Rational approximation in compact Lie groups and their Lie algebras*

Introduction

Let k be a field that is either $\mathbb{R}$ or $\mathbb{C}$. A subgroup H of GL(n,k) is an (affine) algebraic group defined over $\mathbb{Q}$ if there exists a finite set $\{p_1, p_2, \ldots p_N\}$ of polynomials in the variables $\{x_{ij} : 1 \le i,j \le n\}$ with rational coefficients such that $H = \{A \in GL(n,k) : p_i(A) = 0 \text{ for } 1 \le i \le N\}$. Any such group H is a Lie group since it is a closed subgroup of GL(n,k) in its standard Lie group topology.

More generally, let H be a compact, connected Lie group, U a finite dimensional real vector space and $\rho : H \to GL(U)$ a representation of H on U; that is, a Lie group homomorphism. Choosing a basis $\mathfrak{B}$ of U defines an associated homomorphism $\rho_{\mathfrak{B}} : H \to GL(n,\mathbb{R})$, and it is natural to ask the following questions :

<u>Question 1</u> : Can we choose a basis $\mathfrak{B}$ of U so that $\rho_{\mathfrak{B}}(H)$ is an algebraic subgroup of $GL(n,\mathbb{R})$ defined over $\mathbb{Q}$?

<u>Question 2</u> : Can we choose a basis $\mathfrak{B}$ of U so that $H_{\mathfrak{B},\mathbb{Q}} = \{h \in H : \rho_{\mathfrak{B}}(h) \in GL(n,\mathbb{Q})\}$ is a dense subgroup of H in the Lie topology of H ?

We will call a subgroup H' of H <u>rational</u> if $H' = H_{\mathfrak{B},\mathbb{Q}}$ for some basis $\mathfrak{B}$ of U and $\rho : H \to GL(U)$ has discrete kernel.

In fact, it is known that a basis $\mathfrak{B}$ of U that gives an affirmative answer to Question 1 also gives an affirmative answer to Question 2 [S, Cor. 3.5 (iii)]. In the case that H is semisimple, which is sufficient for our purposes, we include an elementary proof in the appendix since no elementary proof seems to exist in the literature.

In this article we show that Questions 1 and 2 have affirmative answers for any finite dimensional real representation $\rho: G_o \to GL(U)$, where G_o is a compact, connected semisimple Lie group. Moreover, the adjoint representation dominates all the others in a sense made precise in Theorem C of [E]. We state a special case of this result in section 5, following Theorem A. To find a satisfactory basis $\mathfrak{B}$ of U we first consider a suitable Chevalley basis $\mathfrak{C}$ of the complexification $\mathfrak{G}_o^{\mathbb{C}}$ of $\mathfrak{G}_o = L(G_o)$. From $\mathfrak{C}$ we pass to a well known real Chevalley basis $\mathfrak{C}_o$ of $\mathfrak{G}_o$, and we show that for any representation $\rho: G_o \to GL(U)$ there exists a basis $\mathfrak{B}$ of U such that $d\rho(\mathfrak{C}_o)$ leaves invariant $\mathbb{Z}$-span ($\mathfrak{B}$). Such a basis gives an affirmative answer to Questions 1 and 2 ,Theorem A of section 5.

---

* Supported in part by NSF grant DMS-9625452

Our second main result, Theorem B of section 5, is a kind of universal rational approximation result. Let $\mathfrak{G}_o$ denote a finite dimensional real Lie algebra that is compact and semisimple ; that is, the Killing form of $\mathfrak{G}_o$ is negative definite. Then there exists a dense subset $\mathfrak{G}_o^{\#}$ of $\mathfrak{G}_o$ with the following properties :

1) If $\sigma : \mathfrak{G}_o \to \mathrm{End}(U)$ is any finite dimensional real representation, then there exists a basis $\mathfrak{B}$ of U such that the matrix of $\sigma(X)$ relative to $\mathfrak{B}$ has entries in $\mathbb{Q}$ for all $X \in \mathfrak{G}_o^{\#}$.
2) If $\sigma : \mathfrak{G}_o \to \mathrm{End}(U)$ is any finite dimensional real representation, then the eigenvalues of $\sigma(X)$ are contained in $i\,\mathbb{Q}$ for all $X \in \mathfrak{G}_o^{\#}$.
3) Let G_o be any compact, connected, semisimple Lie group with Lie algebra $\mathfrak{G}_o$. Then for each $X \in \mathfrak{G}_o^{\#}$ there exists a positive integer m such that $\exp(2\pi mX) = \exp(2\pi X)^m = e$, the identity in G_o.

The set $\mathfrak{G}_o^{\#}$ is explicitly defined in the proof of Theorem B, and the definition depends only on the choice of a real Chevalley basis $\mathfrak{C}_o$ of $\mathfrak{G}_o$. In principle $\mathfrak{G}_o^{\#}$ can be calculated for each of the standard real Chevalley bases $\mathfrak{C}_o$ of the classical simple Lie subalgebras of $M(n,\mathbb{R})$. Each Chevalley basis $\mathfrak{C}_o$ arises from a Cartan subalgebra of $\mathfrak{G}_o^{\mathbb{C}}$ as described in section 1.5. A list of the standard Cartan subalgebras and root space decompositions of the classical complex simple Lie algebras may be found, for example, in [He, pp. 186-190].

There is also an analogous approximation result for compact, connected, semisimple Lie groups G_o. By using the result above we can show that there exists a dense subset $G_o^{\#}$ of G_o such that if $\rho : G_o \to GL(U)$ is any finite dimensional real representation, then the eigenvalues of $\rho(g)$ lie in $F = \mathbb{Q}(i) = \mathbb{Q} + i\,\mathbb{Q}$ for all $g \in G_o^{\#}$.

Modern representation theory focuses primarily on complex or infinite dimensional representations. A large part of this paper involves translating rationality questions about finite dimensional real representations of compact semisimple Lie groups into more familiar questions about finite dimensional complex representations of complex semisimple Lie algebras. The first four sections of the paper are devoted to this translation, and we make no claims of originality for any of the results in these sections, although we are not aware of published references for some of them. The exposition attempts to be self contained and to summarize those basic concepts and results from the representation theory of complex semisimple Lie algebras that are needed here. This paper is aimed at people like the author who are not experts in representation theory but who are interested in applications of it to related fields. Most of the framework for this paper was motivated by the problem of constructing lattices in naturally reductive, simply connected, 2-step nilpotent Lie groups with a left invariant metric.

Throughout this article $\mathfrak{G}_o$ will denote a finite dimensional, compact, semisimple Lie algebra, and U will denote a finite dimensional real $\mathfrak{G}_o$ - module. We let $\mathfrak{G}$ denote a finite dimensional complex semisimple Lie algebra, which in this article typically equals $\mathfrak{G}_o{}^{\mathbb{C}}$, and we let V denote a finite dimensional complex $\mathfrak{G}$ - module.

In section 1 we collect some basic facts that we need about roots, root space decompositions, Weyl groups, Chevalley bases and universal enveloping algebras $\mathfrak{U}(\mathfrak{G})$ of complex semisimple Lie algebras $\mathfrak{G}$. The material is standard but scattered throughout the basic texts.

In section 2 we consider finite dimensional complex $\mathfrak{G}$-modules V, and we describe some basic results about the weight spaces of V determined by a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$. Given a Chevalley basis $\mathbb{C}$ for $\mathfrak{G}$, which arises from the root space decomposition of $\mathfrak{G}$ determined by $\mathfrak{A}$, we describe a standard construction of a basis $\mathbb{B}^*$ of V such that $\mathbb{C}$ leaves invariant $\mathbb{Z}$-span ($\mathbb{B}^*$). There exists a special subring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{G})$ such that $\mathbb{C} \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ and if v is any nonzero highest weight vector in V, then $U(\mathfrak{G})_{\mathbb{Z}}(v)$ is a finitely generated $\mathbb{Z}$-module in V. (Recall that any $\mathfrak{H}$-module is also a $\mathfrak{U}(\mathfrak{H})$-module for any Lie algebra $\mathfrak{H}$.) If $\mathbb{B}^*$ is any $\mathbb{Z}$ -basis of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, then $\mathbb{B}^*$ is also a $\mathbb{C}$-basis of V, and $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$, hence also $\mathbb{C}$, leaves invariant $\mathbb{Z}$-span ($\mathbb{B}^*$) = $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$.

In section 3 we discuss the two types of irreducible, real, finite dimensional $\mathfrak{G}_o$-modules U : 1) $V = U^{\mathbb{C}}$ is an irreducible complex $\mathfrak{G}_o{}^{\mathbb{C}}$-module. 2) There exists an irreducible, complex, finite dimensional $\mathfrak{G}_o{}^{\mathbb{C}}$-module V such that $U = V^{\mathbb{R}}$ (V considered as a real vector space) . Recall that $\mathfrak{G}_o{}^{\mathbb{C}}$ is a complex semisimple Lie algebra . Conversely, if $\mathfrak{G}$ is any finite dimensional complex semisimple Lie algebra, then $\mathfrak{G} = \mathfrak{G}_o{}^{\mathbb{C}}$ for some real, compact , semisimple subalgebra $\mathfrak{G}_o$, and any two choices of $\mathfrak{G}_o$ lie in the same orbit of Aut ($\mathfrak{G}$).

Most of the discussion in section 3 concerns case 1) and relations between the conjugation operator $J : V \to V$ determined by U, the weight spaces of V and certain transformations T_σ of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$, one for each element σ of the Weyl group W, that permute the weight spaces of V according to the action of σ on the weights of V.

Given a real, finite dimensional $\mathfrak{G}_o$-module U and a real Chevalley basis $\mathbb{C}_o$ of $\mathfrak{G}_o$ we define $B(\mathbb{Z},\mathbb{C}_o)$ (respectively $B(\mathbb{Q},\mathbb{C}_o)$) to be the set of bases $\mathbb{B}$ of U such that every element of $\mathbb{C}_o$ leaves invariant $\mathbb{Z}$-span ($\mathbb{B}$) (respectively $\mathbb{Q}$-span ($\mathbb{B}$)). The main result of the first four sections appears in section 4 and states that $B(\mathbb{Z},\mathbb{C}_o)$ is nonempty for any real Chevalley basis $\mathbb{C}_o$ of $\mathfrak{G}_o$. One begins by choosing a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G} = \mathfrak{G}_o{}^{\mathbb{C}}$ and a Chevalley basis $\mathbb{C}$ of $\mathfrak{G}$ that induces the real Chevalley basis $\mathbb{C}_o$ of $\mathfrak{G}_o$ (cf. section 1.5). In case 2) one sets $\mathbb{B} = \mathbb{B}^* \cup i\, \mathbb{B}^*$, where $\mathbb{B}^*$ is the $\mathbb{Z}$-basis of

$\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ described above and v is any nonzero highest weight vector of V. In case 1) one must choose the nonzero highest weight vector v so that J leaves invariant $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. The existence and further properties of such vectors v is given in Proposition 3.2e. Given a nonzero highest weight vector v such that J leaves invariant $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ we again choose $\mathcal{B}^*$ to be a $\mathbb{Z}$-basis of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, and we now define $\mathcal{B}$ to be any $\mathbb{Z}$-basis for $\mathbb{Z}$-span (Re $(\mathcal{B}^*)$, Im $(\mathcal{B}^*)$). In each of the two cases $\mathcal{B} \in B(\mathbb{Z}, \mathcal{C}_o)$.

In section 5 we prove the Theorems A and B stated above.

We would like to give special thanks to Shrawan Kumar, who explained many results about representation theory, and to Gopal Prasad for many helpful communciations about algebraic groups. We are also grateful to Yves Benoist, Tom Brylawski and Marc Burger for helpful direction and information.

Section 1 Notation and preliminaries

1.0 Notation :

We establish the following notation for this article. Definitions will be given below. All vector spaces, Lie groups and Lie algebras are finite dimensional.

