D-p \geq 3$ and $p \geq 3$. By checking the cases $3 \leq k \leq 10$ it is easy to see that cases 1) through
4) are the only ones to satisfy $q^2 - 1 > p(D-p)$ and case 5) is the only one to satisfy
 $q^2 - 1 = p(D-p)$. \square

To make further progress we need an effective way to compute the codimension of
an orbit of $SL(q, \mathbb{R})$ in $G(p, \mathfrak{so}(q, \mathbb{R}))$ or equivalently, to compute the dimension of a
stabilizer $SL(q, \mathbb{R})_W$, where $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$.

5.4c Dimensions of stabilizers of $SL(q, \mathbb{R})$ acting on $G(p, \mathfrak{so}(\mathbb{R}))$

Reduction to complex $\mathfrak{sl}(q, \mathbb{C}) \oplus \mathfrak{sl}(p, \mathbb{C})$ modules

Let $G = SL(q, \mathbb{R})$, and let \mathfrak{G} denote the Lie algebra of G . Let $d_{p,q}$ be the minimum
dimension of a stabilizer G_W , where $W \in G(p, \mathfrak{so}(q, \mathbb{R}))$. By Lemma A of (5.3d) $d_{p,q}$ is

also the minimum dimension of a stabilizer H_φ , where $H = G \times GL(p, \mathbb{R})$ and φ is an element from a Zariski open subset O of $U = \text{Hom}(\mathcal{SO}(q, \mathbb{R})^*, \mathbb{R}^P)$. Since H_φ has the same dimension as its Lie algebra $\mathfrak{H}_\varphi = \{X \in \mathfrak{H} : X\varphi = 0\}$ it follows from (5.3c) that $d_{p,q}$ is the minimum dimension of a stabilizer of $\mathfrak{H}^\mathbb{C} = \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{GL}(p, \mathbb{C})$ acting on $V = U^\mathbb{C} = \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^P)$.

Let $H' = G \times SL(p, \mathbb{R}) = SL(q, \mathbb{R}) \times SL(p, \mathbb{R})$, and let $\mathfrak{H}' = \mathcal{SL}(q, \mathbb{R}) \oplus \mathcal{SL}(p, \mathbb{R})$ denote the Lie algebra of H' . Let $d'_{p,q}$ be the minimum dimension of a stabilizer of \mathfrak{H}' acting on $U = \text{Hom}(\mathcal{SO}(q, \mathbb{R})^*, \mathbb{R}^P)$, which by (5.3c) equals the minimum dimension of a stabilizer of $(\mathfrak{H}')^\mathbb{C}$ acting on $V = \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^P)$. Let $\text{cd}(p, q)$ be the minimum codimension of an orbit of $(\mathfrak{H}')^\mathbb{C}$ acting on V .

By the discussion in (5.3) there exist Zariski open subsets O and O' in V such that the stabilizer $(\mathfrak{H}^\mathbb{C})_\varphi = \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{GL}(p, \mathbb{C})_\varphi$ has dimension $d_{p,q}$ for all φ in O and the stabilizer $(\mathfrak{H}'^\mathbb{C})_\varphi = \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})_\varphi$ has dimension $d'_{p,q}$ for all φ in O' .

Note that $\mathcal{SO}(q, \mathbb{C})$ is an irreducible $\mathcal{SL}(q, \mathbb{C})$ -module relative to the standard action of $SL(q, \mathbb{C})$ on $\mathcal{SO}(q, \mathbb{C})$ given by $g(Z) = gZg^\dagger$ for $g \in SL(q, \mathbb{C})$ and $Z \in \mathcal{SO}(q, \mathbb{C})$. Since \mathbb{C}^P is an irreducible module for $\mathcal{SL}(p, \mathbb{C})$ it follows that $\mathcal{SO}(q, \mathbb{C}) \otimes \mathbb{C}^P \approx \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^P)$ is an irreducible module for $(\mathfrak{H}')^\mathbb{C} = \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$.

Next, following [E1] and the discussion in (5.3d), we shift from considering the action of G on $G(p, \mathcal{SO}(q, \mathbb{R}))$ to considering the linear action of $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ on V . Let $d_{p,q}$, $d'_{p,q}$ and $\text{cd}(p, q)$ be the integers defined above. It is easy to compute from the action of H on $\text{Hom}(\mathcal{SO}(q, \mathbb{R})^*, \mathbb{R}^P)$ that the action of $\mathfrak{H}^\mathbb{C}$ on $V = \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^P)$, is given by $(X, Y)(\varphi) = Y \circ \varphi - \varphi \circ X$ for all $(X, Y) \in \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{GL}(p, \mathbb{C})$ and all $\varphi \in V$.

Proposition Let $G = SL(q, \mathbb{R})$, $H = G \times GL(p, \mathbb{R})$ and $H' = G \times SL(p, \mathbb{R})$. Let \mathfrak{H} and \mathfrak{H}' denote the Lie algebras of H and H' . Let $V = \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^P)$, and let $D = (1/2)q(q-1)$. Then

- 1) If $\text{cd}(p, q) = 0$, then there exists a dense open subset O of $G(p, \mathcal{SO}(q, \mathbb{R}))$ such that the orbit $SL(q, \mathbb{R})(W)$ is open in $G(p, \mathcal{SO}(q, \mathbb{R}))$ for all W in O .
- 2) If $\text{cd}(p, q) \geq 1$, then $d_{p,q} = d'_{p,q}$ and $\dim X(p, q) = \text{cd}(p, q) - 1 = p(D-p) + 1 - q^2 + d'_{p,q}$.

Proof By the discussion above there exist Zariski open subsets O and O' in V such that $\dim(\mathfrak{H}^\mathbb{C})_\varphi$ has a minimal value $d_{p,q}$ for all φ in O and $\dim(\mathfrak{H}'^\mathbb{C})_\varphi$ has a minimal value $d'_{p,q}$ for all φ in O' . Let $A = O \cap O'$.

Since H is an extension of H' by a 1-dimensional central subgroup it follows that $d'_{p,q} \leq d_{p,q} \leq d'_{p,q} + 1$. By Theorem 1 of [E1], $\text{cd}(p, q) = 0 \Leftrightarrow d_{p,q} = d'_{p,q} + 1$.

If $\text{cd}(p, q) = 0$, then $(\mathfrak{H}'^\mathbb{C})(v) = V = \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^P)$ for some $v \in V$, and it follows that $(\mathfrak{H}^\mathbb{C})(v) = V$. Hence by the Proposition in (5.3c) and Lemma A in (5.3d) there

exists $W \in G(p, \mathcal{SO}(q, \mathbb{R}))$ such that $SL(q, \mathbb{R})(W)$ is open in $G(p, \mathcal{SO}(q, \mathbb{R}))$. By 3) of Lemma A in (5.3d) the set O of such elements W in $G(p, \mathcal{SO}(q, \mathbb{R}))$ is dense and open in $G(p, \mathcal{SO}(q, \mathbb{R}))$. This proves 1).

If $cd(p, q) \geq 1$, then $d_{p,q} = d'_{p,q}$ and the maximal dimension of an $\mathfrak{H}^{\mathbb{C}}$ orbit in V is one more than the maximal dimension of an $\mathfrak{H}'^{\mathbb{C}}$ orbit in V . In particular $\dim X(p, q) = \min \{ \text{codim } SL(q, \mathbb{R})(W) : W \in G(p, \mathcal{SO}(q, \mathbb{R})) \} = \min \{ \text{codim } \mathfrak{H}^{\mathbb{C}}(\varphi) : \varphi \in \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^{\mathbb{P}}) \} = cd(p, q) - 1$ by the proposition in (5.3c) and lemma A of (5.3d). The formula for $cd(p, q) - 1$ follows immediately from the facts that $\dim_{\mathbb{C}} \text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^{\mathbb{P}}) = pD$ and $\dim \mathfrak{H}'^{\mathbb{C}} = p^2 + q^2 - 2$. This completes the proof of 2). \square

Generic stabilizers in $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ modules

Elashvili in Table 6 of [EI] has given a complete list of those finite dimensional irreducible complex semisimple modules V whose generic stabilizers have dimension $d'_{p,q} > 0$. See also Tables 2a and 2b of [KL]. Clearly, if $\dim \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C}) > \dim V$, then a generic stabilizer of $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ in V must have positive dimension. Conversely, Table 6 in [EI] shows that these are the only cases where a generic stabilizer of $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ in V has positive dimension. There seems to be no known direct proof of this fact. In any case, the specific values for the dimension $d'_{p,q}$ of a generic stabilizer of $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ in V are needed to compute the dimension of the space $X(p, q) = G(p, \mathcal{SO}(q, \mathbb{R})) / SL(q, \mathbb{R})$.