α, β	arbitrary elements of the finite set Φ of roots in $\mathfrak{A}^*$
$\mathfrak{A}$	a Cartan subalgebra of $\mathfrak{G}$
$\mathfrak{A}_o$	$\mathbb{R}$-span $\{i H_\beta : \beta \in \Delta\}$, a maximal abelian subalgebra of $\mathfrak{G}_o$
A_α	the vector $\xi_\alpha - \xi_{-\alpha}$, an element of a real Chevalley basis $\mathcal{C}_o$ for $\mathfrak{G}_o$
$\mathcal{B}^*$	a basis for a finite dimensional complex vector space V
$\mathcal{B}$	a basis for a finite dimensional real vector space U
B_α	the vector $i\,\xi_\alpha + i\,\xi_{-\alpha}$, an element of a real Chevalley basis $\mathcal{C}_o$ for $\mathfrak{G}_o$
B	the Killing form of $\mathfrak{G}$
B_o	the Killing form of $\mathfrak{G}_o$
$\mathcal{C}$	a (complex) Chevalley basis for $\mathfrak{G}$ whose structure constants lie in $\mathbb{Z}$, $\mathcal{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$
$\mathcal{C}_o$	a real Chevalley basis whose structure constants lie in $\mathbb{Z}$ for a compact real form $\mathfrak{G}_o$ of $\mathfrak{G}$, $\mathcal{C}_o = \{i H_\beta : \beta \in \Delta ; A_\alpha, B_\alpha : \alpha \in \Phi^+\}$
Δ	a $\mathbb{Z}$ - base for Φ
Φ	the finite set of roots in $\mathfrak{A}^* = \mathrm{Hom}(\mathfrak{A}, \mathbb{C})$
Φ^+, Φ^-	the positive, negative roots of Φ as determined by Δ

F	$\mathbb{Q}(i) = \mathbb{Q} + i\mathbb{Q} = \{a + ib \in \mathbb{C} : a \in \mathbb{Q}$ and $b \in \mathbb{Q}\}$.
$\mathfrak{G}$	a complex semisimple Lie algebra
$\mathfrak{G}_{\alpha}$	the 1-dimensional eigenspace of $\mathfrak{G}$ on which ad $A = \alpha(A)$ Id for all A in $\mathfrak{A}$
$\mathfrak{G}_o$	a compact, real, semisimple Lie algebra ; the Lie algebra of G_o
$\mathfrak{G}_o^{\mathbb{C}}$	the complexification of $\mathfrak{G}_o$
G_o	a compact, connected semisimple Lie group
$G_{o,\mathbb{Q}}$	$\{g \in G_o$: Ad(g) leaves invariant $\mathbb{Q}$-span -$\mathcal{C}_o\}$
$G_{o,\mathcal{B},\mathbb{Q}}$	$\{g \in G_o : \rho(g)$ leaves invariant $\mathbb{Q}$-span $(\mathcal{B})\}$
H_{α}	the root vector in $\mathfrak{A}$ determined by a root α in Φ
λ	a highest or lowest weight in $\Lambda(V)$, where V is a complex $\mathfrak{G}$-module
Λ	the set of abstract weights for $\mathfrak{G} = \{\mu \in \mathfrak{A}^* : \mu(H_{\alpha}) \in \mathbb{Z}$ for all $\alpha \in \Phi\}$, a vector lattice in $\mathfrak{A}^*$
Λ_r	the $\mathbb{Z}$- span of Φ in $\mathfrak{A}^*$, the root lattice of finite index in Λ
$\Lambda(V)$	a finite subset of $\mathfrak{A}^*$, the weights determined by $\mathfrak{A}$ and a complex G (or $\mathfrak{G}$) - module V
μ	an arbitrary weight in $\Lambda(V)$, V a finite dimensional, complex $\mathfrak{G}$-module.
τ	the unique element of W such that $\tau^2 = \text{Id}$, $\tau(\Phi^+) = \Phi^-$ and $\tau(\Phi^-) = \Phi^+$
T_{σ}	for $\sigma \in W$, an element of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ such that $T_{\sigma}(V_{\mu}) = V_{\sigma(\mu)}$ for all $\mu \in \Lambda(V)$
U	a real G_o-module or $\mathfrak{G}_o$ - module
$U^{\mathbb{C}}$	the complexification of U
$\mathfrak{U}(\mathfrak{G})$	the universal enveloping algebra of $\mathfrak{G}$
$\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$	the subring of $\mathfrak{U}(\mathfrak{G})$ generated by 1 and $\{(\xi_{\alpha})^n / n! : \alpha \in \Phi$ and $n \in \mathbb{Z}^+\}$
V	a complex G_o - module or $\mathfrak{G}$ - module
$V^{\mathbb{R}}$	the realification of V, V considered as a vector space over $\mathbb{R}$
V_{μ}	the weight space of V on which $A = \mu(A)$ Id for all A in $\mathfrak{A}$
W	the Weyl group, a finite subgroup of $GL(\mathfrak{A})$ or $GL(\mathfrak{A}^*)$
ξ_{α}	a vector that spans $\mathfrak{G}_{\alpha}$ and is an element of a Chevalley basis of $\mathfrak{G}$

Semisimple Lie algebras and groups

1.1 Semisimple Lie algebras and Jordan decompositions

For a field F recall that the Killing form B of a finite dimensional Lie algebra $\mathfrak{H}$ over F is the symmetric bilinear form B : $\mathfrak{H} \times \mathfrak{H} \to F$ given by B(X,Y) = trace (ad X $\circ$ ad Y) for any elements X, Y of $\mathfrak{H}$. A finite dimensional Lie algebra over F is semisimple if its Killing form B is nondegenerate.

If $\mathfrak{G}$ is a complex, finite dimensional, semisimple Lie algebra then for every element X of $\mathfrak{G}$ there exist elements S and N of $\mathfrak{G}$ such that X = S + N, [S,N] = 0, ad S is a semisimple element of End($\mathfrak{G}$) and ad N is a nilpotent element of End($\mathfrak{G}$). Note that ad S and ad N commute by the Jacobi identity since [S,N] = 0. The elements S and N above are uniquely determined. This decomposition of X $\in \mathfrak{G}$ is called the abstract Jordan decomposition of X, and in fact, the decomposition ad X = ad S + ad N is the usual additive Jordan decomposition of ad X in End($\mathfrak{G}$). See, for example, [Hu 1, p.24].

If $\mathfrak{G}$ is a subalgebra of End(V) for some finite dimensional, complex vector space V, then the elements X of $\mathfrak{G}$ have an additive Jordan decomposition as elements of End(V) ; namely, X = S + N, where SN = NS, S is semisimple and N is nilpotent. These two Jordan decompositions, abstract and additive, coincide if $\mathfrak{G}$ is a semisimple subalgebra of End(V) (cf. [Hu 1, p.29]. This implies the following basic result, which indicates the importance of the adjoint representation ad : $\mathfrak{G} \to$ End($\mathfrak{G}$).

Proposition Let $\mathfrak{G}$ be a finite dimensional, complex semisimple Lie algebra, and let X be an element of $\mathfrak{G}$ such that ad X is a semisimple (respectively nilpotent) element of End($\mathfrak{G}$). Let $\sigma : \mathfrak{G} \to$ End(V) be a Lie algebra homomorphism, where V is a finite dimensional, complex vector space. Then σ(X) is a semisimple (respectively nilpotent) element of End(V).

Proof

We show first that if X is any element of $\mathfrak{G}$ such that ad X is a semisimple (respectively nilpotent) element of End($\mathfrak{G}$), then ad (σX) is a semisimple (respectively nilpotent) element of End ($\sigma(\mathfrak{G})$). Note that $\sigma \circ$ ad X = ad (σX) $\circ\, \sigma$ for every element X of $\mathfrak{G}$. If ad X is semisimple and $\mathbb{B}^*$ is any basis of $\mathfrak{G}$ consisting of eigenvectors of ad X, then $\sigma(\mathbb{B}^*)$ spans the semisimple algebra $\sigma(\mathfrak{G})$, and $\sigma(\mathbb{B}^*)$ consists of eigenvectors of ad (σX). Hence ad (σX) is a semisimple element of End ($\sigma(\mathfrak{G})$). If ad X is nilpotent, then since $\sigma \circ (\text{ad } X)^n = (\text{ad } (\sigma X))^n \circ \sigma$ for all positive integers n (induction on n) it follows that ad (σX) is a nilpotent element of End ($\sigma(\mathfrak{G})$).

Next, let X be an arbitrary element of $\mathfrak{G}$, and let X = S+N be its abstract Jordan decomposition. The discussion in the previous paragraph shows that $\sigma(X) = \sigma(S)+\sigma(N)$ is the abstract Jordan decomposition of $\sigma(X)$ in the semisimple Lie algebra $\sigma(\mathfrak{G})$. Since $\sigma(\mathfrak{G}) \subseteq \mathrm{End}(V)$ the discussion above shows that $\sigma(X) = \sigma(S)+\sigma(N)$ is also the additive Jordan decomposition of $\sigma(X)$ in End(V). In particular, if ad X is semisimple, then N = 0 and $\sigma(X) = \sigma(S)$ is a semisimple element of End(V). Similarly, if ad X is nilpotent, then S = 0 and $\sigma(X) = \sigma(N)$ is a nilpotent element of End(V).□

1.2 Cartan subalgebras and root space decompositions

A Cartan subalgebra of a finite dimensional, complex semisimple Lie algebra $\mathfrak{G}$ is a maximal abelian subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ such that ad A : $\mathfrak{G} \to \mathfrak{G}$ is a semisimple linear transformation for all A in $\mathfrak{A}$. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are any two Cartan subalgebras of $\mathfrak{G}$, then there exists an element φ of Aut ($\mathfrak{G}$) such that $\varphi(\mathfrak{A}_1) = \mathfrak{A}_2$.

If $\mathfrak{A}$ is a Cartan subalgebra of a finite dimensional, complex semisimple Lie algebra $\mathfrak{G}$, then by 1.1 ad($\mathfrak{A}$) is a family of commuting semisimple linear transformations in End ($\mathfrak{G}$). The decomposition of $\mathfrak{G}$ into a direct sum of common eigenspaces of the elements of ad($\mathfrak{A}$) is called the root space decomposition of $\mathfrak{G}$. Specifically, there is a finite subset Φ of $\mathfrak{A}^* = \mathrm{Hom}(\mathfrak{A},\mathbb{C})$, the roots of $\mathfrak{G}$, such that

$$\mathfrak{G} = \mathfrak{A} + \sum_{\alpha\in\Phi} \mathfrak{G}_\alpha \qquad \text{(direct sum)}$$

where ad A = $\alpha(A)$ Id on the subspace $\mathfrak{G}_\alpha$ for each A in $\mathfrak{A}$ and each α in Φ.

Proposition

1) Each subspace $\mathfrak{G}_\alpha$ is a 1-dimensional complex subspace of $\mathfrak{G}$.
2) If α, β are any roots, then $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] \subseteq \mathfrak{G}_{\alpha+\beta}$ if $\alpha+\beta \in \Phi$ and $[\mathfrak{G}_\alpha, \mathfrak{G}_\beta] = \{0\}$ otherwise.

Proof

See, for example, [Hu 1, p.39].□

1.3 Root vectors and Weyl group

If $\mathfrak{A}$ is a Cartan subalgebra of a finite dimensional, complex semisimple Lie algebra $\mathfrak{G}$, then the restriction of the Killing form B to $\mathfrak{A}$ is nondegenerate on $\mathfrak{A}$ (cf. [Hu 1, p.37]). Hence we may define a $\mathbb{C}$ - linear isomorphism between $\mathfrak{A}^*$ and $\mathfrak{A}$ by

$$\lambda \to t_\lambda \qquad \text{, where } \lambda(A) = B(t_\lambda, A) \text{ for all A in } \mathfrak{A}.$$

Next, we define a symmetric, bilinear form (,) : $\mathfrak{A}^* \times \mathfrak{A}^* \to \mathbb{C}$ by

$$(\lambda, \mu) = B(t_\lambda, t_\mu)$$

Lemma 1.3a

1) If $\mathfrak{A}_{\mathbb{R}} = \mathbb{R}$ - span $\{t_\alpha : \alpha \in \Phi\}$, then $B : \mathfrak{A}_{\mathbb{R}} \times \mathfrak{A}_{\mathbb{R}} \to \mathbb{C}$ has values in $\mathbb{R}$, and B is positive definite on $\mathfrak{A}_{\mathbb{R}}$.

2) If $\mathfrak{A}^*_{\mathbb{R}} = \mathbb{R}$ - span (Φ), then (,) : $\mathfrak{A}^*_{\mathbb{R}} \times \mathfrak{A}^*_{\mathbb{R}} \to \mathbb{C}$ has values in $\mathbb{R}$, and (,) is positive definite on $\mathfrak{A}^*_{\mathbb{R}}$.

Proof

Assertion 2) follows immediately from 1). For a proof of 1) see [Hu 1, p.40] or [He, p.145] □

Remark : We also use (,) to denote the positive definite inner product on $\mathfrak{A}_{\mathbb{R}}$ given by

$$(A, A^*) = B(A, A^*) \qquad \text{for all } A, A^* \text{ in } \mathfrak{A}.$$

Definitions and notation :

For each root α in Φ we define a corresponding <u>root vector</u> H_α in $\mathfrak{A}$ by

$$H_\alpha = 2\, t_\alpha / (\alpha,\alpha) = 2\, t_\alpha / (t_\alpha, t_\alpha)$$

Note that $\alpha(H_\alpha) = 2$ for all α in Φ.

For each α in Φ define the <u>Weyl reflection</u> σ_α in Hom $(\mathfrak{A}^*, \mathfrak{A}^*)$ by

$$\sigma_\alpha(\lambda) = \lambda - \{2\,(\lambda,\alpha) / (\alpha,\alpha)\}\, \alpha \qquad \text{for all } \lambda \text{ in } \mathfrak{A}^*.$$

Note that $\sigma_\alpha^{\,2}$ = Id and σ_α is the reflection in the hyperplane

$$P_\alpha^* = \{\lambda \in \mathfrak{A}^* : (\lambda,\alpha) = 0\}$$

Let W denote the subgroup of GL($\mathfrak{A}^*$) generated by the Weyl reflections $\{\sigma_\alpha : \alpha \in \Phi\}$. The group W is finite and is called the <u>Weyl group</u> of $\mathfrak{G}$ (relative to the choice of Cartan subalgebra $\mathfrak{A}$). Any two Weyl groups are isomorphic since any two Cartan subalgebras lie in a single orbit of Aut ($\mathfrak{G}$).

Action of W on $\mathfrak{A}$ and $\mathfrak{A}^*$

We let W act on $\mathfrak{A}$ by invertible linear transformations by defining

$$\lambda(\sigma(A)) = (\sigma^{-1}\lambda)(A) \quad \text{for all } \sigma \text{ in W, all } \lambda \text{ in } \mathfrak{A}^* \text{ and all A in } \mathfrak{A}.$$

We list some basic facts.

Lemma 1.3b

1) $\sigma_\alpha(A) = A - \alpha(A)H_\alpha$ for all A in $\mathfrak{A}$ and all $\alpha \in \Phi$. In particular, σ_α is the reflection in $\mathfrak{A}$ in the hyperplane orthogonal to H_α since $\alpha(H_\alpha) = 2$.

2) The Weyl group W preserves the bilinear form (,) on $\mathfrak{A}$ and $\mathfrak{A}^*$.

a) $(\sigma(A), \sigma(A^*)) = (A, A^*)$ for all $\sigma \in W$ and all $A, A^* \in \mathfrak{A}$.

b) $(\sigma(\mu), \sigma(\lambda)) = (\mu, \lambda)$ for all $\sigma \in W$ and all $\mu, \lambda \in \mathfrak{A}^*$.

3) $\sigma(t_\mu) = t_{\sigma(\mu)}$ for all $\sigma \in W$ and all $\mu \in \mathfrak{A}^*$.

4) $\sigma(H_\alpha) = H_{\sigma(\alpha)}$ for all $\sigma \in W$ and all $\alpha \in \Phi$.

5) $\sigma(\mu)(H_\alpha) = \mu(H_{\sigma^{-1}(\alpha)})$ for all $\sigma \in W$, all $\alpha \in \Phi$ and all $\mu \in \mathfrak{A}^*$.

Proof

These assertions follow routinely from the definitions. In 1) it suffices to prove that $\lambda(\sigma_\alpha(A)) = \lambda(A - \alpha(A)H_\alpha)$ for all $\lambda \in \mathfrak{A}^*$. In 2) and 3) it suffices to consider the generators $\{\sigma_\alpha : \alpha \in \Phi\}$ for W and in 3) to prove that $(\sigma(t_\mu), A) = (t_{\sigma(\mu)}, A)$ for all $A \in \mathfrak{A}$. Assertion 4) follows from 2b) and 3), and 5) follows from 4).□

Some useful facts

We now collect some results that will be helpful later. The assertions 2) through 4) in the next result say that Φ satisfies the axioms for a <u>root system</u> (cf. [Hu 1, p. 42])

Proposition 1.3c Using the notation above one has

1) $\mathfrak{A} = \mathbb{C}$ - span $\{H_\alpha : \alpha \in \Phi\}$.

2) $\beta(H_\alpha) \in \mathbb{Z}$ for all α, β in Φ.

3) If $\alpha \in \Phi$, then $-\alpha \in \Phi$. If $m\alpha \in \Phi$ for some $m \in \mathbb{Z}$, then $m = 1$ or -1.

4) The Weyl group W leaves Φ invariant. In particular for α, β in Φ we have

$$\sigma_\alpha(\beta) = \beta - \beta(H_\alpha)\,\alpha$$

5) If $\mathfrak{A}^*_{\mathbb{R}} = \mathbb{R}$ - span $(\Phi) \subseteq \mathfrak{A}^*$, then W leaves $\mathfrak{A}^*_{\mathbb{R}}$ invariant and defines a finite group of isometries of $\mathfrak{A}^*_{\mathbb{R}}$ with the inner product (,). Similarly, if $\mathfrak{A}_{\mathbb{R}} = \mathbb{R}$ - span $\{t_\alpha : \alpha \in \Phi\} = \mathbb{R}$ - span $\{H_\alpha : \alpha \in \Phi\}$, then W leaves $\mathfrak{A}_{\mathbb{R}}$ invariant and defines a finite group of isometries of $\mathfrak{A}_{\mathbb{R}}$ with the inner product (,) = B(,).

Proof

1) If A is any nonzero element of $\mathfrak{A}$, then ad $A : \mathfrak{G} \to \mathfrak{G}$ is nonzero since $\mathfrak{G}$ is semisimple, and hence, by the root space decomposition, $\alpha(A)$ is nonzero for some $\alpha \in \Phi$. It follows that $B(A, H_\alpha) = 2\,\alpha(A) / (\alpha, \alpha) \neq 0$, and hence no nonzero element of $\mathfrak{A}$ is B - orthogonal to $\mathbb{C}$- span $\{H_\alpha : \alpha \in \Phi\}$. Since B is nondegenerate on $\mathfrak{A}$ we are done.

For proofs of 2), 3) and 4) see [Hu 1, p.39]. The second assertion in 4) follows from the definitions of H_α and the action of σ_α on $\mathfrak{A}^*$. The group W leaves $\mathfrak{A}^*_{\mathbb{R}}$ and $\mathfrak{A}_{\mathbb{R}}$ invariant by 4) and 3) of Lemma 1.3b. Assertion 5) now follows from this remark and 2) of Lemma 1.3b.□

1.4 Root basis, positive and negative roots and Chevalley bases

A good reference for this section is [Hu 1, pp. 47-50 and 143-146.]

An element γ of $\mathfrak{A}^*_{\mathbb{R}}$ = is regular if (γ, α) is nonzero for all $\alpha \in \Phi$. For a regular element γ of $\mathfrak{A}^*_{\mathbb{R}}$ let $\Phi^+(\gamma) = \{\alpha \in \Phi : (\alpha,\gamma) > 0\}$ and let $\Delta = \Delta(\gamma) = \{\alpha \in \Phi^+(\gamma) : \alpha$ is indecomposable, i.e α cannot be written as a sum of two elements of $\Phi^+(\gamma)\}$. Then

1) Δ is an $\mathbb{R}$ - basis for $\mathfrak{A}^*_{\mathbb{R}}$.

2) If $\Delta = \{\alpha_1, ..., \alpha_n\}$ and α is any element of Φ, then there exist integers $\{m_1, ..., m_n\}$ such that

a) $\alpha = \Sigma\, m_i\, \alpha_i$

b) Either $m_i \geq 0$ for every i or $m_i \leq 0$ for every i.

Define the positive roots Φ^+ and the negative roots Φ^- (as determined by Δ)by
$\Phi^+ = \{\alpha \in \Phi : \alpha = \Sigma\, m_i\, \alpha_i$, where $m_i \geq 0$ for every i $\}$ and
$\Phi^- = \{\alpha \in \Phi : \alpha = \Sigma\, m_i\, \alpha_i$, where $m_i \leq 0$ for every i $\}$. Clearly $\Phi = \Phi^+ \cup \Phi^-$.

Properties 1) and 2) say that Δ is a base for the roots Φ. Any base Δ for Φ can be obtained as $\Delta = \Phi^+(\gamma)$ for some regular element γ of $\mathfrak{A}^*_{\mathbb{R}}$. The Weyl group acts simply transitively on the set of all bases for Φ.

If Δ is a base for the roots Φ, then the root vectors $\{H_\alpha : \alpha \in \Delta\}$ are a $\mathbb{C}$-basis for the Cartan subalgebra $\mathfrak{A}$ by 1) of Proposition 1.3c. We extend the root vectors to a Chevalley basis $\mathfrak{C}$ by appropriately choosing elements ξ_α from the 1-dimensional subspaces $\mathfrak{G}_\alpha$ that occur in the root space decomposition of $\mathfrak{G}$. The Chevalley basis $\mathfrak{C}$ is not uniquely defined by $\mathfrak{A}$ and Δ. See [Hu 1, p.146] for further discussion.

Proposition Let $\mathfrak{A}$ be a Cartan subalgebra in a complex, semisimple Lie algebra $\mathfrak{G}$, and let Δ be a base of the roots $\Phi \subseteq \mathfrak{A}^*$. For each $\alpha \in \Phi$ one can choose a nonzero vector $\xi_\alpha \in \mathfrak{G}_\alpha$ such that the set $\mathfrak{C} = \{H_\beta : \beta \in \Delta\ ;\ \xi_\alpha : \alpha \in \Phi\}$ has the following properties :

1) $[\xi_\alpha, \xi_{-\alpha}] = H_\alpha$; $[H_\alpha, \xi_\alpha] = 2\,\xi_\alpha$; $[H_\alpha, \xi_{-\alpha}] = -\,2\,\xi_{-\alpha}$ for all $\alpha \in \Phi$.

2) $[H_\alpha, \xi_\beta] = \beta(H_\alpha)\,\xi_\beta$ for all $\alpha, \beta \in \Phi$, where $\beta(H_\alpha) \in \mathbb{Z}$.

3) Given linearly independent roots $\alpha, \beta \in \Phi$ there exist integers $r \geq 0$ and $s \geq 0$ such that

a) $\beta + k\alpha \in \Phi$ for $-\,r \leq k \leq s$

b) $[\xi_\alpha, \xi_\beta] = \pm(r+1)\,\xi_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$

c) $[\xi_\alpha, \xi_\beta] = 0$ if $\alpha + \beta \notin \Phi$

4) For each $\alpha \in \Phi$, $H_\alpha \in \mathbb{Z}$-span $\{H_\beta : \beta \in \Delta\}$

Proof

See for example [Hu 1, p. 145].□

Example Let $\mathfrak{G} = s\ell(2,\mathbb{C}) = \mathbb{C}$-span $\{\xi_1, \xi_2, \xi_3\}$, where $\xi_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\xi_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\xi_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ with the standard bracket relations $[\xi_1, \xi_2] = 2\xi_2$, $[\xi_1, \xi_3] = -2\xi_3$ and $[\xi_2, \xi_3] = \xi_1$. In this case $\mathfrak{C} = \{\xi_1, \xi_2, \xi_3\}$ is a Chevalley basis of $s\ell(2,\mathbb{C})$ determined by the root space decomposition $s\ell(2,\mathbb{C}) = \mathfrak{A} + \mathfrak{G}_2 + \mathfrak{G}_{-2}$, where $\mathfrak{A} = \mathbb{C}\xi_1$, $\Phi = \{\alpha, -\alpha\}$, $\Delta = \{\alpha\}$, $\mathfrak{G}_\alpha = \mathbb{C}\xi_2$ and $\mathfrak{G}_{-\alpha} = \mathbb{C}\xi_3$, where $\alpha \in \mathfrak{A}^*$ satisfies $\alpha(\xi_1) = 2$.

1.5 Compact real forms and real Chevalley bases

A good reference for this section is [He, pp.152-159].

A finite dimensional, real Lie algebra $\mathfrak{G}_o$ is <u>compact</u> and <u>semisimple</u> if its Killing form B_o is negative definite. If $\mathfrak{G}_o$ is compact and semisimple, and if G_o is any connected Lie group with Lie algebra $\mathfrak{G}_o$, then G_o is compact (cf. [He, Theorem 6.9, p.123])

A <u>compact real form</u> of a finite dimensional, complex semisimple Lie algebra $\mathfrak{G}$ is a compact, semisimple Lie algebra $\mathfrak{G}_o$ whose complexification $\mathfrak{G}_o^{\mathbb{C}}$ is $\mathfrak{G}$. Every finite dimensional, complex semisimple Lie algebra $\mathfrak{G}$ admits a compact real form $\mathfrak{G}_o$, and the set of all compact real forms for $\mathfrak{G}$ is the orbit of $\mathfrak{G}_o$ under Aut $(\mathfrak{G})$.

Every Chevalley basis $\mathfrak{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ for $\mathfrak{G}$ defines a compact real form $\mathfrak{G}_o$ as follows. For each positive root α (i.e. each $\alpha \in \Phi^+$) define vectors $A_\alpha = \xi_\alpha - \xi_{-\alpha}$ and $B_\alpha = i\,\xi_\alpha + i\,\xi_{-\alpha}$. Define $\mathfrak{C}_o = \{i\,H_\beta : \beta \in \Delta ; A_\alpha, B_\alpha : \alpha \in \Phi^+\}$ and define $\mathfrak{G}_o = \mathbb{R}$–span $(\mathfrak{C}_o)$. Then $\mathfrak{G}_o$ is a compact real form for $\mathfrak{G}$ (cf. [He, pp.155-156]).

If $\mathfrak{G}_o$ is any given compact real form for $\mathfrak{G}$, then the fact that all compact real forms of $\mathfrak{G}$ lie in an orbit of Aut$(\mathfrak{G})$ implies that one can choose a Cartan subalgebra $\mathfrak{A}$, a basis Δ for Φ and a Chevalley basis $\mathfrak{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ for $\mathfrak{G}$ such that $\mathfrak{G}_o = \mathbb{R}$–span $(\mathfrak{C}_o)$, where $\mathfrak{C}_o$ is defined as above. We call a basis $\mathfrak{C}_o$ for $\mathfrak{G}_o$ that is constructed in this manner a <u>real Chevalley basis</u> for $\mathfrak{G}_o$.

Alternatively, we can make a direct construction of the Chevalley basis $\mathfrak{C}$ for $\mathfrak{G}$ such that $\mathfrak{G}_o = \mathbb{R}$–span $(\mathfrak{C}_o)$. Given $\mathfrak{G}_o$, we let $\mathfrak{A}_o$ be a maximal abelian subalgebra of $\mathfrak{G}_o$ and let $\mathfrak{A} = \mathfrak{A}_o^{\mathbb{C}} \subseteq \mathfrak{G}_o^{\mathbb{C}}$, the complexification of $\mathfrak{A}_o$. Then $\mathfrak{A}$ is a Cartan subalgebra of $\mathfrak{G} = \mathfrak{G}_o^{\mathbb{C}}$ and $\mathfrak{A}_o = \mathbb{R}$-span $\{i\,H_\beta : \beta \in \Phi\}$. If $\mathfrak{C} =$

$\{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ is a Chevalley basis of $\mathfrak{G}$, then $\mathfrak{G}_o = \mathbb{R}$–span $(\mathfrak{C}_o)$.

Example Let $\mathfrak{G}_o = su(2) = \mathbb{R}$-span $\{Z_1, Z_2, Z_3\}$, where $Z_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $Z_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $Z_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$. If $\mathfrak{G} = \mathfrak{G}_o{}^{\mathbb{C}}$, then $\mathfrak{G} \approx s\ell(2,\mathbb{C}) = \mathbb{C}$-span $\{\xi_1, \xi_2, \xi_3\}$ as above in 1.4. Note that $Z_1 = i\,\xi_1$, $Z_2 = \xi_2 - \xi_3$ and $Z_3 = i\,\xi_2 + i\,\xi_3$. Hence if we consider the Chevalley basis $\mathfrak{C} = \{\xi_1, \xi_2, \xi_3\}$ for $\mathfrak{G}$, then $\mathfrak{C}_o = \{Z_1, Z_2, Z_3\}$ is the corresponding real Chevalley basis for $\mathfrak{G}_o$.