For the complex semisimple Lie algebra $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ we summarize the results of Table 6 of [EI] and Tables 2a, 2b of [KL]. We use the reduction from the case that $SL(q, \mathbb{R})$ acts on $G(p, \mathcal{SO}(q, \mathbb{R}))$ to the case that $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ acts on $\text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^{\mathbb{P}})$. We use also the duality result from (5.1) that $d_{p,q} = d_{D-p,q}$ for all positive integers p, q , where $d_{p,q} = \min \{ \dim SL(q, \mathbb{R})_W : W \in G(p, \mathcal{SO}(q, \mathbb{R})) \}$. The proof of the proposition in (5.4c) shows that $d'_{p,q} \leq d_{p,q} \leq d'_{p,q} + 1$ and $d_{p,q} = d'_{p,q} \Leftrightarrow cd(p, q) = 0$,

Spaces $X(p, q)$ where $d'_{p,q}$ is positive.

Table 1

Proposition Let p, q be positive integers with $1 \leq p \leq D = (1/2)q(q-1)$. Let $d'_{p,q} = \min \{ \dim \mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})_{\varphi} : \varphi \in \text{Hom}(\mathcal{SO}(q, \mathbb{R})^*, \mathbb{C}^{\mathbb{P}}) \}$. Then $d'_{p,q} = 0$ except in the following cases :

	(p, q)		$d'_{p,q}$
1)	$(1, q)$	$q \geq 2$	$(1/2)q(q+1) - 1$
	$(D-1, q)$	$q \geq 2$	
2)	(D, q)	$q \geq 2$	$q^2 - 2$ free 2-step

3)	(2, 2k+1)		2k+3
	(D-2, 2k+1)		2k+3
4)	(2, 2k),	$k \geq 3$	3k
	(D-2, 2k)	$k \geq 3$	3k
5)	(2, 4)		7
	(4, 4)		7
6)	(3, 4)	self dual	6
7)	(3, 5)		3
	(7, 5)		3
8)	(3,6)		1
	(12,6)		1

(5.4d) Dimensions of the spaces $X(p,q)$

Spaces $X(p,q)$ with dimension zero

Table 2

Proposition A The space $X(p,q)$ has dimension zero in precisely the following cases :

- 1) $p = 1$ $q \geq 2$
 $p = D-1$ $q \geq 2$
- 2) $p = D$ $q \geq 2$ free 2-step on q generators
- 3) $p = 2$ $q = 2k+1, k \geq 1$
 $p = D-2$ $q = 2k+1, k \geq 1$
- 4) $p = 2$ $q = 4$
 $p = 4$ $q = 4$
- 5) $p = 2$ $q = 6$
 $p = 13$ $q = 6$
- 6) $p = 3,$ $q = 4$ self dual
- 7) $p = 3,$ $q = 5$
 $p = 7,$ $q = 5$
- 8) $p = 4,$ $q = 5$
 $p = 6,$ $q = 5$

Spaces $X(p,q)$ with positive dimension

Proposition B If $(p,q) \neq (2, 2k)$ for $k \geq 4$ and (p,q) does not appear in Table 1 or Table 2, then the space $X(p,q)$ has dimension $p(D-p) + 1 - q^2 > 0$. The dual spaces $X(3,6)$ and

$X(12,6)$ have dimension 2. The dual spaces $X(2,2k)$ and $X(2k^2-k-2, 2k)$ have dimension $k-3$ for $k \geq 4$.

Proof of Proposition A If \mathfrak{N} is a 2-step nilpotent Lie algebra such that $[\mathfrak{N}, \mathfrak{N}]$ has dimension p and codimension q , then we showed in Proposition 2.2f or the corollary in (3.3) that $p \leq D$ with equality $\Leftrightarrow \mathfrak{N}$ is a free 2-step nilpotent Lie algebra on q generators. We saw in Proposition 2.2a that a free 2-step nilpotent Lie algebra on $q \geq 2$ generators is unique up to isomorphism. Hence $X(D,q)$ is a point, and in particular it has dimension zero.

We have reduced our consideration to the case that $D \geq p+1$. With this restriction all spaces $X(p,q)$ that are candidates to have dimension zero are listed in the proposition of (5.4b). For these candidates we use the proposition in (5.4c) and Table 1 to compute $\dim X(p,q) = p(D-p) + 1 - q^2 + d'_{p,q}$.

We observed in (3.6) that $X(1,q)$ has dimension zero for $q \geq 2$. It is well known that $X(2, 2k+1)$ has dimension zero for $k \geq 1$, and one can deduce this also from the value of $d'_{2,2k+1}$ that arises from Table 6 of [E1], or Tables 2a and 2b of [KL].

We consider now the cases $(2,2k)$, $k \geq 2$. From Table 1 it is routine to compute in the case $(p,q) = (2, 4)$ that a generic orbit of $\mathcal{SL}(4, \mathbb{C}) \oplus \mathcal{SL}(2, \mathbb{C})$ in $\text{Hom}(\mathcal{SO}(4, \mathbb{C})^*, \mathbb{C}^2)$ has codimension one. Hence $X(2, 4)$ and its dual $X(4, 4)$ have dimension zero by Lemma B in (5.3d) or the Proposition in (5.4c).

Table 1 above shows that $d'_{2,2k} = 3k$ for $k \geq 3$, and hence for $k \geq 3$ the generic codimension $\text{cd}(2, 2k)$ of an orbit of $\mathcal{SL}(2k, \mathbb{C}) \oplus \mathcal{SL}(2, \mathbb{C})$ in $\text{Hom}(\mathcal{SO}(2k, \mathbb{C})^*, \mathbb{C}^2)$ is $k-2$. If $k = 3$ then $\text{cd}(2, 2k) = 1$, and by Lemma B in (5.3d) or the Proposition in (5.4c) it follows that $X(2, 6)$ and its dual space $X(13, 6)$ have dimension zero.

If $k \geq 4$, then $\text{cd}(2,2k) = k-2 \geq 2$. By the proposition in (5.4c) we conclude that $d'_{2,2k} = d'_{2,2k} = 3k$, and $\dim X(2,2k) = \text{cd}(2, 2k) - 1 = k-3$.

If $p \geq 3$, then only the values $(3, 4), (3, 5), (4, 5)$ and their dual pairs appear in (5.4b). From Table 1 we verify in all of these cases that a generic orbit of $\mathcal{SL}(q, \mathbb{C}) \oplus \mathcal{SL}(p, \mathbb{C})$ in $\text{Hom}(\mathcal{SO}(q, \mathbb{C})^*, \mathbb{C}^p)$ has codimension one. Hence either Lemma B of (5.3d) or the Proposition in (5.4c) shows that the spaces $X(3, 4), X(3, 5), X(4, 5)$ and their dual spaces $X(7, 5), X(6, 5)$ all have dimension zero.

This completes the verification of Table 2 and Proposition A.

Proof of proposition B If $(p,q) = (2,2k)$ for $k \geq 4$, then $\dim X(2,2k) = k-3$ by the discussion above in the proof of Proposition A. If $(p,q) = (3,6)$, then $\text{cd}(3,6) = 3$ since $d'_{3,6} = 1$ by Table 1. By the Proposition in (5.4c) we obtain $\dim X(3,6) = \text{cd}(3,6) - 1 = 2$.

If (p,q) is not listed in Table 1, then $d'_{p,q} = 0$. If (p,q) is also not listed in Table 2, then $\dim X(p,q) > 0$ and $\text{cd}(p,q) \geq 2$ by Lemma B of (5.3d). Hence $d'_{p,q} = d'_{p,q} = 0$ by the Proposition in (5.4c). It follows from the discussion at the beginning of (5.4) that \dim

$X(p,q) = p(D-p) + 1 - q^2$ if (p,q) does not appear in Table 1 or 2. This completes the proof of Proposition B. \square

Section 6 Proofs of the results in section 4

Proof of proposition 4.1a

By the definition in section 1, $\mathfrak{N}_2(n) = \{\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) : \text{Im}(\rho) \subseteq Z(\rho)\}$, where $Z(\rho) = \{x \in \mathbb{R}^n : \rho(x \wedge y) = 0 \text{ for all } y \in \mathbb{R}^n\}$. For each $x \in \mathbb{R}^n$ define $I_x : \mathbb{R}^n \rightarrow \Lambda^2(\mathbb{R}^n)$ by $I_x(y) = x \wedge y$. If $\text{ad}_\rho x = \rho \circ I_x$ for $x \in \mathbb{R}^n$, then $\text{ad}_\rho x(y) = \rho(x \wedge y) = [x, y]$ in the standard bracket notation.