Proposition

Let $\mathfrak{G}_o$ be a finite dimensional, compact, semisimple real Lie algebra, and let $\mathfrak{C}_o = \{X_1, \dots, X_n\}$ be a real Chevalley basis for $\mathfrak{G}_o$. Then $[X_i, X_j] = \sum_{j=1}^{n} C_{ij}^k X_k$ where $\{C_{ij}^k\} \subseteq \mathbb{Z}$ for all i, j, k.

Proof

This follows routinely from the definition of $\mathfrak{C}_o$ and the structure of complex Chevalley bases $\mathfrak{C}$ as described in the Proposition of 1.4. □

1.6 The universal enveloping algebra $\mathfrak{U}(\mathfrak{G})$ and the subring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$

For a formal treatment of universal enveloping algebras see for example [Hu 1, pp. 89-101].

The universal enveloping algebra $\mathfrak{U}(\mathfrak{H})$ of a Lie algebra $\mathfrak{H}$ is an associative algebra defined by a universal mapping property, and $\mathfrak{U}(\mathfrak{H})$ can be realized as $\mathfrak{T}(\mathfrak{H})$ / J, where $\mathfrak{T}(\mathfrak{H})$ denotes the tensor algebra of $\mathfrak{H}$ and J denotes the two sided ideal of $\mathfrak{T}(\mathfrak{H})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$, where x and y are arbitrary elements of $\mathfrak{H}$.

The algebra $\mathfrak{U}(\mathfrak{H})$ has, among others, the following properties :

1) If F is the field over which $\mathfrak{H}$ is defined, then $\mathfrak{U}(\mathfrak{H})$ is the set of finite F - linear combinations of formal expressions $X_1X_2 \dots X_n$, where n is any positive integer and the X_i are arbitrary elements of $\mathfrak{H}$. The only relations in $\mathfrak{U}(\mathfrak{H})$ are those that come from the bracket in $\mathfrak{H}$: $XY - YX = [X,Y]$ for elements X,Y of $\mathfrak{H}$.

2) $\mathfrak{U}(\mathfrak{H})$ becomes a Lie algebra by defining $[\xi,\eta] = \xi\eta - \eta\xi$ for all elements ξ, η of $\mathfrak{U}(\mathfrak{H})$. The Lie algebra $\mathfrak{H}$ is isomorphic to a Lie subalgebra of $\mathfrak{U}(\mathfrak{H})$.

3) Every representation of $\mathfrak{H}$ on a finite dimensional vector space V over F also defines a representation of $\mathfrak{U}(\mathfrak{H})$ on V. In fact, if $\mathfrak{H}$ and $\mathfrak{U}(\mathfrak{H})$ also denote their images in End(V) under the representation, then $\mathfrak{U}(\mathfrak{H})$ is the subalgebra of End(V) generated by the Lie subalgebra $\mathfrak{H}$.

Now let $\mathfrak{G}$ be a finite dimensional, complex semisimple Lie algebra, and let $\mathfrak{C}$ = $\{H_\beta : \beta \in \Delta \,;\, \xi_\alpha : \alpha \in \Phi\}$ be a Chevalley basis for $\mathfrak{G}$ determined by a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$ and a basis Δ of the roots Φ. Let $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ denote the subring of $\mathfrak{U}(\mathfrak{G})$ generated by 1 and the elements $\{(\xi_\alpha)^n / n! : n$ is any positive integer and $\alpha \in \Phi\}$. Then

1) $\mathfrak{C} \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$.

The vectors $\{\xi_\alpha : \alpha \in \Phi\}$ lie in $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ by the definition of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$. The root vectors $\{H_\beta : \beta \in \Delta\}$ lie in $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ since $H_\beta = [\xi_\beta, \xi_{-\beta}] = \xi_\beta\xi_{-\beta} - \xi_{-\beta}\xi_\beta$ by the Proposition in 1.4.

2) The ring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ is an infinitely generated $\mathbb{Z}$ - module with a particularly nice explicit basis due to Kostant. See [Hu 1, pp. 149-154] for details.

Section 2 Finite dimensional complex $\mathfrak{G}$ - modules

2.1 Weights of a representation

Let $\mathfrak{G}$ be a finite dimensional, complex semisimple Lie algebra, and let V be a finite dimensional, complex $\mathfrak{G}$ - module; i.e. V is a finite dimensional, complex vector space and there is given a Lie algebra homomorphism $\sigma : \mathfrak{G} \to \mathrm{End}(V)$, where End(V) has the usual Lie algebra structure given by $[X,Y] = XY - YX$. In the sequel we suppress the notation σ when the context is clear, and we let $\mathfrak{G}$ also denote its image in End(V).

Let $\mathfrak{A}$ be a Cartan subalgebra of $\mathfrak{G}$. It follows from the definition of Cartan subalgebra and the Proposition in 1.1 that $\mathfrak{A}$ is an abelian subalgebra of End(V) and consists of semisimple linear transformations of V. Hence we may decompose V into a direct sum of common eigenspaces of $\mathfrak{A}$ to obtain the <u>weight space decomposition of V</u>. More precisely, there is a finite subset $\Lambda(V)$ of $\mathfrak{A}^* = \mathrm{Hom}(\mathfrak{A},\mathbb{C})$, the <u>weights</u> of the representation such that

$$V = V_o + \sum_{\mu \in \Lambda(V)} V_\mu \quad \text{(direct sum)}$$

where $V_o = \{v \in V : A(v) = 0 \text{ for all } A \in \mathfrak{A}\}$ and $A = \mu(A)\,\mathrm{Id}$ on the subspace V_μ for each A in $\mathfrak{A}$ and each μ in $\Lambda(V)$.

Remark : The roots and root space decomposition are just the weights and weight space decomposition if we are considering the adjoint representation
$\mathrm{ad} : \mathfrak{G} \to \mathrm{End}(\mathfrak{G})$, where $V = \mathfrak{G}$.

The root space and weight space decompositions are related by the following elementary but important result.

Proposition Let $\mathfrak{A}$ be a Cartan subalgebra of $\mathfrak{G}$, and let V be a finite dimensional complex $\mathfrak{G}$ - module. Let $\mathfrak{G} = \mathfrak{A} + \sum_{\alpha \in \Phi} \mathfrak{G}_\alpha$ (direct sum) and $V = V_o + \sum_{\mu \in \Lambda(V)} V_\mu$ (direct sum) denote the root space and weight space decompositions determined by $\mathfrak{A}$. Then for any root $\alpha \in \Phi$ and any weight $\mu \in \Lambda(V)$ one has

$\mathfrak{G}_\alpha(V_\mu) \subseteq V_{\mu+\alpha}$ if $\mu + \alpha \in \Lambda(V)$ and

$\mathfrak{G}_\alpha(V_\mu) = \{0\}$ if $\mu + \alpha \notin \Lambda(V)$

Proof

See, for example, [Hu 1, p.107].$\square$

2.2 Abstract weights and Weyl group invariance

A good reference for this section and the next is [Hu, pp. 67-71 and 108-116].

For the remainder of Section 2 we let $\mathfrak{G}$ be a complex semisimple Lie algebra and V a finite dimensional, complex $\mathfrak{G}$ - module. We fix a Cartan subalgebra $\mathfrak{A}$, a base Δ for the roots Φ and a Chevalley basis $\mathbb{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ for $\mathfrak{G}$. We regard $\mathfrak{G}$, $\mathfrak{U}(\mathfrak{G})$ and $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ as subsets of End(V). Let $W \subseteq GL(\mathfrak{A}^*)$ be the Weyl group and $\{H_\alpha : \alpha \in \Delta\} \subseteq \mathfrak{A}$ the root vectors defined above in 1.3.

An element $\mu \in \mathfrak{A}^*$ is said to be an <u>abstract weight</u> of $\mathfrak{A}$ if $\mu(H_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$. Note that by 2) of Proposition 1.3c every root $\alpha \in \Phi$ is an abstract weight .

Let Λ denote the set of all abstract weights of $\mathfrak{A}$. Then Λ is a vector lattice in $\mathfrak{A}^*$ (i.e. a $\mathbb{Z}$ - module whose rank is dim $\mathfrak{A}$), and Λ is the union of the weights $\Lambda(V)$ as V ranges over all finite dimensional, complex $\mathfrak{G}$ -modules V.

The Weyl group W leaves Λ invariant by 5) of Lemma 1.3b since W leaves Φ invariant. W leaves $\Lambda(V)$ invariant for all finite dimensional, complex $\mathfrak{G}$ -modules V [Hu 1, p. 113].

The <u>root lattice</u> Λ_r is defined to be the $\mathbb{Z}$ - span of the roots Φ, and W clearly leaves the root lattice invariant since W leaves Φ invariant. The root lattice has finite index in the abstract weight lattice, which indicates one again the importance of the adjoint representation of $\mathfrak{G}$.

2.3 Highest and lowest weights

Let V be a finite dimensional complex $\mathfrak{G}$ - module. Let $\mathfrak{A}$ be a Cartan subalgebra with corresponding weights $\Lambda(V) \subseteq \mathfrak{A}^*$ and weight space decomposition

$V = V_o + \sum_{\mu\in\Lambda(V)} V_\mu$ (direct sum). A choice of basis Δ for the roots Φ determines a partition of Φ into positive roots Φ^+ and negative roots Φ^- as explained in 1.4.

Relative to the choice of Δ, a weight $\mu \in \Lambda(V)$ is said to be a

<u>highest weight</u> if $\mathfrak{G}_\alpha(V_\mu) = \{0\}$ for all roots $\alpha \in \Phi^+$

<u>lowest weight</u> if $\mathfrak{G}_\alpha(V_\mu) = \{0\}$ for all roots $\alpha \in \Phi^-$

Proposition 2.3a

1) For a given choice of Δ, a highest weight $\lambda \in \Lambda(V)$ always exists for any $\mathfrak{G}$–module V , and the highest weight is unique if V is irreducible.
2) For a given choice of Δ, if two irreducible $\mathfrak{G}$–modules V_1 and V_2 have the same highest weight, then they are equivalent as $\mathfrak{G}$–modules.
3) For a given choice of Δ, if V is irreducible, $\lambda \in \Lambda(V)$ is a highest weight and $\mu \in \Lambda(V)$ is any weight distinct from λ, then $\lambda - \mu = \sum_{i=1}^{N} \alpha_i$, where $\alpha_i \in \Phi^+$ for every i.
4) For a given choice of Δ, if V is irreducible, $\lambda \in \Lambda(V)$ is a lowest weight and $\mu \in \Lambda(V)$ is any weight distinct from λ, then $\lambda - \mu = \sum_{i=1}^{N} \alpha_i$, where $\alpha_i \in \Phi^-$ for every i.
5) If V is irreducible and if $\lambda \in \Lambda(V)$ is a highest or lowest weight for a given choice of Δ, then the weight space V_λ is 1- dimensional.

Proof

See for example [Hu 1, pp.108-109] for the proofs of 1), 2), 3) and 5). The proof of 4) is an obvious modification of the proof of 3).□

For later use we prove the following

Proposition 2.3b

Let V be an irreducible, finite dimensional, complex $\mathfrak{G}$ - module. Let $\mathfrak{A}$, Δ and $\Lambda(V) \subseteq \mathfrak{A}^*$ be as above. Then

1) There exists a unique element τ of W such that $\tau^2 = \mathrm{Id}$, $\tau(\Phi^+) = \Phi^-$ and $\tau(\Phi^-) = \Phi^+$.
2) If λ^+ and λ^- are the highest and lowest weights in $\Lambda(V)$, then $\tau(\lambda^+) = \lambda^-$ and $\tau(\lambda^-) = \lambda^+$.

Proof

1) If one replaces the base Δ by the base $-\Delta$ for the roots Φ, then one interchanges Φ^+ and Φ^-. Since the Weyl group acts transitively on the set of bases of Φ (cf. 1.4) there exists $\tau \in$ W such that $\tau(\Phi^+) = \Phi^-$ and $\tau(\Phi^-) = \Phi^+$. To prove that $\tau^2 =$

Id and τ is unique it suffices to prove that if $\sigma(\Phi^+) = \Phi^+$ for $\sigma \in W$, then $\sigma = Id$. Note that Δ is the set of indecomposable elements of Φ^+ by the two defining properties of a base for Φ as stated in 1.4. Hence, if $\sigma(\Phi^+) = \Phi^+$ for $\sigma \in W$, then $\sigma(\Delta) = \Delta$ and $\sigma = Id$ since the Weyl group acts simply transitively on the set of bases for Φ (cf. (e) of [Hu 1, p.51]).

2) It suffices to prove that $\tau(\lambda^+) = \lambda^-$ since $\tau^2 = Id$. This fact follows from 3) and 4) of the previous result and the observation that the zero element of $\mathfrak{A}^*$ cannot be written as a sum of elements in Φ^+ or a sum of elements in Φ^-. $\square$

2.4 More about $\mathfrak{U}(\mathfrak{G})$ and the subring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$

By the discussion in 1.6 the subring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ is generated in End(V) by Id and the elements $\{(\xi_\alpha)^n / n!$, where n is any positive integer and $\alpha \in \Phi\}$. It follows from 2) of the proposition in 1.2 that ad ξ_α is a nilpotent operator in End($\mathfrak{G}$) for all $\alpha \in \Phi$. Hence each ξ_α is a nilpotent operator in End(V) by the proposition in 1.1. In particular the elements $\exp(\xi_\alpha) = \sum_{n=1}^{\infty} (\xi_\alpha{}^n / n!)$ belong to $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ for every $\alpha \in \Phi$ since the power series expansion is finite.

The first result of this section extends the observation above.

Proposition 2.4a

For every element σ of W there exists an element T_σ of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ such that $T_\sigma(V_o) = V_o$ and $T_\sigma(V_\mu) = V_{\sigma(\mu)}$ for all weights μ in $\Lambda(V)$, where $W \subseteq GL(\mathfrak{A}^*)$ denotes the Weyl group. The definition of T_σ depends only on σ and not on V.

Proof

We define T_σ first for the generating elements of W, the reflections $\{\sigma_\alpha : \alpha \in \Phi\}$ as defined in 1.3, and then treat the general case.