Fix a basis $\{x_1, x_2, \dots, x_n\}$ for \mathbb{R}^n . For $1 \leq i, j \leq n$ let $\varphi_{ij} : \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ be the polynomial map given by $\varphi_{ij}(\rho) = (\rho \circ I_{x_i}) \circ (\rho \circ I_{x_j})$. Then $\rho \in \mathfrak{N}_2(n) \Leftrightarrow \varphi_{ij}(\rho) = 0$ for all $1 \leq i, j \leq n$. \square

Proof of Proposition 4.1b

Let $n = p+q$. Let $\varphi : \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) \rightarrow \text{Hom}(\mathbb{R}^n, \text{Hom}(\mathbb{R}^n, \mathbb{R}^n))$ be the injective linear mapping given by $\varphi(\rho)(x) = \text{ad}_\rho x = \rho \circ I_x$. Since $Z(\rho) = \text{Ker} \varphi(\rho)$ it follows from Proposition 1 of the Appendix that $\Sigma(p,q) = \{\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) : \text{nullity} \varphi(\rho) \geq p\}$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$. Hence $Z_2(p,q) = \Sigma(p,q) \cap \mathfrak{N}_2(n)$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$ by Proposition 4.1a. \square

Proof of Proposition 4.1c

Let $n = p+q$. If $\Sigma'(p,q) = \{\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n) : \text{rank}(\rho) \leq p\}$, then $\Sigma'(p,q)$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$ by the corollary to Proposition 1 of the Appendix. Hence $R_2(p,q) = \Sigma'(p,q) \cap \mathfrak{N}_2(n)$ is an algebraic variety in $\text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$. \square

Notation

We standardize some notation that will be used in the discussion and proofs of Propositions 4.2a, 4.2b and 4.2c. Let $G(k,n)$ denote the Grassmann manifold of k -dimensional subspaces of \mathbb{R}^n . Let $q = n-p$ and $D = (1/2)q(q-1)$. Let U be the vector space $\mathfrak{so}(q, \mathbb{R}) \times \dots \times \mathfrak{so}(q, \mathbb{R})$ (p times). Recall from section 1 that $\mathfrak{N}_2(q)$ denotes the set of 2-step nilpotent Lie algebras that are free on q generators.

We first make some observations that will be useful for the proofs of Propositions 4.2a, 4.2b and 4.2c.

Lemma

Let $n = p+q$. Let $\rho \in R_2(p,q) \cap Z_2(p,q)$. Let $W \in G(p,n)$ be a subspace such that $\text{Im}(\rho) \subseteq W \subseteq Z(\rho)$, and let V be a q -dimensional subspace of \mathbb{R}^n such $V \oplus W = \mathbb{R}^n$. Let

$\{x_1, \dots, x_q, z_1, \dots, z_p\}$ be a basis of \mathbb{R}^n such that $\{x_1, \dots, x_q\}$ is a basis of V and $\{z_1, \dots, z_p\}$ is a basis of W . Define matrices $\{C^1(\rho), \dots, C^p(\rho)\}$ in $\mathfrak{so}(q, \mathbb{R})$ by the equations

$$\rho(x_i \wedge x_j) = \sum_{\alpha=1}^p C_{ij}^{\alpha}(\rho) z_{\alpha} \text{ for } 1 \leq i < j \leq p. \text{ Then}$$

$$1) \quad Z(\rho) = W \Leftrightarrow \bigcap_{\alpha=1}^p \text{Ker}(C^{\alpha}(\rho)) = \{0\}.$$

$$2) \quad \text{Im}(\rho) = W \Leftrightarrow \{C^1(\rho), \dots, C^p(\rho)\} \text{ are linearly independent in } \mathfrak{so}(q, \mathbb{R}).$$

Proof of the lemma

1) Since $W \subseteq Z(\rho)$ it suffices to show

$$(*) \quad a = (a_1, \dots, a_q) \text{ lies in } \bigcap_{\alpha=1}^p \text{Ker}(C^{\alpha}(\rho)) \Leftrightarrow x = \sum_{i=1}^q a_i x_i \text{ lies in } V \cap Z(\rho)$$

Note that $a \in \bigcap_{\alpha=1}^p \text{Ker}(C^{\alpha}(\rho)) \Leftrightarrow 0 = \sum_{j=1}^q C_{ij}^{\alpha}(\rho) a_j = - \sum_{j=1}^q C_{ji}^{\alpha}(\rho) a_j$ for $1 \leq i \leq q$ and

$1 \leq \alpha \leq p$. On the other hand $\rho(x \wedge x_i) = \sum_{j=1}^q a_j \rho(x_j \wedge x_i) = \sum_{\alpha=1}^p \left\{ \sum_{j=1}^q a_j C_{ji}^{\alpha}(\rho) \right\} z_{\alpha}$. The assertion (*) now follows since $x \in V \cap Z(\rho) \Leftrightarrow \rho(x \wedge x_i) = 0$ for $1 \leq i \leq q$.

2) Define a positive definite inner product $\langle \cdot, \cdot \rangle$ in $\mathfrak{so}(q, \mathbb{R})$ by $\langle A, B \rangle = -\frac{1}{2} \text{trace } AB$. Define a linear isomorphism $\xi : \mathfrak{so}(q, \mathbb{R}) \rightarrow \Lambda^2 V$ by $\xi(A) = \sum_{i < j} A_{ij} x_i \wedge x_j$.

It is routine to verify that $\rho(\xi(A)) = \sum_{\beta=1}^p \langle A, C^{\beta}(\rho) \rangle z_{\beta}$.

Suppose first that the elements $\{C^1(\rho), \dots, C^p(\rho)\}$ are linearly independent in $\mathfrak{so}(q, \mathbb{R})$, and let $X = \text{span} \{C^1(\rho), \dots, C^p(\rho)\} \subseteq \mathfrak{so}(q, \mathbb{R})$. Let $\{\varphi_1, \dots, \varphi_p\} \subseteq X^*$ be the dual basis of $\{C^1(\rho), \dots, C^p(\rho)\}$, and choose $\{A^1, \dots, A^p\} \subseteq X$ so that $\varphi_{\alpha}(B) =$

$$\langle A^{\alpha}, B \rangle \text{ for all } B \text{ in } X \text{ and } 1 \leq \alpha \leq p. \text{ Then } \rho(\xi(A^{\alpha})) = \sum_{\beta=1}^p \langle A^{\alpha}, C^{\beta}(\rho) \rangle z_{\beta} = z_{\alpha}$$

since $\langle A^{\alpha}, C^{\beta}(\rho) \rangle = \delta_{\alpha\beta}$. Hence $\text{Im}(\rho) = W$ since $\text{Im}(\rho)$ contains the basis $\{z_1, \dots, z_p\}$ of W .

Conversely, suppose that $\text{Im}(\rho) = W$. For $1 \leq \alpha \leq p$ choose an element A^{α} in $\mathfrak{so}(q, \mathbb{R})$ such that $z_{\alpha} = \rho(\xi(A^{\alpha})) = \sum_{\beta=1}^p \langle A^{\alpha}, C^{\beta}(\rho) \rangle z_{\beta}$, which implies that

$$\langle A^{\alpha}, C^{\beta}(\rho) \rangle = \delta_{\alpha\beta}. \text{ If } 0 = \sum_{\beta=1}^p a_{\beta} C^{\beta}(\rho) \text{ for some real numbers } \{a_{\beta}\}, \text{ then } 0 =$$

$\langle \sum_{\beta=1}^p a_{\beta} C^{\beta}(\rho), A^{\alpha} \rangle = a_{\alpha}$ for $1 \leq \alpha \leq p$. This proves that the set $\{C^1(\rho), \dots, C^p(\rho)\}$ is linearly independent in $\mathfrak{so}(q, \mathbb{R})$. \square

Proof of Proposition 4.2a

We now begin the proof of Proposition 4.2a. It is clear from Proposition 4.1b that $Z_2^0(p, q) = Z_2(p, q) - Z_2(p+1, q-1)$ is a Zariski open subset of $Z_2(p, q)$. Recall that $n = p+q$.

Recall that $U = so(q, \mathbb{R}) \times \dots \times so(q, \mathbb{R})$ (p times), and let $F = \{C = (C^1, \dots, C^p) \in U : \bigcap_{i=1}^p \text{Ker}(C^i) = \{0\}\}$. The proof of the Proposition is now a consequence of the

following statements :

- 1) F is an arc connected Zariski open subset of U , and $U - F$ has codimension ≥ 2 .
- 2) If $Z : Z_2^0(p, q) \rightarrow G(p, n)$ denotes the map that sends $\rho \in Z_2^0(p, q)$ to the center $Z(\rho) \in G(p, n)$, then $Z^{-1}(W)$ is bijectively equivalent to F for every $W \in G(p, n)$.
- 3) $Z_2^0(p, q)$ with the quotient topology determined by Z is a connected C^∞ manifold of dimension $pq + pD$. Moreover, $Z_2^0(p, q)$ is a fiber bundle over $G(p, n)$ with fiber F .

Assertion 1) follows immediately from assertion 2) and the remark thereafter in Proposition 4 of Appendix 1. Assertion 2) above follows immediately from 1) of the lemma above. We prove 3). Let $\rho \in Z_2^0(p, q)$ be given, and let $W = Z(\rho) \in G(p, n)$. As in the lemma above we let V be a q -dimensional subspace of \mathbb{R}^n such $V \oplus W = \mathbb{R}^n$, and we let $\{x_1, \dots, x_q, z_1, \dots, z_p\}$ be a basis of \mathbb{R}^n such that $\{x_1, \dots, x_q\}$ is a basis of V and $\{z_1, \dots, z_p\}$ is a basis of W .

Since $V \oplus W = \mathbb{R}^n$ every small neighborhood O of the zero map in $\text{Hom}(W, V)$ defines a coordinate neighborhood O' of W in $G(p, n)$. Given $T \in O$ we define $W' = (\text{Id}+T)(W)$ and let $O' = \{(\text{Id}+T)(W) : T \in O\} \subseteq G(p, n)$. Every element W' of O' also comes equipped with a distinguished basis $\{z_1', \dots, z_p'\}$; if $W' = (\text{Id}+T)(W)$ for $T \in O$, then define $z_i' = (\text{Id}+T)(z_i)$ for $1 \leq i \leq p$.

Fix a coordinate neighborhood O' of W in $G(p, n)$ that arises from an open set O of $\text{Hom}(W, V)$ as above. Choose O so small that $V \oplus W' = \mathbb{R}^n$ for every $W' \in O'$. We construct a bijection $\psi : Z^{-1}(O') \rightarrow O' \times F$ that will define the desired manifold and fiber bundle structures on $Z_2^0(p, q)$.

Given $\rho' \in Z^{-1}(O')$ let $\{z_1', \dots, z_p'\}$ be the distinguished basis of $W' = Z(\rho')$ constructed as above. Define $C(\rho') = (C^1(\rho'), \dots, C^p(\rho'))$ in F by the equations

$$\rho'(x_i \wedge x_j) = \sum_{\alpha=1}^p C_{ij}^\alpha(\rho') z_\alpha' \text{ for all } 1 \leq i < j \leq p. \text{ Define } \psi : Z^{-1}(O') \rightarrow O' \times F \text{ by } \psi(\rho')$$

$= (Z(\rho'), C(\rho'))$. It is routine to verify that ψ is a homeomorphism if $Z_2^0(p, q)$ has the quotient topology defined by Z , F has the Euclidean topology from $U = so(q, \mathbb{R}) \times \dots \times so(q, \mathbb{R})$ (p times), and $O' \times F$ has the product topology. Moreover, the pairs $\{\psi, Z^{-1}(O')\}$ constructed in the manner above define coordinate charts and strip maps that give $Z_2^0(p, q)$ the structure of a C^∞ manifold and a fiber bundle over $G(p, n)$ with fiber F . Since F is open in the Euclidean topology of U it follows that $\dim Z_2^0(p, q) = \dim F + \dim G(p, n) = pD + pq$, where $q = n-p$ and $D = (1/2)q(q-1)$. This completes the proof of 3).

Proof of Proposition 4.2b

It is clear from Proposition 4.1c that $\mathfrak{N}(p,q) = R_2(p,q) - R_2(p-1,q+1)$ is a Zariski open subset of $R_2(p,q)$.

Let $F = \{C = (C^1, \dots, C^p) \in U = so(q, \mathbb{R})^p : \{C^1, \dots, C^p\} \text{ are linearly independent in } so(q, \mathbb{R})\}$. Note that F is empty unless $p \leq D = \dim so(q, \mathbb{R})$. We treat separately the cases that $p \neq D$ and $p = D = (1/2)q(q-1)$. Again, recall that $n = p+q$.

If $p \neq D$, then the proof is a consequence of the following statements :

- 1) F is an arc connected Zariski open subset of U , and $U - F$ has codimension ≥ 2 .
- 2) If $R : \mathfrak{N}(p,q) \rightarrow G(p,n)$ denotes the map that sends $\rho \in \mathfrak{N}(p,q)$ to $\text{Im } \rho \in G(p,n)$, then $R^{-1}(W)$ is bijectively equivalent to F for every $W \in G(p,n)$.
- 3) $\mathfrak{N}(p,q)$ with the quotient topology determined by R is a connected C^∞ manifold of dimension $pq + pD$. Moreover, $\mathfrak{N}(p,q)$ is a fiber bundle over $G(p,n)$ with fiber F .

Assertion 1) follows immediately from assertion 3) of Proposition 2 in Appendix 1. Assertion 2) follows immediately from assertion 2) of the lemma above. We omit the details of the proof of assertion 3), which is essentially the same as the proof of assertion 3) of Proposition 4.2a.

Suppose now that $p = D$. Then the three assertions above must be modified as follows. In assertion 1) the fiber F is a Zariski open subset of U , but F has two arc components by Proposition 2 and the remark thereafter in Appendix 1. Assertion 2) remains true. Except for the connectedness of $\mathfrak{N}(p,q)$, assertion 3) remains true as well, and in this case $\dim \mathfrak{N}(p,q) = pn$.

We prove that $\mathfrak{N}(p,q)$ has two connected components in the case that $p = D$. By Proposition 2.2f or the Corollary in (3.3) it follows that $\mathfrak{N}(p,q) = \mathfrak{N}_2(q)$, the set of 2-step nilpotent Lie algebra structures on \mathbb{R}^N , $N = (1/2)q(q+1)$, that are free on q generators. Note that $GL(N, \mathbb{R})$ acts transitively on $\mathfrak{N}_2(q)$ since the elements of $\mathfrak{N}_2(q)$ are all isomorphic by Proposition 2.2a. By the discussion at the beginning of section (2.2) the two connected components of $\mathfrak{N}_2(q)$ are the orbits of $GL(N, \mathbb{R})^+$, the connected, index 2 subgroup of $GL(N, \mathbb{R})$ consisting of those elements with positive determinant. \square

Proof of Proposition 4.2c

Let $F = \{C = (C^1, \dots, C^p) \in U = so(q, \mathbb{R})^p : \{C^1, \dots, C^p\} \text{ are linearly independent in } so(q, \mathbb{R}) \text{ and } \bigcap_{i=1}^p \text{Ker}(C^i) = \{0\}\}$.

If $p \neq D$, then the proof is a consequence of the following statements :

- 1) F is an arc connected Zariski open subset of U , and $U - F$ has codimension ≥ 2 .

2) If $R : \mathfrak{N}(p,q) \rightarrow G(p,n)$ denotes the map that sends $\rho \in \mathfrak{N}(p,q)$ to $\text{Im } \rho = Z(\rho) \in G(p,n)$, then $R^{-1}(W)$ is bijectively equivalent to F for every $W \in G(p,n)$.

3) $\mathfrak{N}(p,q)$ with the quotient topology determined by R is a connected C^∞ manifold of dimension $pq + pD$. Moreover, $\mathfrak{N}(p,q)$ is a fiber bundle over $G(p,n)$ with fiber F .

The fiber F in this situation is the intersection of the fibers F that occur in Propositions 4.2a and 4.2b. Hence assertion 1) in this proof follows immediately from the assertions 1) in the proofs of Propositions 4.2a and 4.2b. Assertion 2) follows immediately from assertions 1) and 2) of the lemma above. We again omit the details of the proof of assertion 3), which is essentially the same as the proof of assertion 3) of Proposition 4.2a.

If $p = D$, then $\mathfrak{N}_0(p,q) \subseteq \mathfrak{N}(p,q) = \mathfrak{N}_2(q)$ by Proposition 4.2b. Equality holds since the commutator ideal and the center are equal for a free 2-step nilpotent Lie algebra on q generators. \square

Appendix 1

Some results on real algebraic varieties

Proposition 1

Let U, V and W be finite dimensional real vector spaces, and let $\varphi : U \rightarrow \text{Hom}(V,W)$ be a polynomial mapping. Let $s = \min \{\dim V, \dim W\}$ and let p be an integer with $1 \leq p \leq s - 1$. Let $X_p = \{u \in U : \text{rank } \varphi(u) \leq p\}$, and $Y_p = \{u \in U : \text{nullity } \varphi(u) \geq p\}$. Then X_p and Y_p are real algebraic varieties in U .