Given $\alpha \in \Phi$ we define $T_\alpha = \exp(\xi_\alpha) \exp(-\xi_{-\alpha}) \exp(\xi_\alpha)$. By the discussion above each T_α is clearly an element of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$. We need some intermediate results to prove that T_α permutes the weight spaces V_μ by σ_α as desired.

Lemma 1

Let $X \in End(V)$ be any nilpotent transformation. Then for any element Y of End(V)

$$\exp(X) \circ Y \circ \exp(-X) = \exp(ad\, X)(Y)$$

where $\exp(T) = \sum_{n=1}^{\infty} (T^n / n!)$ for a linear transformation T.

Proof

[Hu 1, p.9].□

Lemma 2

Let $\alpha \in \Phi$ be given. Then

1) As elements of End(V), $T_\alpha \circ A \circ T_\alpha^{-1} = \sigma_\alpha(A) = A - \alpha(A) H_\alpha$ for all $A \in \mathfrak{A}$.

2) As elements of End(V), $T_\alpha \circ H_\alpha = - H_\alpha \circ T_\alpha$.

3) $T_\alpha(V_o) = V_o$ and $T_\alpha(V_\mu) = V_{\sigma_\alpha(\mu)}$ for all weights μ in $\Lambda(V)$.

Proof

1) For each $\beta \in \Phi$ it follows from Lemma 1 above that $\exp(\xi_\beta) \circ Y \circ \exp(-\xi_\beta) = \exp(\text{ad } \xi_\beta)(Y)$ for all $Y \in \mathfrak{G}$. If we define $\tau_\alpha = \exp(\text{ad } \xi_\alpha) \circ \exp(\text{ad } -\xi_{-\alpha}) \circ \exp(\text{ad } \xi_\alpha)$, then since $T_\alpha = \exp(\xi_\alpha) \exp(-\xi_{-\alpha}) \exp(\xi_\alpha)$ we obtain

(*) $T_\alpha \circ Y \circ T_\alpha^{-1} = \tau_\alpha(Y)$ for all $Y \in \mathfrak{G}$.

Note that $\tau_\alpha \in \text{Aut}(\mathfrak{G})$ since ad ξ_α is a derivation of $\mathfrak{G}$. The root vectors $\{H_\beta : \beta \in \Phi\}$ span $\mathfrak{A}$ by Proposition 1.3c, and hence to prove 1) it suffices by (*) to show that $\tau_\alpha(H_\beta) = \sigma_\alpha(H_\beta) = H_\beta - \alpha(H_\beta) H_\alpha$ for all $\beta \in \Phi$.

We use the bracket relations of the Chevalley basis vectors as stated in 1.4. Now for $\beta \in \Phi$ we have $\text{ad } \xi_\alpha(H_\beta) = -[H_\beta, \xi_\alpha] = -\alpha(H_\beta)\, \xi_\alpha$, which shows that $(\text{ad } \xi_\alpha)^m(H_\beta) = 0$ for all $m \geq 2$. Hence

(a) $\exp(\text{ad } \xi_\alpha)(H_\beta) = H_\beta + \text{ad } \xi_\alpha(H_\beta) = H_\beta - \alpha(H_\beta)\, \xi_\alpha$.

Similarly $\text{ad}(-\xi_{-\alpha})(H_\beta - \alpha(H_\beta)\, \xi_\alpha) = [H_\beta, \xi_{-\alpha}] - \alpha(H_\beta)\, [\xi_\alpha, \xi_{-\alpha}] = - \alpha(H_\beta)\, \xi_{-\alpha} - \alpha(H_\beta)\, H_\alpha$. It follows that $\text{ad } (-\xi_{-\alpha})^2(H_\beta - \alpha(H_\beta)\, \xi_\alpha) = \text{ad}(-\xi_{-\alpha})(- \alpha(H_\beta)\, \xi_{-\alpha} - \alpha(H_\beta)\, H_\alpha) = - \alpha(H_\beta)\, [H_\alpha, \xi_{-\alpha}] = 2\, \alpha(H_\beta)\, \xi_{-\alpha}$. This proves that $\text{ad } (-\xi_{-\alpha})^m(H_\beta - \alpha(H_\beta)\, \xi_\alpha) = 0$ for all $m \geq 3$, and from (a) we obtain

(b) $\exp(\text{ad } -\xi_{-\alpha})\exp(\text{ad } \xi_\alpha)(H_\beta) = H_\beta - \alpha(H_\beta)\, H_\alpha - \alpha(H_\beta)\, \xi_\alpha$

Finally, $\text{ad}(\xi_\alpha)(H_\beta - \alpha(H_\beta)\, H_\alpha - \alpha(H_\beta)\, \xi_\alpha) = - [H_\beta, \xi_\alpha] + \alpha(H_\beta)\, [H_\alpha, \xi_\alpha] = - \alpha(H_\beta)\, \xi_\alpha + 2\, \alpha(H_\beta)\, \xi_\alpha = \alpha(H_\beta)\, \xi_\alpha$. This proves that $\text{ad}(\xi_\alpha)^m(H_\beta - \alpha(H_\beta)\, H_\alpha - \alpha(H_\beta)\, \xi_\alpha) = 0$ for $m \geq 2$, and we conclude from (b) that $\tau_\alpha(H_\beta) = \exp(\text{ad } \xi_\alpha)(H_\beta - \alpha(H_\beta)\, H_\alpha - \alpha(H_\beta)\, \xi_\alpha) = (\text{Id} + \text{ad } \xi_\alpha)(H_\beta - \alpha(H_\beta)\, H_\alpha - \alpha(H_\beta)\, \xi_\alpha) = H_\beta - \alpha(H_\beta)\, H_\alpha$. By (*) above the proof of (1) is complete.

2) This follows immediately from 1) by setting $A = H_\alpha$ and using the fact that $\alpha(H_\alpha) = 2$.

3). We prove that $T_\alpha(V_o) = V_o$. From 1) and 2) we obtain $A \circ T_\alpha = T_\alpha \circ A + \alpha(A)\,(H_\alpha \circ T_\alpha) = T_\alpha \circ A - \alpha(A)\,(T_\alpha \circ H_\alpha)$ for all $A \in \mathfrak{A}$. If $v \in V_o$ is arbitrary, then

$A(T_\alpha v) = 0$ by the equation above, which proves that $T_\alpha(V_o) \subseteq V_o$, and equality holds since T_α is invertible.

Given $\mu \in \Lambda(V)$ we know that $A = \mu(A)$ Id on V_μ. If we set $A^* = \sigma_\alpha(A)$, then from (1) and the discussion in 1.3 we see that on the subspace $T_\alpha(V_\mu)$ one has $A^* = T_\alpha \circ A \circ T_\alpha^{-1} = \mu(A)\,\mathrm{Id} = (\mu \circ \sigma_\alpha^{-1})(A^*)\,\mathrm{Id} = \sigma_\alpha(\mu)(A^*)\,\mathrm{Id}$. Hence $T_\alpha(V_\mu) \subseteq V_{\sigma_\alpha(\mu)}$ and $\dim(V_\mu) \leq \dim(V_{\sigma_\alpha(\mu)})$. Equality holds by symmetry, which completes the proof of 3).□

We now complete the proof of Proposition 2.4a. Let σ be any element of W and write $\sigma = \sigma_1 \circ \sigma_2 \circ \ldots \circ \sigma_N$, where each σ_k is a Weyl reflection as above. We define $T_\sigma = T_{\sigma_1} \circ T_{\sigma_2} \circ \ldots \circ T_{\sigma_N}$, and the assertion of Proposition 2.4a now follows from 3) of Lemma 2 and induction on N.□

Proposition 2.4b

Let V be a $\mathfrak{G}$-module, and let $\pi_o : V \to V_o$ and $\pi_\mu : V \to V_\mu$ denote the projections of V onto the weight spaces V_o and V_μ respectively, where $\mu \in \Lambda(V)$ is arbitrary. Then π_o and π_μ can be realized as elements of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ for every $\mu \in \Lambda(V)$.

Proof

See [Hu 1, p.156].□

The next result is essential for the methods of this article.

Proposition 2.4c

Let V be an irreducible $\mathfrak{G}$ - module, and let $\lambda \in \Lambda(V)$ be the unique highest weight. Let v be any nonzero vector in V_λ. Then

1) $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ is a finitely generated $\mathbb{Z}$ - module.
2) If $\mathbb{B}^*$ is any $\mathbb{Z}$ - basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, then $\mathbb{B}^*$ is a $\mathbb{C}$ - basis for V. Any element of End(V) that lies in $\mathbb{C}$ or $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ has a $\mathbb{Z}$ - matrix with respect to $\mathbb{B}^*$.
3) There exists a $\mathbb{Z}$ - basis $\mathbb{B}^*$ for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ such that $\mathbb{B}^*$ is a union of sets $\{\mathbb{B}_o^*, \mathbb{B}_\mu^*\}$ where $\mathbb{B}_o^*$ is a $\mathbb{C}$-basis for V_o and $\mathbb{B}_\mu^*$ is a $\mathbb{C}$-basis for V_μ for all $\mu \in \Lambda(V)$.

Proof

For a proof of 1) see [Hu 1, p.156]. The ring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ clearly leaves invariant $\mathbb{Z}$ - span $(\mathbb{B}^*) = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. Recall that $\mathbb{C} \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ by the discussion in 1.6. Since $\mathbb{C}$ is a basis for $\mathfrak{G}$ and $\mathbb{C}$ leaves invariant $\mathbb{Z}$ - span $(\mathbb{B}^*)$ it follows that $\mathfrak{G}$ leaves invariant $\mathbb{C}$-span $(\mathbb{B}^*) = V'$. Hence $V' = V$ since V is an irreducible $\mathfrak{G}$-module. The discussion in [Hu 1, p.156] shows that the $\mathbb{Z}$ - rank of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ equals dim V, and hence $\mathbb{B}^*$ is a $\mathbb{C}$ - basis for V. This proves 2)

3) Let $\overline{\mathcal{B}}$ be any $\mathbb{Z}$ - basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. Let $\pi_o : V \to V_o$ and $\pi_\mu : V \to V_\mu$ denote the projections of V onto the weight spaces V_o and V_μ respectively, where $\mu \in \Lambda(V)$ is arbitrary. Define $\mathcal{B}' = \mathcal{B}_o' \cup \bigcup_{\mu \in \Lambda(V)} \mathcal{B}_\mu'$, where $\mathcal{B}_o' = \pi_o(\overline{\mathcal{B}})$ and $\mathcal{B}_\mu' = \pi_\mu(\overline{\mathcal{B}})$. By Proposition 2.4b, $\mathcal{B}' \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) = \mathbb{Z}$ - span $(\overline{\mathcal{B}})$. If $\xi \in \overline{\mathcal{B}}$, then $\pi_o(\xi) \in \mathcal{B}_o' \subseteq \mathcal{B}'$ and $\pi_\mu(\xi) \in \mathcal{B}_\mu' \subseteq \mathcal{B}'$ for all $\mu \in \Lambda(V)$. Hence $\overline{\mathcal{B}} \subseteq \mathbb{Z}$-span $(\mathcal{B}')$ and it follows that $\mathbb{Z}$-span $(\overline{\mathcal{B}}) = \mathbb{Z}$-span $(\mathcal{B}') = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$.

Now let $\mathcal{B}^*$ be a $\mathbb{Z}$- linearly independent subset of $\mathcal{B}'$ such that $\mathbb{Z}$-span $(\mathcal{B}^*) = \mathbb{Z}$-span $(\mathcal{B}') = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, and let $\mathcal{B}_o^* = \mathcal{B}^* \cap \mathcal{B}_o'$ and $\mathcal{B}_\mu^* = \mathcal{B}^* \cap \mathcal{B}_\mu'$ for all $\mu \in \Lambda(V)$. Since both $\mathcal{B}^*$ and $\overline{\mathcal{B}}$ are $\mathbb{Z}$-bases for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ they have the same cardinality, and it follows from 2) that $\mathcal{B}^*$ is a $\mathbb{C}$-basis for V. Hence $\mathcal{B}_o^*$ is a $\mathbb{C}$-basis for V_o and $\mathcal{B}_\mu^*$ is a $\mathbb{C}$-basis for V_μ for all $\mu \in \Lambda(V)$. We conclude that $\mathcal{B}^*$ satisfies the assertions of 3). □

Proposition 2.4d

Let τ be the unique element of W such that $\tau^2 = \text{Id}$, $\tau(\Phi^+) = \Phi^-$ and $\tau(\Phi^-) = \Phi^+$. Let T_τ be the transformation in $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ constructed in Proposition 2.4a. Then

1) $(T_\tau)^2$ leaves invariant V_o and V_μ for all $\mu \in \Lambda(V)$.
2) The restriction of $(T_\tau)^2$ to V_μ has determinant 1 or – 1 for all $\mu \in \Lambda(V)$. The restriction of T_τ to V_o has determinant 1.
3) If V is irreducible, then $(T_\tau)^2 = \text{Id}$ or – Id on the highest and lowest weight spaces.

Proof

1) This follows immediately from Proposition 2.4a and the fact that $\tau^2 = \text{Id}$.

2) By Proposition 2.4c there exists a $\mathbb{C}$ -basis $\mathcal{B}^*$ of V such that $\mathcal{B}^*$ is the union of bases $\mathcal{B}_o^*$ for V_o and $\mathcal{B}_\mu^*$ for V_μ, $\mu \in \Lambda(V)$, and furthermore every element of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ has a $\mathbb{Z}$-matrix relative to $\mathcal{B}^*$. It follows that the restriction of $(T_\tau)^2$ to V_μ has a $\mathbb{Z}$-matrix relative to $\mathcal{B}_\mu^*$, and we conclude that the determinant of $(T_\tau)^2$ restricted to V_μ is an integer m_μ. Similarly, the determinant of $(T_\tau)^2$ restricted to V_o is a positive integer m_o since T_τ leaves V_o invariant by Proposition 2.4a. Hence the determinant of $(T_\tau)^2$ on V is an integer m, the product of the integers m_o and m_μ, $\mu \in \Lambda(V)$.