Proof

If $n = \dim V$, then $X_p = Y_{n-p}$ since $\text{rank } \varphi(u) + \text{nullity } \varphi(u) = \dim V$. Hence it suffices to prove that X_p is a real algebraic variety for each p . Fix a basis $\{v_1, v_2, \dots, v_n\}$ for V , and let p be an integer as above. Define $\Phi(p) = \{(\alpha_1, \alpha_2, \dots, \alpha_{p+1}) \in \mathbb{Z}^{p+1} : 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{p+1} \leq n\}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{p+1}) \in \Phi(p)$ let $\varphi^\alpha : \text{Hom}(V,W) \rightarrow \Lambda^{p+1}(W)$ be given by $\varphi^\alpha(T) = T(v_{\alpha_1}) \wedge T(v_{\alpha_2}) \wedge \dots \wedge T(v_{\alpha_{p+1}})$ for $T \in \text{Hom}(V,W)$. Then $\text{rank } T \leq p \Leftrightarrow \varphi^\alpha(T) = 0$ for all $\alpha \in \Phi(p)$. Since the maps $\{\varphi^\alpha\}$ are polynomial and compositions of polynomial maps are polynomial maps it follows that X_p is a real algebraic variety in U . \square

If U is a subspace of $\text{Hom}(V,W)$ and φ is the inclusion map in the result above, then we obtain the following

Corollary

Let V and W be finite dimensional real vector spaces, and let U be a subspace of $\text{Hom}(V,W)$. Let $s = \min \{\dim V, \dim W\}$ and let p be an integer with $1 \leq p \leq s - 1$. If $X_p = \{u \in U : \text{rank } u \leq p\}$ and $Y_p = \{u \in U : \text{nullity } u \geq p\}$, then X_p and Y_p are real algebraic varieties in $\text{Hom}(V,W)$.

Proposition 2

Let $p \geq 2$ and $D \geq 2$ be integers, and let $s = \min \{p, D\}$. Let V be a real vector space of dimension D , and let $V^p = V \times V \times \dots \times V$ (p times). For $v = (v_1, v_2, \dots, v_p) \in V^p$ define $\text{rank}(v) = \dim \text{span} \{v_1, v_2, \dots, v_p\}$. For $1 \leq k \leq s-1$ let $\Sigma'_k = \{v \in V^p : \text{rank}(v) \leq k\}$ and for $1 \leq k \leq s$ let $\Sigma_k = \{v \in V^p : \text{rank}(v) = k\}$. Then

$$1) \Sigma'_k \text{ is an algebraic variety for } 1 \leq k \leq s-1, \text{ and } \Sigma'_k = \{\text{zero}\} \cup \bigcup_{i=1}^k \Sigma_i.$$

2) If $1 \leq k \leq s-1$, then Σ_k is an arc connected smooth manifold of dimension $pk + k(D-k)$ that is a fiber bundle over $G(k, V)$, the Grassmann manifold of k -dimensional subspaces of V . Moreover, $V^p - \Sigma_k$ has codimension ≥ 2 .

3) The set Σ_s is a Zariski open subset of V^p , and Σ_s is arc connected if $p \neq D$. In addition, $V^p - \Sigma_s$ has codimension ≥ 2 .

Remark

If $p = D = s$, then Σ_s can be identified with the set of bases of V in an obvious fashion. Hence Σ_s has two arc components in this case, each of which corresponds to the bases of V with a fixed orientation.

Proof

1) It suffices to prove that Σ'_k is an algebraic variety for each k since the second statement of 1) is obvious. For an integer k with $1 \leq k \leq s-1$ let $\Phi(k) = \{(\alpha_1, \alpha_2, \dots, \alpha_{k+1}) \in \mathbb{Z}^{k+1} : 1 \leq \alpha_1 < \alpha_2 < \dots < \alpha_{k+1} \leq s\}$. For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{k+1}) \in \Phi(k)$ let $\varphi^\alpha : V^p \rightarrow \Lambda^{k+1}(V)$ be the polynomial map defined by $\varphi^\alpha(v) = v_{\alpha_1} \wedge v_{\alpha_2} \wedge \dots \wedge v_{\alpha_{k+1}}$ for $v = (v_1, v_2, \dots, v_p) \in V^p$. Then $\text{rank}(v) \leq k \Leftrightarrow \varphi^\alpha(v) = 0$ for all $\alpha \in \Phi(k)$, which proves that Σ'_k is an algebraic variety.

2) Fix an integer k with $1 \leq k \leq s-1$ and define $\pi : \Sigma_k \rightarrow G(k, V)$ by $\pi(v) = \text{span} \{v_1, v_2, \dots, v_p\}$. The map π is clearly surjective. For $W \in G(k, V)$ note that $\pi^{-1}(W) = W^p - A$, where $A = \{w \in W^p : \text{rank}(w) \leq k-1\}$. The set A is an algebraic variety in W^p by 1), and hence $\pi^{-1}(W)$ is a Zariski open subset of W^p . The diffeomorphism type of $\pi^{-1}(W)$ is independent of W since any two real vector spaces of dimension k are isomorphic.

Next, we construct local trivializations of $\pi : \Sigma_k \rightarrow G(k, V)$. Let $W \in G(k, V)$ be given and let W' be a subspace of V such that $W \oplus W' = V$. If $U \subseteq \text{Hom}(W, W')$ is any small neighborhood of the zero map, then U defines a coordinate chart for W in $G(k, V)$; namely, given $\varphi \in U$ let $W(\varphi) = (\text{Id} + \varphi)(W) \in G(k, V)$. For such a neighborhood U let $O = \{W(\varphi) : \varphi \in U\} \subseteq G(k, V)$, and let $F = \pi^{-1}(O)$.

We define a bijection $\psi : F \times O \rightarrow \pi^{-1}(O)$. For $w = (w_1, w_2, \dots, w_p) \in F \subseteq W^p$ and $\varphi \in U$ let $(\text{Id} + \varphi)(w) = ((\text{Id} + \varphi)(w_1), (\text{Id} + \varphi)(w_2), \dots, (\text{Id} + \varphi)(w_p)) \in V^p$. Given

$(w, W^*) \in F \times O$ let $\varphi \in U$ be the unique element such that $W^* = W(\varphi)$ and let $\psi(w, W^*) = (\text{Id} + \varphi)(w) \in \pi^{-1}(W^*)$. Clearly $\psi(F \times O) \subseteq \pi^{-1}(O)$. Since $\text{Id} + \varphi : W \rightarrow W^*$ is invertible for all $\varphi \in U$ it follows immediately that $\psi : F \times O \rightarrow \pi^{-1}(O)$ is a bijection.

The pairs $\{(\psi, O)\}$ define coordinate charts on Σ_k and local trivializations of $\pi : \Sigma_k \rightarrow G(k, V)$ that give Σ_k the structure of a smooth manifold that is also a fiber bundle over $G(k, V)$ with fiber F . Hence $\dim \Sigma_k = \dim F + \dim G(k, V) = pk + k(D-k)$ since F is a Zariski open subset of W^p , where $W \in G(k, V)$.

To show that Σ_k is arc connected it suffices to prove that the fiber $F = \pi^{-1}(W)$ is arc connected for $W \in G(k, V)$. Note that $\pi^{-1}(W) - \{\text{zero}\} = W^{p-k-1} \cup_{i=1}^{k-1} A_i$, where $A_i = \{w \in W^p : \text{rank}(w) = i\}$. By the previous paragraph A_i is a smooth submanifold of W^p with dimension $pi + i(k-i)$, where $i \leq k-1 \leq s-2 \leq p-2$. Hence A_i has codimension $(p-i)(k-i) \geq 2$ in W^p , and it follows that $\pi^{-1}(W)$ is an arc connected subset of W^p .

3) For any integer k with $1 \leq k \leq s-1$ it follows from 2) that Σ_k has codimension $(p-k)(D-k)$ in V^p . Since $1 \leq k \leq s-1$ and $s = \min\{p, D\}$ it follows that $(p-k)(D-k) \geq 2$ unless $p = D = s$ and $k = s-1$. Hence if $p \neq D$, then $V^p - \Sigma_s = \Sigma_{s-1}$ has codimension ≥ 2 , and $\Sigma_s = V^p - \Sigma_{s-1}$ is an arc connected Zariski open subset of V^p .

Proposition 3

Let V be a real vector space of dimension q , and let G be a Lie subgroup of $GL(V)$. Let W be a finite dimensional G -module and let U be a G -submodule of $\text{Hom}(V, W)$. For $1 \leq k \leq q-1$ let $U_k = \{u \in U : \text{nullity}(u) = k\}$. If U_k is nonempty and G acts transitively on the Grassmann manifold $G(k, V)$, then U_k is a smooth manifold and a fiber bundle over $G(k, V)$.

Proof

We recall that if V and W are finite dimensional G -modules, then $\text{Hom}(V, W)$ becomes a G -module under the action $(gT)(v) = g(T(g^{-1}v))$ for $g \in G$, $v \in V$ and $T \in \text{Hom}(V, W)$.

Fix an integer k with $1 \leq k \leq q-1$ and suppose that G acts transitively on $G(k, V)$. Define $\pi : U_k \rightarrow G(k, V)$ by $\pi(u) = \text{Ker}(u)$. Since U is a G -submodule of $\text{Hom}(V, W)$ it follows that $\pi(gu) = g(\pi u)$ for all $g \in G$ and $u \in U$. Hence $g(\pi^{-1}X) = \pi^{-1}(gX)$ for $g \in G$ and $X \in G(k, V)$, and we conclude that all fibers of π are the same, modulo the G -action on U , since G acts transitively on $G(k, V)$.

Next, we show that each fiber of π is a Zariski open subset of some vector subspace of U . For $X \in G(k, V)$ let $U_X = \{T \in U : X \subseteq \text{Ker}(T)\}$, a subspace of U . Let $\Sigma = \{T \in \text{Hom}(V, W) : \dim \text{Ker}(T) \geq 1 + \dim X\}$. Then Σ is Zariski closed in $\text{Hom}(V, W)$ by the

corollary to Proposition 1 in this appendix. Hence $\pi^{-1}(X) = \{ T \in U : X = \text{Ker}(T) \} = U_X - \Sigma$ is Zariski open in U_X .

Let G be the Lie subgroup of $GL(V)$ that occurs in the statement of Proposition 3. Fix $X \in G(k, V)$ and let $F = \pi^{-1}(X)$. For a suitable neighborhood O of X in $G(k, V)$ we construct a bijection $\psi : F \times O \rightarrow \pi^{-1}(O)$. As in the proof of 2) in Proposition 2, the pairs $\{(\psi, O)\}$ define coordinate charts on U_k and local trivializations of $\pi : U_k \rightarrow G(k, V)$ that give U_k the structure of a smooth manifold that is also a fiber bundle over $G(k, V)$ with fiber F .

It is convenient to construct a submanifold Y of G such that 1) Y contains the identity of G 2) $Y(X) = \{g(X) : g \in Y\}$ is a neighborhood of X in $G(k, V)$ and 3) if $g(X) = g'(X)$ for elements g, g' in Y , then $g = g'$. Assuming for the moment that we have constructed Y , we let $O = Y(X)$ and define $\psi : F \times O \rightarrow \pi^{-1}(O)$ as follows. Given (u, X^*) in $F \times O$ we let g be the unique element of Y such that $g(X) = X^*$ and we define $\psi(u, X^*) = g(u) \in \pi^{-1}(X^*)$. It is routine to check that $\psi(F \times O) \subseteq \pi^{-1}(O)$ and $\psi : F \times O \rightarrow \pi^{-1}(O)$ is a bijection.

We complete the proof of Proposition 3 by constructing the submanifold Y of G with the properties 1), 2) and 3) above. Given $X \in G(k, V)$ we define $H = \{g \in G : g(X) = X\}$. Note that H is a closed subgroup of G and $G(k, V)$ is diffeomorphic to G/H under the map $gH \rightarrow g(X)$ since G acts transitively on $G(k, V)$. Let \mathfrak{G} and \mathfrak{H} denote the Lie algebras of G and H respectively, and let $\exp : \mathfrak{G} \rightarrow G$ denote the Lie group exponential map. Let \mathfrak{K} denote a vector subspace of \mathfrak{G} such that $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{H}$ (direct sum). If A and B are small neighborhoods of the origin in \mathfrak{K} and \mathfrak{H} respectively, then $A \times B$ is a neighborhood of the origin in \mathfrak{G} and the map $f : A \times B \rightarrow G$ given by $f(a, b) = \exp(a) \cdot \exp(b)$ is a diffeomorphism onto an open neighborhood of the identity in G . If $Y = \exp(A)$, then by making A still smaller if necessary it follows that Y has the properties 1), 2) and 3) stated above.

Next we apply Proposition 3 in a way that is useful in section 6.

Proposition 4

Let $U = so(q, \mathbb{R}) \times \dots \times so(q, \mathbb{R})$ (p times), where $p \geq 2$ and $q \geq 3$. For $0 \leq k \leq q-1$ let $U_k = \{C = (C^1, \dots, C^p) \in U : \dim \bigcap_{i=1}^p \text{Ker}(C^i) = k\}$. Then

1) For $1 \leq k \leq q-2$ the set U_k is a nonempty smooth manifold of dimension $(1/2)(q-k)\{k(2-p) + p(q-1)\}$ and is also a fiber bundle over $G(k, q)$. Each manifold U_k has codimension ≥ 2 in U .

2) $U_0 = U - \{\bigcup_{1 \leq k \leq q-2} U_k\}$ is an arc connected Zariski open subset of U , and $U - U_0$ has codimension ≥ 2 in U .

Remark

If $q = 2$, then U_1 is empty and $U_0 = U - \{\text{zero}\}$ is arc connected since $p \geq 2$. More generally, U_{q-1} is empty for all $q \geq 2$ since the kernel of a skew symmetric linear transformation of \mathbb{R}^q has even codimension. Elementary arguments show that U_k is nonempty for $1 \leq k \leq q-2$ since $p \geq 2$.

Proof

Let $V = \mathbb{R}^q$ and $W = V^p$. We may regard U as a subspace of $\text{Hom}(V, W)$; given $C = (C^1, \dots, C^p) \in U$ and $v \in V$ we define $C(v) = (C^1(v), \dots, C^p(v)) \in W$. Let $G = \text{SO}(q, \mathbb{R}) \subseteq \text{GL}(V)$ and note that G acts transitively on $G(k, V)$ for $1 \leq k \leq q-1$. Let G act on W by $g(v) = (g(v_1), \dots, g(v_p))$ for $v = (v_1, \dots, v_p) \in W$. The natural G -action on $\text{Hom}(V, W)$ is given by $(gT)(v) = g(T(g^{-1}v))$ for $g \in G$, $v \in V$ and $T \in \text{Hom}(V, W)$. In particular, if $C = (C^1, \dots, C^p) \in U$, then $gC = (gC^1g^{-1}, \dots, gC^pg^{-1}) \in U$ for all $g \in G$, i.e. U is a G -submodule of $\text{Hom}(V, W)$.

1) If $C = (C^1, \dots, C^p) \in U$, then $\text{Ker}(C) = \bigcap_{i=1}^p \text{Ker}(C^i)$. Fix an integer k with $1 \leq k \leq q-1$. As we observed above, U_k is nonempty, and by Proposition 3 in this appendix the set U_k is a smooth manifold and a fiber bundle over $G(k, V)$. Let $\pi_k : U_k \rightarrow G(k, V)$ be the projection given by $\pi_k(C) = \text{Ker}(C)$. To compute the dimension of U_k it suffices to compute the dimension of a fiber of π_k .

Fix $W \in G(k, V)$. The proof of Proposition 3 showed that $\pi_k^{-1}(W) = \{C \in U : W = \text{Ker}(C)\}$ is a Zariski open subset of $U_W = \{C \in U : W \subseteq \text{Ker}(C)\}$. Define $O_W = \{A \in \mathcal{SO}(q, \mathbb{R}) : W \subseteq \text{Ker}(A)\}$, and $O_W^0 = \{A \in O_W : \dim \text{Ker}(A) \leq \dim \text{Ker}(A') \text{ for all } A' \in O_W\}$. Then O_W^0 is Zariski open in O_W , and it follows that $(O_W^0)^p$ is Zariski open in $O_W^p = U_W$. Since both $(O_W^0)^p$ and $\pi_k^{-1}(W)$ are Zariski open subsets of U_W it follows that $\dim \pi_k^{-1}(W) = \dim (O_W^0)^p = p \dim O_W^0$.

We show that for each $W \in G(k, V)$ the set O_W^0 is a smooth manifold of dimension $(1/2)(q-k)(q-k-1)$. It then follows immediately that $\dim U_k = \dim \pi_k^{-1}(W) + \dim G(k, V) = (1/2)p(q-k)(q-k-1) + k(q-k) = (1/2)(q-k)\{k(2-p) + p(q-1)\}$, which proves 1).

We need to consider separately the cases that $q-k$ is even or odd.