For the first assertion in 2) it suffices to prove that m = 1. The definition of T_τ in 2.4 shows that T_τ is a composition of transformations of the form T_α, where $\alpha \in \Phi$ and $T_\alpha = \exp(\xi_\alpha) \exp(-\xi_{-\alpha}) \exp(\xi_\alpha)$. Hence it suffices to show that each T_α has determinant 1 on V. By the discussion at the beginning of 2.4 each element ξ_α, $\alpha \in \Phi$, is a nilpotent element of End(V), and hence all eigenvalues of ξ_α equal zero. It follows that all eigenvalues of $\exp(\xi_\alpha)$ equal 1, and we conclude that T_α has determinant 1 on V.

The proof of the second assertion in 2) is similar. The transformation T_τ leaves V_o invariant by Proposition 2.4a, and its determinant is 1 by the argument above.

3) This an immediate consequence of 2) since the highest and lowest weight spaces are 1-dimensional if V is irreducible by 5) of Proposition 2.3a.□

2.5 Invariant Hermitian inner products on V

We continue the conventions of 2.2 for $\mathfrak{A}, \mathfrak{G}, \mathfrak{C}$, V.

Proposition 2.5a

Let V be a finite dimensional, complex $\mathfrak{G}$ - module, and let $\mathfrak{G}_o$ be a compact real form for $\mathfrak{G}$. Then there exists a $\mathfrak{G}_o$ - invariant Hermitian inner product < , > on V; that is, $X^* = -X$ for all $X \in \mathfrak{G}_o$, where X* denotes the metric adjoint of X.

Proof

Let $\tilde{G}_o$ be a simply connected Lie group with Lie algebra $\mathfrak{G}_o$. Then $\tilde{G}_o$ is compact by the discussion at the beginning of 1.5. If $\sigma : \mathfrak{G} \to \mathrm{End}(V)$ is the Lie algebra homomorphism that defines V as a $\mathfrak{G}$-module, then since $\tilde{G}_o$ is simply connected there exists a unique Lie group homomorphism $\rho : \tilde{G}_o \to GL(V)$ such that $d\rho = \sigma$ on $\mathfrak{G}_o$. Since $\tilde{G}_o$ is compact there exists a Hermitian inner product < , > on V such that the elements of $\rho(\tilde{G}_o)$ leave < , > invariant; that is, $< \rho(g)v, \rho(g)w > = < v, w >$ for all v,w in V and all $g \in \tilde{G}_o$. If $X \in \mathfrak{G}_o$ is given arbitrarily, then by differentiating at t = 0 the equation $0 \equiv < \rho(e^{tX})v, \rho(e^{tX})w > = < e^{t\sigma(X)} v, e^{t\sigma(X)} w >$, we obtain $0 = < X(v), w > + < v, X(w) > = < (X+X^*)(v), w >$ for all vectors v,w in V.□

Proposition 2.5b

Let V be a finite dimensional, complex $\mathfrak{G}$ - module, and let $\mathfrak{G}_o$ be a compact real form for $\mathfrak{G}$. Let $\mathfrak{A}$ be a Cartan subalgebra of $\mathfrak{G}$ and let $\mathfrak{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ be a Chevalley basis for $\mathfrak{G}$ such that $\mathfrak{G}_o = \mathbb{R}$–span $(\mathfrak{C}_o)$, where $\mathfrak{C}_o$ is the real Chevalley basis constructed from $\mathfrak{C}$ as described in 1.5. Let < , > be a $\mathfrak{G}_o$ - invariant Hermitian inner product on V. Then

1) $\xi_\alpha^* = \xi_{-\alpha}$ for all $\alpha \in \Phi$.

2) $H_\alpha^* = H_\alpha$ for all $\alpha \in \Phi$.

3) The weight spaces $\{V_o, V_\mu : \mu \in \Lambda(V)\}$ from the weight space decomposition of V determined by $\mathfrak{A}$ are orthogonal relative to < , >.

4) For each element σ of the Weyl group W let $T_\sigma \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ be the transformation constructed in Proposition 2.4a such that $T_\sigma(V_\mu) = V_{\sigma(\mu)}$ for all $\mu \in \Lambda(V)$. Then

a) $T_\sigma^*(V_o) = V_o$ and $T_\sigma^*(V_\mu) = V_{\sigma^{-1}(\mu)}$ for all $\mu \in \Lambda(V)$.

b) $T_\sigma^* T_\sigma$ and $T_\sigma T_\sigma^*$ leave V_μ invariant, and the restrictions of $T_\sigma^* T_\sigma$ and $T_\sigma T_\sigma^*$ to V_μ have determinant 1 for all $\mu \in \Lambda(V)$.

c) T_σ and T_σ^* leave V_o invariant, and the restrictions of T_σ and T_σ^* to V_o have determinant 1.

d) If V is irreducible and $\lambda \in \Lambda(V)$ is the highest or lowest weight, then $T_\sigma^* T_\sigma$ and $T_\sigma T_\sigma^*$ are the identity transformations on V_λ.

Proof

1) By hypothesis $\mathfrak{G}_o = \mathbb{R}$–span $(\mathfrak{C}_o)$, where $\mathfrak{C}_o = \{i H_\beta : \beta \in \Delta ; A_\alpha, B_\alpha : \alpha \in \Phi\}$ and $A_\alpha = \xi_\alpha - \xi_{-\alpha}$, $B_\alpha = i\,\xi_\alpha + i\,\xi_{-\alpha}$ for all $\alpha \in \Phi$. Hence $\xi_\alpha = (1/2)\,(A_\alpha - i\,B_\alpha)$ and $\xi_{-\alpha} = -(1/2)\,(A_\alpha + i\,B_\alpha)$. The assertion 1) now follows immediately since $A_\alpha^* = -A_\alpha$ and $B_\alpha^* = -B_\alpha$ by hypothesis.

2) By the Proposition in 1.4, $H_\alpha = [\,\xi_\alpha, \xi_{-\alpha}\,] = \xi_\alpha \circ \xi_{-\alpha} - \xi_{-\alpha} \circ \xi_\alpha$ as an element of End(V). The result now follows from 1).

3) Let μ, μ^* be distinct weights in $\Lambda(V)$, and let v_μ, v_{μ^*} be arbitrary vectors in V_μ, V_{μ^*} respectively. Since $\{H_\beta : \beta \in \Phi\}$ spans $\mathfrak{A}$ there exists $\alpha \in \Phi$ such that $\mu(H_\alpha) \neq \mu^*(H_\alpha)$. Using 2) we compute $\mu(H_\alpha) < v_\mu, v_{\mu^*} > \, = \, < H_\alpha(v_\mu), v_{\mu^*} > \, = \, < v_\mu, H_\alpha(v_{\mu^*}) > \, = \overline{\mu^*(H_\alpha)} < v_\mu, v_{\mu^*} > \, = \mu^*(H_\alpha) < v_\mu, v_{\mu^*} >$ since $\mu^*(H_\alpha)$ is an integer. We conclude that $< v_\mu, v_{\mu^*} > = 0$ since since $\mu(H_\alpha) \neq \mu^*(H_\alpha)$.

We show that V_o is orthogonal to V_μ for all $\mu \in \Lambda(V)$. Given $\mu \in \Lambda(V)$ choose $\alpha \in \Phi$ such that $\mu(H_\alpha) \neq 0$; this can be done since $\mu \neq 0$ and $\{H_\alpha : \alpha \in \Phi\}$ spans $\mathfrak{A}$. Given arbitrary elements $v_o \in V_o$ and $v_\mu \in V_\mu$ we compute $0 = \, < H_\alpha(v_o), v_\mu > \, = \, < v_o, H_\alpha(v_\mu) > \, = \mu(H_\alpha) < v_o, v_\mu >$ by 2), which completes the proof of 3).

4)

a) Let σ be any element of W and let μ be any element of $\Lambda(V)$. Given arbitrary elements $v_o \in V_o$ and $v_\mu \in V_\mu$ we compute $< T_\sigma^*(v_o), v_\mu > \, = \, < v_o, T_\sigma(v_\mu) > \, = 0$ by 3) and Proposition 2.4a. Hence $T_\sigma^*(V_o) \subseteq V_o$ by 3), and equality holds since T_σ and T_σ^* are invertible.

Let σ be any element of W, and let μ, μ^* be weights in $\Lambda(V)$ such that $\mu \neq \sigma(\mu^*)$. Let v_μ, v_{μ^*} be arbitrary vectors in V_μ, V_{μ^*} respectively. Since $T_\sigma(V_{\mu^*}) = V_{\sigma(\mu^*)}$ it follows from 3) that $0 = \, < T_\sigma(v_{\mu^*}), v_\mu > \, = \, < v_{\mu^*}, T_\sigma^*(v_\mu) >$. This shows that $T_\sigma^*(V_\mu)$ is orthogonal to V_{μ^*} if $\mu^* \neq \sigma^{-1}(\mu)$, and 4a) now follows from 3) and the weight space decomposition.

b) The fact that $T_\sigma{}^*T_\sigma$ and $T_\sigma T_\sigma{}^*$ leave V_μ invariant for every $\mu \in \Lambda(V)$ is an immediate consequence of 4a) and the fact that $T_\sigma(V_\mu) = V_{\sigma(\mu)}$ by Proposition 2.4a.

Note that $T_\alpha{}^* = T_{-\alpha}$ for all $\alpha \in \Phi$ by 1) of Proposition 2.5b since $T_\alpha = \exp(\xi_\alpha)\exp(-\xi_{-\alpha})\exp(\xi_\alpha)$. It follows that $T_\sigma{}^* \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ for all $\sigma \in W$ by the definition of T_σ, and we conclude that $T_\sigma{}^*T_\sigma$ and $T_\sigma T_\sigma{}^*$ belong to $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ for all $\sigma \in W$.

The proof that the restrictions of $T_\sigma{}^*T_\sigma$ and $T_\sigma T_\sigma{}^*$ to each V_μ have determinant 1 is virtually identical to the proof of 2) in Proposition 2.4d. To modify the proof appropriately we note that the integers m_o and m_μ from the earlier proof are positive in this case since $T_\sigma{}^*T_\sigma$ and $T_\sigma T_\sigma{}^*$ are positive definite on V_o and each V_μ. The argument showing that each T_α has determinant 1 on V for all $\alpha \in \Phi$ implies that $T_\alpha{}^* = T_{-\alpha}$ has determinant 1 on V for all $\alpha \in \Phi$. Hence T_σ and $T_\sigma{}^*$ have determinant 1 on V for all $\sigma \in W$. The proof of b) is now complete.

c) T_σ leaves V_o invariant and has determinant 1 on V_o by 2) of Proposition 2.4d. For each $\sigma \in W$ there exists $\sigma' \in W$ such that $T_\sigma{}^* = T_{\sigma'}$ since $T_\alpha{}^* = T_{-\alpha}$ for all $\alpha \in \Phi$. This completes the proof of c).

d) If V is irreducible, then the highest or lowest weight space V_λ is 1-dimensional (cf. 5) of Proposition 2.3a). The assertion now follows from b).□

Section 3 Real $\mathfrak{G}_o$ - modules

3.1 Irreducible real $\mathfrak{G}_o$ - modules

Proposition 3.1a Let $\mathfrak{G}_o$ be a compact semisimple Lie algebra, and let U be a finite dimensional, irreducible real $\mathfrak{G}_o$ - module. Then, up to module equivalence, one of the following occurs :

1) There exists an irreducible complex $\mathfrak{G}_o{}^{\mathbb{C}}$–module V such that U is a real subspace of V and $V = U^{\mathbb{C}}$.
2) There exists an irreducible, complex $\mathfrak{G}_o{}^{\mathbb{C}}$ - module V such that $U = V^{\mathbb{R}}$.

Here $U^{\mathbb{C}}$ and $\mathfrak{G}_o{}^{\mathbb{C}}$ denote the complexifications of U and $\mathfrak{G}_o$ respectively, and $V^{\mathbb{R}}$ denotes V regarded as a real vector space with $\dim_{\mathbb{R}} V^{\mathbb{R}} = 2 \dim_{\mathbb{C}} V$.

Proof

If $\tilde{G}_o$ is the simply connected Lie group with Lie algebra $\mathfrak{G}_o$, then for any Lie algebra homomorphism $\sigma : \mathfrak{G}_o \to End(U)$ there exists a unique Lie group homomorphism $\rho : \tilde{G}_o \to GL(U)$ such that $d\rho = \sigma$. Hence the $\tilde{G}_o$-modules and the $\mathfrak{G}_o$-modules are the same. By 1.5 the group $\tilde{G}_o$ is compact, and the result above now becomes an immediate consequence of the following somewhat more general result :

Proposition 3.1b

Let G be a compact, connected Lie group, and let U be an irreducible, finite dimensional, real G-module. If $V = U^{\mathbb{C}}$, then one of the following occurs :

1) V is an irreducible, complex G-module.

2) $V = W \oplus J(W)$ for any irreducible, complex G-module W, where $J : V \to V$ denotes the conjugation induced by U. In this case the map $\varphi : W^{\mathbb{R}} \to U$ given by $\varphi(w) = w + J(w)$ is a G-isomorphism between the real G-modules $W^{\mathbb{R}}$ and U.

The following preliminary result will be useful.

Lemma

Let $W \subseteq V$ be a nonzero complex subspace of V that is invariant under J. Then

1) $W = U_o^{\mathbb{C}}$, where $U_o = W \cap U$.

2) If W is also invariant under G, then W = V.