Case 1 $q-k$ is even

In this case $O_W^0 = \{A \in \mathcal{so}(q, \mathbb{R}) : \text{Ker } A = W\}$. Let W^\perp denote the orthogonal complement of W in \mathbb{R}^q , and let $\mathcal{so}(W^\perp)$ denote the skew symmetric transformations of W^\perp . The subset $\mathcal{so}(W^\perp)_0$ of invertible elements of $\mathcal{so}(W^\perp)$ is Zariski open in $\mathcal{so}(W^\perp)$ since $\dim W^\perp = q-k$ is even. Note that O_W^0 may be identified with $\mathcal{so}(W^\perp)_0$ under the

homeomorphism that sends $A^* \in so(W^\perp)_O$ to the unique element A of $so(q, \mathbb{R})$ such that $A = A^*$ on W^\perp and $A \equiv 0$ on W . Hence O_W^O is a manifold with dimension $(1/2)(q-k)(q-k-1)$.

Case 2 $q-k$ is odd

In this case $O_W^O = \{A \in so(q, \mathbb{R}) : W \text{ is a codimension 1 subspace of Ker } A\}$. As above, for $W^* \in G(q-k-1, W^\perp)$ we let $so(W^*)_O$ denote the Zariski open subset of invertible elements of $so(W^*)$. We identify an element A^* of $so(W^*)_O$ with the unique element A of O_W^O such that $A = A^*$ on W^* and $A \equiv 0$ on $(W^*)^\perp$. If $\pi : O_W^O \rightarrow G(q-k-1, W^\perp)$ is the projection map defined by $\pi(A) = (\text{Ker } A)^\perp = W^*$, then π is surjective and $\pi^{-1}(W^*)$ may be identified as above with $so(W^*)_O$. The diffeomorphism type F of a fiber $\pi^{-1}(W^*) \approx so(W^*)_O$ is clearly independent of W^* , and hence O_W^O becomes a smooth fiber bundle over $G(q-k-1, W^\perp)$ with fiber F . In particular $\dim O_W^O = \dim F + \dim G(q-k-1, W^\perp) = (1/2)(q-k-1)(q-k-2) + (q-k-1) = (1/2)(q-k)(q-k-1)$, the same as in case 1. The fact that U_k has codimension ≥ 2 is proved below in 2). This completes the proof of 1).

2) Note that $U_O = U - \Sigma$, where $\Sigma = \{C \in U : \dim \text{Ker } C \geq 1\}$. By the corollary to Proposition 1 of this appendix, Σ is an algebraic variety in U , and hence U_O is a Zariski open subset of U .

To prove that U_O is arc connected and that $U - U_O = \Sigma$ has codimension ≥ 2 it suffices to prove that each manifold U_k has codimension ≥ 2 for $1 \leq k \leq q-2$. By 1) we have $\dim U_k = f(k)$, where $f(x) = (1/2)(q-x)\{x(2-p) + p(q-1)\}$ for $1 \leq x \leq q-2$. Note that $f(x) < 0$ on $[1, q-2]$, and hence $f(x)$ is strictly decreasing on $[1, q-2]$. We conclude that $\text{codim } U_k = \dim U - f(k) \geq \dim U - f(1) = (p-1)(q-1) \geq 2$ since $p \geq 2$ and $q \geq 3$ by hypothesis. \square

Appendix 2 Free 2-step nilpotent Lie algebras and their quotients

In this appendix we present the omitted proofs of the results on free 2-step nilpotent Lie algebras stated in section 2.

Proof of Proposition 2.2a

1) We prove uniqueness first. Suppose that \mathfrak{N} and \mathfrak{N}^* are free 2-step nilpotent Lie algebras on q generators, and let $\{x_1, \dots, x_q\}$ and $\{x_1^*, \dots, x_q^*\}$ be admissible generating sets for \mathfrak{N} and \mathfrak{N}^* respectively. By definition there exist unique Lie algebra homomorphisms $T : \mathfrak{N} \rightarrow \mathfrak{N}^*$ and $T^* : \mathfrak{N}^* \rightarrow \mathfrak{N}$ such that $T(x_i) = x_i^*$ and $T^*(x_i^*) = x_i$ for $1 \leq i \leq q$. Then $T^* \circ T$ and $T \circ T^*$ are the identity maps on \mathfrak{N} and \mathfrak{N}^* respectively since they fix all elements of the generating sets $\{x_1, \dots, x_q\}$ and $\{x_1^*, \dots, x_q^*\}$.

We prove existence. Let $q \geq 2$ be given and let $D = (1/2)q(q-1)$. Let $F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$, and let $[\cdot, \cdot]$ be the 2-step nilpotent Lie algebra structure on $F_2(q)$ such that $\Lambda^2(\mathbb{R}^q)$ is in the center of $F_2(q)$ and $[v, w] = v \wedge w$ for all $v, w \in \mathbb{R}^q$.

Now let $\{\mathfrak{N}^*, [\cdot, \cdot]^*\}$ be any 2-step nilpotent Lie algebra, and let $\{x_1^*, \dots, x_q^*\}$ be any subset of q elements in \mathfrak{N}^* . Let $\{v_1, \dots, v_q\}$ be any basis of \mathbb{R}^q , and let $T : F_2(q) \rightarrow \mathfrak{N}^*$ be a linear map such that (a) $T(v_i) = x_i^*$ for $1 \leq i \leq q$. In order for T to be a Lie algebra homomorphism we must have (b) $T(v_i \wedge v_j) = T([v_i, v_j]) = [Tv_i, Tv_j]^* = [x_i^*, x_j^*]^*$ for all i, j . The map $T : F_2(q) \rightarrow \mathfrak{N}^*$ is uniquely defined by (a) and (b) since $\{v_i : 1 \leq i \leq q\} \cup \{v_i \wedge v_j : 1 \leq i < j \leq q\}$ is a basis for $F_2(q)$. Condition (b) ensures that T is a Lie algebra homomorphism.

Finally, $\dim F_2(q) = q + D = q + (1/2)q(q-1) = (1/2)q(q+1)$.

2) We shall need a preliminary result.

Lemma Let \mathfrak{N} be a finite dimensional real Lie algebra, and let $q = \dim \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$. Then

1) Every generating set for \mathfrak{N} contains at least q elements.

2) Let $\{\xi_1, \dots, \xi_q\}$ be a generating set for \mathfrak{N} with q elements. Then $\{\pi(\xi_1), \dots, \pi(\xi_q)\}$ is a basis for $\mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$, where $\pi : \mathfrak{N} \rightarrow \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$ is the projection.

3) Let \mathfrak{N} be 2-step nilpotent. Let $\{\xi_1, \dots, \xi_q\}$ be any subset of \mathfrak{N} with q elements such that $\{\pi(\xi_1), \dots, \pi(\xi_q)\}$ is a basis for $\mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$. Then $\{\xi_1, \dots, \xi_q\}$ is a generating set for \mathfrak{N} .

Proof Let $\pi : \mathfrak{N} \rightarrow \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$ be the projection homomorphism. Let $\{\xi_1, \dots, \xi_N\}$ be a generating set for \mathfrak{N} . Then $\mathfrak{S} = \{\pi(\xi_1), \dots, \pi(\xi_N)\}$ is a generating set for $\mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$, and \mathfrak{S} must contain a basis of the abelian Lie algebra $\mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$. This proves 1) and 2).

3) Let \mathfrak{N} be 2-step nilpotent, and let $\{\xi_1, \dots, \xi_q\}$ be any subset of \mathfrak{N} such that $\{\pi(\xi_1), \dots, \pi(\xi_q)\}$ is a basis for $\mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$. Clearly $\{\xi_1, \dots, \xi_q\}$ is a linearly independent subset of \mathfrak{N} . If $\mathcal{V} = \text{span} \{\xi_1, \dots, \xi_q\}$, then $\pi : \mathcal{V} \rightarrow \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}]$ is an isomorphism, and we conclude that $\mathfrak{N} = \mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}]$ since $q = \dim \mathcal{V} = \dim \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}] = \dim \mathfrak{N} - \dim [\mathfrak{N}, \mathfrak{N}]$. Let \mathfrak{N}' be the subalgebra of \mathfrak{N} generated by $\{\xi_1, \dots, \xi_q\}$. Then $\{[\xi_i, \xi_j] : 1 \leq i, j \leq q\}$ generates $[\mathcal{V}, \mathcal{V}]$, which equals $[\mathfrak{N}, \mathfrak{N}]$ since \mathfrak{N} is 2-step nilpotent. Hence \mathfrak{N}' contains $\mathcal{V} \oplus [\mathfrak{N}, \mathfrak{N}] = \mathfrak{N}$. \square

Let \mathfrak{N}^* be a 2-step nilpotent Lie algebra such that $r = \dim \mathfrak{N}^* / [\mathfrak{N}^*, \mathfrak{N}^*] \leq q$. By 3) of the lemma above \mathfrak{N}^* has a generating set with r elements, and hence \mathfrak{N}^* admits a generating set $\{x_1^*, \dots, x_q^*\}$ with exactly q elements. Let \mathfrak{N} be a free 2-step nilpotent Lie algebra on q generators, and let $\{x_1, \dots, x_q\}$ be an admissible generating set. Let $T : \mathfrak{N} \rightarrow \mathfrak{N}^*$ be the unique Lie algebra homomorphism such that $T(x_i) = x_i^*$ for $1 \leq i \leq q$. It follows that $T(\mathfrak{N}) = \mathfrak{N}^*$ since $T(\mathfrak{N})$ is a Lie algebra that contains $\{x_1^*, \dots, x_q^*\}$.