Proof of the Lemma

1) Since W is J-invariant it is the direct sum of $W \cap U$ and $W \cap iU$, the +1 and −1 eigenspaces of J. Since W is a complex subspace, $W \cap iU = i\,(W \cap U)$, which completes the proof of 1).

2) If W is G-invariant, then so is $W \cap U$, and hence $W \cap U = U$ by the irreducibility of U as a G-module. Therefore $W = U_o^{\mathbb{C}} = V$ by 1).

Proof of Proposition 3.1b

Suppose that V is not irreducible as a complex G-module, and let W be any proper, irreducible, complex G-submodule of V. We break the proof into steps.

Since G commutes with J, $W \cap J(W)$ is a complex G-module that is also invariant under J. If $W \cap J(W) \neq \{0\}$, then $W \cap J(W) = V$ by 2) of the lemma, which contradicts the fact that W is a proper G-submodule of V. Hence we conclude

(a) $W \cap J(W) = \{0\}$

The complex subspace $W \oplus J(W)$ is invariant under G and J. Hence by 2) of the Lemma we obtain

(b) $W \oplus J(W) = V$

Define $\varphi : W^{\mathbb{R}} \to V$ by $\varphi(w) = w + J(w)$. Note that $\varphi(W^{\mathbb{R}}) \subseteq U$, the +1 eigenspace of J. The map φ is $\mathbb{R}$-linear, injective by (a) and hence also surjective since $\dim_{\mathbb{R}} W^{\mathbb{R}} = \dim_{\mathbb{R}} U$ by (b). Since φ commutes with G the real $\mathbb{R}$-modules $W^{\mathbb{R}}$ and U are equivalent.$\square$

3.2 Irreducible real $\mathfrak{G}_o$ - modules of type 1

The discussion in this section will be limited to irreducible $\mathfrak{G}_o$ - modules U of type 1); that is, $V = U^{\mathbb{C}}$ is a complex irreducible $\mathfrak{G}_o{}^{\mathbb{C}}$-module. We let $J : V \to V$ be the conjugate linear map that is defined by conjugation with respect to the real subspace U. We let $\mathfrak{G}$ denote $\mathfrak{G}_o{}^{\mathbb{C}}$, and we fix a Cartan subalgebra $\mathfrak{A}$ of $\mathfrak{G}$, a basis Δ for the roots Φ and a Chevalley basis $\mathfrak{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ such that $\mathfrak{G}_o = \mathbb{R}$-span $(\mathfrak{C}_o)$, where $\mathfrak{C}_o$ is the real Chevalley basis for $\mathfrak{G}_o$ defined by $\mathfrak{C}_o = \{i\, H_\beta : \beta \in \Delta ; A_\alpha, B_\alpha : \alpha \in \Phi^+\}$.

Proposition 3.2a

The conjugation operator $J : V \to V$ has the following properties :

1) $J^2 = Id$ and hence $J = J^{-1}$.

2) $J \circ \xi_\alpha \circ J^{-1} = -\xi_{-\alpha}$ for all $\alpha \in \Phi$.

3) $J \circ A \circ J^{-1} = -A$ for all $A \in \mathfrak{A}$.

Proof

1) Obvious.

2) By definition $A_\alpha = \xi_\alpha - \xi_{-\alpha}$ and $B_\alpha = i\,\xi_\alpha + i\,\xi_{-\alpha}$, and hence $\xi_\alpha = (1/2)\,(A_\alpha - i\,B_\alpha)$ and $\xi_{-\alpha} = -(1/2)(A_\alpha + i\,B_\alpha)$. Since $\mathfrak{G}_o$ leaves U invariant and $\mathfrak{G}_o = \mathbb{R}$-span $(\mathfrak{C}_o)$ it follows that J commutes with all elements of $\mathfrak{C}_o$ and in particular with A_α and B_α. The assertion now follows since J is conjugate linear on V.

3) For any $\alpha \in \Phi$, J commutes with $i\,H_\alpha$ since $i\,H_\alpha \in \mathfrak{C}_o$. This, together with the conjugate linearity of J, shows that $J \circ H_\alpha \circ J^{-1} = -H_\alpha$ for all $\alpha \in \Phi$. The assertion 3) now follows since $\mathfrak{A} = \mathbb{C}$-span $\{H_\alpha : \alpha \in \Phi\}$.□

Proposition 3.2b

1) The conjugation operator J normalizes $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ in End(V).

2) If $\mu \in \Lambda(V)$, then $-\mu \in \Lambda(V)$. If $\{V_o, V_\mu : \mu \in \Lambda(V)\}$ are the weight spaces in the weight space decomposition of V, then $J(V_o) = V_o$ and $J(V_\mu) = V_{-\mu}$ for all $\mu \in \Lambda(V)$.

Proof

1) This assertion follows from 2) of the result above since $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ is generated in End(V) by Id and $\{(\xi_\alpha)^n / n! : \alpha \in \Phi$ and $n \in \mathbb{Z}^+\}$.

2) Let $\mu \in \Lambda(V)$ and $v \in V_\mu$ be given. For every $\alpha \in \Phi$, $H_\alpha v = \mu(H_\alpha)\, v$, and hence $H_\alpha Jv = -JH_\alpha v = -J\,(\mu(H_\alpha)\, v) = -\overline{\mu(H_\alpha)}\, Jv = -\mu(H_\alpha)\, Jv$ by 3) in 3.2a , the fact that $\mu(H_\alpha)$ is an integer and the conjugate linearity of J. It follows that $A\,(Jv) =$

$-\mu(A)$ (Jv) for all $A \in \mathfrak{A}$ since $\{H_\alpha : \alpha \in \Phi\}$ spans $\mathfrak{A}$ by Proposition 1.3c. This proves that $J(V_\mu) \subseteq V_{-\mu}$ for all $\mu \in \Lambda(V)$, and equality holds by applying J to both sides of the inclusion.

The fact that $J(V_o) = V_o$ follows from 3) of Proposition 3.2a. □

Proposition 3.2c

Let < , > be a $\mathfrak{G}_o$ - invariant Hermitian inner product on V. For each element σ in the Weyl group W let $T_\sigma \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ be the transformation defined in Proposition 2.4a such that $T_\sigma(V_\mu) = V_{\sigma(\mu)}$ for all $\mu \in \Lambda(V)$. Then for all $\sigma \in W$

1) $J \circ T_\sigma \circ J^{-1} = (T_\sigma^*)^{-1}$
2) If V is irreducible, then J commutes with T_σ^+ and T_σ^-, where T_σ^+ and T_σ^- denote the restrictions of T_σ to the highest and lowest weight spaces respectively.

Proof Let $\sigma \in W$ be given.

1) The map $C : g \to (g^*)^{-1}$ is an isomorphism of GL(V). By its definition in Proposition 2.4a, the transformation T_σ is a composition of transformations of the form $T_\alpha = \exp(\xi_\alpha) \exp(-\xi_{-\alpha}) \exp(\xi_\alpha)$, where $\alpha \in \Phi$. Hence to prove 1) it suffices to prove that $J \circ T_\alpha \circ J^{-1} = C(T_\alpha)$ for all $\alpha \in \Phi$.

Using properties of the matrix exponential map and 2) of Proposition 3.2a we compute $J \circ T_\alpha \circ J^{-1} = (J \circ \exp(\xi_\alpha) \circ J^{-1}) \circ (J \circ \exp(-\xi_{-\alpha}) \circ J^{-1}) \circ (J \circ \exp(\xi_\alpha) \circ J^{-1})$ $= (\exp(J \circ \xi_\alpha \circ J^{-1}) \circ (\exp(-J \circ \xi_{-\alpha} \circ J^{-1}) \circ (\exp(J \circ \xi_\alpha \circ J^{-1}) =$ $\exp(-\xi_{-\alpha}) \circ \exp(\xi_\alpha) \circ \exp(-\xi_{-\alpha})$. A similar argument together with 1) of Proposition 2.5b shows that $T_\alpha^* = \exp(\xi_{-\alpha}) \circ \exp(-\xi_\alpha) \circ \exp(\xi_{-\alpha}) = (J \circ T_\alpha \circ J^{-1})^{-1}$, which completes the proof of 1).

2) By 4d) of Proposition 2.5b we know that $(T_\sigma^+)^* = (T_\sigma^+)^{-1}$ and $(T_\sigma^-)^* = (T_\sigma^-)^{-1}$. Now apply 1). □

Proposition 3.2d

Let τ be the element of W such that $\tau(\Phi^+) = \Phi^-$ and $\tau(\Phi^-) = \Phi^+$. Let V be an irreducible $\mathfrak{G}$ - module, and let T_τ be the element of $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}} \cap GL(V)$ constructed in Proposition 2.4a. Let λ denote the highest weight. Then

1) The lowest weight is $-\lambda$.
2) $J \circ T_\tau$ leaves invariant both V_λ and $V_{-\lambda}$.
3) $(T_\tau)^2 = Id$ on V_λ and $(T_\tau)^2 = Id$ on $V_{-\lambda}$.
4) $(J \circ T_\tau)^2$ is the identity on V_λ, and $(J \circ T_\tau)^2$ is the identity on $V_{-\lambda}$.

Proof

1) It suffices to prove that $J(V_\lambda)$ is the lowest weight space since $J(V_\lambda) = V_{-\lambda}$ by 2) of Proposition 3.2b. Recall that $J \circ \xi_\alpha = - \xi_{-\alpha} \circ J$ for all $\alpha \in \Phi$ by 2) of Proposition 3.2a. If α is any positive root, then $\xi_{-\alpha}(J(V_\lambda)) = - J(\xi_\alpha(V_\lambda)) = 0$ since V_λ is the highest weight space. Since each subspace $\mathfrak{G}_\alpha$ of $\mathfrak{G}$ is 1-dimensional it is spanned by ξ_α, and it follows from the definition at the beginning of 2.3 that $J(V_\lambda)$ is the lowest weight space.

2) By 2) of Proposition 2.3b and 1) above we see that $\tau(\lambda) = -\lambda$. It follows from Proposition 2.4a that $T_\tau(V_\lambda) = V_{\tau(\lambda)} = V_{-\lambda}$ and $T_\tau(V_{-\lambda}) = V_{\tau(-\lambda)} = V_\lambda$. This completes the proof of 2) since $J(V_\mu) = V_{-\mu}$ for all $\mu \in \Lambda(V)$ by Proposition 3.2b.

3) and 4) We prove these assertions simultaneously. By 3) of Proposition 2.4d it follows that $(T_\tau)^2 = Id$ or $- Id$ on each of V_λ and $V_{-\lambda}$. We prove only that $(T_\tau)^2 = Id$ on V_λ since the proof in the other case is similar. Note that $(J \circ T_\tau)^2 = (T_\tau)^2$ since J commutes with the restrictions of T_τ to V_λ and $V_{-\lambda}$ by 2) of Proposition 3.2c. Hence $(J \circ T_\tau)^2$ has determinant 1 or -1 on V_λ. It suffices to prove the first possibility to prove 3) and 4) since V_λ is 1-dimensional by 5) of Proposition 2.3a.

Let v be any nonzero vector in V_λ. Since V_λ is 1-dimensional it follows from 2) above that $(J \circ T_\tau)(v) = cv$ for some $c \in \mathbb{C}$. Because $J \circ T_\tau$ is a conjugate linear map of V we conclude that $(J \circ T_\tau)^2(v) = (J \circ T_\tau)(cv) = \bar{c}(J \circ T_\tau)(v) = |c|^2 v$. Hence the determinant of $(J \circ T_\tau)^2$ on $V_\lambda = |c|^2 = \pm 1$ by the previous paragraph. We conclude that $(J \circ T_\tau)^2$ has determinant 1 on V_λ, which completes the proof of 3) and 4).

Proposition 3.2e

Let V, τ, λ and T_τ be as in Proposition 3.2d. Let $V_\lambda^+ = \{v \in V_\lambda : (J \circ T_\tau)(v) = v\}$ and $V_\lambda^- = v \in V_\lambda : (J \circ T_\tau)(v) = - v\}$. Then

1) V_λ^+ and V_λ^- are 1-dimensional real subspaces of V_λ.

2) Let $v \in V_\lambda$.Then the following statements are equivalent :

a) $v \in V_\lambda^+$ or V_λ^-.

b) $J(v) \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$

c) $J(\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)) = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$

d) There exists a $\mathbb{C}$-basis $\mathbb{B}^*$ for V such that $\mathbb{B}^*$ is also a $\mathbb{Z}$-basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ and J has a $\mathbb{Z}$ - matrix relative to $\mathbb{B}^*$.

If any of the equivalent statements a) through d) hold, then J has a $\mathbb{Z}$ - matrix relative to any $\mathbb{C}$-basis $\mathbb{B}^*$ of V that is also a $\mathbb{Z}$-basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$.

Proof

1) By 2) and 4) of the previous result $J \circ T_\tau$ leaves V_λ invariant and $(J \circ T_\tau)^2 = Id$ on V_λ. Hence for all vectors w in V_λ, V_λ^+ contains $(J \circ T_\tau)(w) + w$ and V_λ^- contains

$(J \circ T_\tau)$ (w) – w. Clearly both V_λ^+ and V_λ^- are nonzero real subspaces of V_λ since $J \circ T_\tau$ is a conjugate linear map on V_λ, and the real dimension of both subspaces must be 1 since the complex dimension of V_λ is 1.

2) We prove the equivalences in cyclic order. If $v \in V_\lambda^+$ or V_λ^-, then $v = \pm(J \circ T_\tau)(v) \in J\,(\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v))$ since T_τ is an element of the ring $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$. Applying J again, we obtain $J(v) \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, which proves that a) $\Rightarrow$ b). The map J normalizes $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ in End(V) by 1) of Proposition 3.2b and hence $J\,(\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)) = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}\,(Jv) \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ if b) holds. The reverse inclusion then follows by applying J to both sides, which proves that b) $\Rightarrow$ c). By Proposition 2.4c there exists a $\mathbb{C}$-basis $\mathfrak{B}^*$ for V that is also a $\mathbb{Z}$-basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. In any such case $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) = \mathbb{Z}$-span $(\mathfrak{B}^*)$, which implies that $J(\mathfrak{B}^*) \subseteq \mathbb{Z}$-span $(\mathfrak{B}^*)$ if c) holds. Hence c) $\Rightarrow$ d).