Proof of Proposition 2.2b

1) If $\{y_1, \dots, y_r\}$ is a generating set for \mathfrak{N} , then \mathfrak{N} is the linear span of the set $S = \{y_1, \dots, y_r, [y_i, y_j] : 1 \leq i < j \leq r\}$, which has at most $r + (1/2)r(r-1) = (1/2)r(r+1)$ distinct elements. Hence $r \geq s$ since $(1/2)r(r+1) \geq \text{card}(S) \geq n = (1/2)q(q+1)$ by Proposition 2.2a.

2) Let $\{x_1, \dots, x_q\}$ be an admissible generating set for \mathfrak{N} , and let $\{y_1, \dots, y_q\}$ be any generating set for \mathfrak{N} with q elements. If $T : \mathfrak{N} \rightarrow \mathfrak{N}$ is the unique Lie algebra homomorphism such that $T(x_i) = y_i$ for $1 \leq i \leq q$, then $T([x_i, x_j]) = [y_i, y_j]$ for $1 \leq i < j \leq q$. Hence T is an isomorphism since $\text{Im } T$ contains the linear spanning set $S = \{y_1, \dots, y_q, [y_i, y_j] : 1 \leq i < j \leq q\}$ of \mathfrak{N} . It is clear from the definition of an admissible generating set A for \mathfrak{N} that any isomorphic image of A is also an admissible generating set for \mathfrak{N} . \square

Proof of Proposition 2.2c

It is easy to verify the properties 1) and 2) for the example $\mathfrak{N} = F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$ constructed in the existence proof of Proposition 2.2a. Hence properties 1) and 2) hold in general since by Propositions 2.2a and 2.2b \mathfrak{N} is unique and all generating sets with q elements are the same up to isomorphism. \square

Proof of Proposition 2.2d

We use the explicit model $F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$ from the proof of Proposition 2.2a.

1) Since $\rho' : F_2(q) \rightarrow \mathfrak{N}'$ is surjective there exist elements $\{\xi_i : 1 \leq i \leq q\}$ of $F_2(q)$ such that $\rho'(\xi_i) = (T \circ \rho)(e_i)$ for all i , where $\{e_i : 1 \leq i \leq q\}$ is the natural basis of \mathbb{R}^q . There exists a unique Lie algebra homomorphism $\tilde{T} : F_2(q) \rightarrow F_2(q)$ such that $\tilde{T}(e_i) = \xi_i$ for all i since $F_2(q)$ is a free 2-step nilpotent Lie algebra. The Lie algebra homomorphisms $\rho' \circ \tilde{T}$ and $T \circ \rho$ are equal since they have the same values on the set $\{e_i : 1 \leq i \leq q\}$, which generates $F_2(q)$. The proof of 1) is complete.

2) Suppose that $T : \mathfrak{N} \rightarrow \mathfrak{N}'$ is an isomorphism and $q = \dim \mathfrak{N} / [\mathfrak{N}, \mathfrak{N}] = \dim \mathfrak{N}' / [\mathfrak{N}', \mathfrak{N}']$. Using the notation from the proof of 1) let \mathfrak{X} be the subalgebra of $F_2(q)$ generated by $\{\xi_i : 1 \leq i \leq q\}$. It suffices to show that $\mathfrak{X} = F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$.

The set $\{\rho'(\xi_i) = (T \circ \rho)(e_i), 1 \leq i \leq q\}$ generates \mathfrak{N}' since $\{e_i, 1 \leq i \leq q\}$ generates $F_2(q)$ and $T \circ \rho : F_2(q) \rightarrow \mathfrak{N}'$ is surjective. If $\pi' : \mathfrak{N}' \rightarrow \mathfrak{N}' / [\mathfrak{N}', \mathfrak{N}']$ is the projection homomorphism, then $\mathfrak{S} = \{(\pi' \circ \rho')(\xi_i), 1 \leq i \leq q\}$ is a basis of $\mathfrak{N}' / [\mathfrak{N}', \mathfrak{N}']$ by 2) of the lemma in Proposition 2.2a. Note that $\Lambda^2(\mathbb{R}^q) = [F_2(q), F_2(q)] \subseteq \rho'^{-1}([\mathfrak{N}', \mathfrak{N}'])$, and hence $[F_2(q), F_2(q)] \subseteq \text{Ker}(\pi' \circ \rho')$. If $\sigma : F_2(q) \rightarrow F_2(q) / [F_2(q), F_2(q)] \approx \mathbb{R}^q$ is the projection

homomorphism, then there exists a Lie algebra homomorphism $\varphi : F_2(q) / [F_2(q), F_2(q)] \rightarrow \mathfrak{N}' / [\mathfrak{N}', \mathfrak{N}']$ such that $\varphi \circ \sigma = \pi' \circ \rho'$. The map φ is surjective since $\varphi(\sigma(\xi_i)) = (\pi' \circ \rho')(\xi_i)$ for $1 \leq i \leq q$, and hence φ is an isomorphism since its domain and range both have dimension q . It follows that $\{\sigma(\xi_i) : 1 \leq i \leq q\}$ is a basis for $F_2(q) / [F_2(q), F_2(q)]$. By 3) of the lemma in Proposition 2.2a we conclude that $\{\xi_i : 1 \leq i \leq q\}$ generates $F_2(q)$. \square

Proof of Proposition 2.2e

Let \mathcal{V} be a subspace of \mathbb{R}^n such that $\mathcal{V} \oplus \text{Im } \rho = \mathbb{R}^n$, and let $\{x_1, \dots, x_q\}$ be a basis for \mathcal{V} . Hence $\{(x_i + \text{Im } \rho) \wedge (x_j + \text{Im } \rho) : 1 \leq i < j \leq q\}$ is a basis for $\Lambda^2(\mathbb{R}^n / \text{Im } \rho)$. It follows that $\{\rho(x_i \wedge x_j) : 1 \leq i < j \leq q\} = \{\rho'((x_i + \text{Im } \rho) \wedge (x_j + \text{Im } \rho)) : 1 \leq i < j \leq q\}$ is a linear spanning set for $\text{Im } \rho$ with at most $(1/2)q(q-1)$ linearly independent elements. We conclude that $n = q + \dim \text{Im } \rho \leq q + (1/2)q(q-1) = (1/2)q(q+1)$, with equality $\Leftrightarrow \rho' : \Lambda^2(\mathbb{R}^n / \text{Im } \rho) \rightarrow \text{Im } \rho$ is an isomorphism.

If $\rho \in \text{Hom}(\Lambda^2(\mathbb{R}^n), \mathbb{R}^n)$ is a free 2-step nilpotent Lie algebra with q generators, then $n = (1/2)q(q+1)$ by Proposition 1. It follows that $\rho' : \Lambda^2(\mathbb{R}^n / \text{Im } \rho) \rightarrow \text{Im } \rho$ is an isomorphism by the discussion of the previous paragraph. Conversely, if ρ' is an isomorphism, then $n = (1/2)q(q+1)$ and $\{\rho(x_i \wedge x_j) : 1 \leq i < j \leq q\}$ is a basis for $\text{Im } \rho$ by the previous paragraph. Let $\{v_1, \dots, v_q\}$ be any basis for \mathbb{R}^q , and let $T : \mathbb{R}^n \rightarrow F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$ be the linear isomorphism such that a) $T(x_i) = v_i$ for $1 \leq i \leq q$ and b) $T(\rho(x_i \wedge x_j)) = v_i \wedge v_j$ for $1 \leq i < j \leq q$. By the definition of the bracket in $F_2(q) = \mathbb{R}^q \oplus \Lambda^2(\mathbb{R}^q)$ we obtain $[T(x_i), T(x_j)] = [v_i, v_j] = v_i \wedge v_j = T(\rho(x_i \wedge x_j)) = T([x_i, x_j])$ for $1 \leq i < j \leq q$. Hence T is a Lie algebra isomorphism since $\text{Im } \rho$ lies in the center of $\{\mathbb{R}^n, \rho\}$. \square

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