To prove that d) $\Rightarrow$ a) we use the following

Lemma

If v is any nonzero vector in V_λ, then $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) \cap V_\lambda = \mathbb{Z}v$.

Proof

By 3) of Proposition 2.4c we may choose a $\mathbb{Z}$ - basis $\mathfrak{B}^*$ for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ such that $\mathfrak{B}^*$ is a union of sets $\{\mathfrak{B}_o{}^*, \mathfrak{B}_\mu{}^*\}$, where $\mathfrak{B}_o{}^*$ is a $\mathbb{C}$-basis for V_o and $\mathfrak{B}_\mu{}^*$ is a $\mathbb{C}$-basis for V_μ for all $\mu \in \Lambda(V)$. Since V_λ is 1-dimensional there is a unique vector v* in $\mathfrak{B}^* \cap V_\lambda = \mathfrak{B}_\lambda{}^*$ and by the structure of $\mathfrak{B}^*$ it follows that $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) \cap V_\lambda = \mathbb{Z}$ - span$(\mathfrak{B}^*) \cap V_\lambda = \mathbb{Z}$ - span$(\mathfrak{B}_\lambda{}^*) = \mathbb{Z}v^*$. In particular v = mv* for some $m \in \mathbb{Z}$ since $v \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) \cap V_\lambda$. Since $v^* \in \mathfrak{B}^* \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ it follows that $\mathbb{Z}$ - span$(\mathfrak{B}^*) = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) = m\,\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v^*) \subseteq m\,\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) = m\,\mathbb{Z}$ - span$(\mathfrak{B}^*)$. We conclude that m = ± 1, which implies that $\mathbb{Z}v = \mathbb{Z}v^* = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) \cap V_\lambda$. □

We now prove that d) $\Rightarrow$ a). Let $\mathfrak{B}^*$ satisfy d). It follows that J leaves invariant $\mathbb{Z}$ - span$(\mathfrak{B}^*) = \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. Since $T_\tau \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ we conclude that $J \circ T_\tau$ leaves invariant $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. Since $v \in V_\lambda$ it follows from the previous line, 2) of Proposition 3.2d and the Lemma above that $(J \circ T_\tau)(v) \in \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) \cap V_\lambda = \mathbb{Z}v$. Hence $(J \circ T_\tau)(v)$ = mv for some $m \in \mathbb{Z}$, and it follows that m = ± 1 since $(J \circ T_\tau)^2$ = Id on V_λ by 4) of Proposition 3.2d. Hence d) $\Rightarrow$ a).

Finally, suppose that any one of the equivalent statements a) through d) holds, and let $\mathfrak{B}^*$ be any basis of V that is also a $\mathbb{Z}$-basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$. Since J leaves invariant $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) = \mathbb{Z}$ - span$(\mathfrak{B}^*)$ by c), it follows that $J(\mathfrak{B}^*) \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v) = \mathbb{Z}$ - span$(\mathfrak{B}^*)$, or equivalently, that J has a $\mathbb{Z}$ - matrix relative to $\mathfrak{B}^*$.

Section 4 Existence of $\mathbb{Z}$ - bases for real $\mathfrak{G}_o$ - modules

Theorem Let $\mathfrak{G}_o$ be a compact, semisimple, finite dimensional real Lie algebra, and let U be a finite dimensional real $\mathfrak{G}_o$ - module. Let $\mathfrak{C}_o = \{X_1, \ldots, X_n\}$ be a real Chevalley basis for $\mathfrak{G}_o$ (cf. 1.5) Then

1) $[X_i, X_j] = \sum_{j=1}^{n} C_{ij}^k X_k$ where $\{C_{ij}^k\} \subseteq \mathbb{Z}$ for all i,j,k.

2) There exists an $\mathbb{R}$ - basis $\mathfrak{B}$ for U such that $\mathfrak{C}_o$ leaves invariant $\mathbb{Z}$ - span($\mathfrak{B}$).

Proof of the Theorem

It suffices to prove this result for irreducible $\mathfrak{G}_o$ - modules U since every $\mathfrak{G}_o$ - module U is a direct sum of irreducible $\mathfrak{G}_o$ - modules . We break the discussion into the two cases described in Proposition 3.1a, using the same numbering.

Case 2 This case is by far the easier, and we treat it first. Let $\mathfrak{G} = \mathfrak{G}_o^{\mathbb{C}}$. Let $\mathfrak{A}$ be a Cartan subalgebra of $\mathfrak{G}$ and Δ a basis for the roots $\Phi \subseteq \mathfrak{A}^*$ such that $\mathfrak{G}_o =$ $\mathbb{R}$ - span ($\mathfrak{C}_o$), where $\mathfrak{C}_o = \{i\, H_\beta : \beta \in \Delta\, ;\, A_\alpha, B_\alpha : \alpha \in \Phi^+\}$ is the real Chevalley basis determined by the Chevalley basis $\mathfrak{C} = \{H_\beta : \beta \in \Delta\, ;\, \xi_\alpha : \alpha \in \Phi\}$. This choice of $\mathfrak{A}$, Δ and $\mathfrak{C}$ can be made by the discussion in 1.5. Regard $\mathfrak{G}$ as a subalgebra of End(V).

Assertion 1) of the theorem is satisfied according to the Proposition in 1.5. By Proposition 2.4c there exists a $\mathbb{C}$ - basis $\mathfrak{B}^*$ for V such that the Chevalley basis $\mathfrak{C}$ for $\mathfrak{G}$ leaves invariant $\mathbb{Z}$ - span($\mathfrak{B}^*$). If $\mathfrak{B} = \mathfrak{B}^* \cup i\, \mathfrak{B}^*$, then clearly $\mathfrak{B}$ is an $\mathbb{R}$ - basis for U $= V^{\mathbb{R}}$, and it follows routinely from the definition of $\mathfrak{C}_o$ that $\mathfrak{C}_o$ leaves invariant $\mathbb{Z}$ - span($\mathfrak{B}$).

Case 1 In this case we need to apply the results from section 3, and for the convenience of the reader we outline the proof. Regard $\mathfrak{G}$ and $\mathfrak{U}(\mathfrak{G})$ as subalgebras of End(V), and, as usual, let $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ be the subring of $\mathfrak{U}(\mathfrak{G})$ generated by 1 and $\{(\xi_\alpha)^n / n! : \alpha \in \Phi$ and $n \in \mathbb{Z}^+ \}$. Let $\mathfrak{C}_o$ be exactly as in Case 1. Let λ denote the unique highest weight in Λ(V), and let V_λ^+ and V_λ^- be the 1-dimensional real subspaces of V_λ defined in Proposition 3.2e. Let v be any nonzero vector in $V_\lambda^+ \cup V_\lambda^-$, and let $\mathfrak{B}$ be any $\mathbb{Z}$ - basis for the finitely generated $\mathbb{Z}$ - module $U_v = \mathbb{Z}$ - span (Re $\{\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)\}$, Im $\{\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)\}$). We show $\mathfrak{B}$ is an $\mathbb{R}$-basis of U and that the pair $\{\mathfrak{C}_o, \mathfrak{B}\}$ satisfies the two assertions of the theorem.

We now begin filling in the details of the outline. Assertion 1) follows exactly as in case 2, so we consider only assertion 2). Let v be any nonzero vector in $V_\lambda^+ \cup V_\lambda^-$. Since $v \in V_\lambda$ we know that $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$ is a finitely generated $\mathbb{Z}$ - module by 1) of Propositiion 2.4c. Hence if $\mathfrak{B}^*$ is a finite $\mathbb{Z}$ - basis for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, then

$\mathbb{Z}$ - span {Re ($\mathcal{B}$*), Im ($\mathcal{B}$*)}= $\mathbb{Z}$-span {Re ($\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$(v), Im ($\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$(v)}= U_V . Hence U_V is a finitely generated $\mathbb{Z}$ - module and admits a finite $\mathbb{Z}$ - basis $\mathcal{B}$. The set $\mathcal{B}$* is a $\mathbb{C}$-basis for V by 2) of Proposition 2.4c, and hence $\mathbb{R}$-span ($\mathcal{B}$) = $\mathbb{R}$-span {Re ($\mathcal{B}$*), Im ($\mathcal{B}$*)} = U since $\mathbb{Z}$-span ($\mathcal{B}$) = $\mathbb{Z}$-span {Re ($\mathcal{B}$*), Im ($\mathcal{B}$*)}.

To prove that $\mathcal{B}$ is an $\mathbb{R}$ - basis for U it suffices to prove that $|\mathcal{B}| = \dim_{\mathbb{R}}$ U. This is an immediate consequence of the next result whose proof we postpone temporarily.

Lemma 1

Rank $_{\mathbb{Z}}$ (U_V) = $\dim_{\mathbb{C}}$ V = $\dim_{\mathbb{R}}$ U.

Next, observe that $\mathcal{C} \subseteq \mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ by the discussion at the end of 1.6, and $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$ leaves invariant $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$(v) = $\mathbb{Z}$-span ($\mathcal{B}$*). Hence $\mathcal{C}$ leaves invariant $\mathbb{Z}$-span ($\mathcal{B}$*). Assertion 2) and the theorem will now follow immediately from the next result :

Lemma 2

Let $\mathcal{B}$ be an $\mathbb{R}$-basis for U and $\mathcal{B}$* a $\mathbb{C}$-basis for V = $U^{\mathbb{C}}$ such that $\mathbb{Z}$-span ($\mathcal{B}$) = $\mathbb{Z}$-span {Re ($\mathcal{B}$*), Im ($\mathcal{B}$*)}. If $\mathcal{C}$ leaves invariant $\mathbb{Z}$-span ($\mathcal{B}$*), then $\mathcal{C}_o$ leaves invariant $\mathbb{Z}$-span ($\mathcal{B}$).

Proof of Lemma 1

We proceed by constructing a special $\mathbb{Z}$ - basis $\mathcal{B}$ for U_V. By 3) of Proposition 2.4c there exists a $\mathbb{Z}$ - basis $\mathcal{B}$* for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}$(v) such that $\mathcal{B}$* is a union of sets {$\mathcal{B}_o$*, $\mathcal{B}_\mu$*} where $\mathcal{B}_o$* is a $\mathbb{C}$-basis for V_o and $\mathcal{B}_\mu$* is a $\mathbb{C}$-basis for V_μ for all $\mu \in \Lambda$(V).

By 2) of Proposition 3.2b we know that $-\mu \in \Lambda$(V) if $\mu \in \Lambda$(V). Hence we can find a subset Λ^+(V) of Λ(V) that contains exactly one of {μ, $-\mu$} for each $\mu \in \Lambda$(V) ; that is, Λ(V) is the disjoint union of Λ^+(V) and $-\Lambda^+$(V).

Let I, R be subsets of $\mathcal{B}_o$* with the following properties :

i) Im (I) is a $\mathbb{Z}$ - basis for the finitely generated $\mathbb{Z}$-module, $\mathbb{Z}$ - span {Im ($\mathcal{B}_o$*)}.

. ii) Re (R) is a $\mathbb{Z}$ - basis for the finitely generated $\mathbb{Z}$-module, $\mathbb{Z}$ - span {Re ($\mathcal{B}_o$*)}

Lemma 1 will be a consequence of the following assertions :

a) $|R| + |I| = \dim_{\mathbb{C}} V_o$

b) $\mathbb{Z}$ - span (Re ($\mathcal{B}_\mu$*)) = $\mathbb{Z}$ - span (Re ($\mathcal{B}_{-\mu}$*)) and
$\mathbb{Z}$ - span (Im ($\mathcal{B}_\mu$*)) = $\mathbb{Z}$ - span (Im ($\mathcal{B}_{-\mu}$*)) for every $\mu \in \Lambda^+$(V).

Assuming a) and b) we complete the proof of Lemma 1. Let $\mathcal{B}_o$ = {Im (I), Re (R)}, and define $\mathcal{B} = \mathcal{B}_o \cup \{ \bigcup_{\mu\in\Lambda^+(V)}$ Re ($\mathcal{B}_\mu$*)} $\cup \{ \bigcup_{\mu\in\Lambda^+(V)}$ Im ($\mathcal{B}_\mu$*)}. This will be the special $\mathbb{Z}$ - basis $\mathcal{B}$ for U_V referred to in the first paragraph of the proof of the lemma.

From the definition of I and R we see that $\mathbb{Z}$ - span ($\mathcal{B}_o$) = $\mathbb{Z}$ - span (Re ($\mathcal{B}_o$*), Im ($\mathcal{B}_o$*)). From b) we see that $\mathbb{Z}$ - span ({ $\bigcup_{\mu\in\Lambda^+(V)}$ Re ($\mathcal{B}_\mu$*)} $\cup$ { $\bigcup_{\mu\in\Lambda^+(V)}$ Im ($\mathcal{B}_\mu$*)})

$= \mathbb{Z}$ - span (Re ($\mathcal{B}_\mu$*), Im ($\mathcal{B}_\mu$*) : $\mu \in \Lambda(V)$). Hence $\mathbb{Z}$ - span ($\mathcal{B}$) = $\mathbb{Z}$ - span (Re ($\mathcal{B}$*), Im ($\mathcal{B}$*)) = U_v . We conclude that $\mathcal{B}$ is a $\mathbb{Z}$ - spanning set for U_v, and it follows that $\mathcal{B}$ is an $\mathbb{R}$-spanning set for U by the discussion preceding the lemma above. Hence

(*) $|\mathcal{B}| \geq \dim_{\mathbb{R}} U = \dim_{\mathbb{C}} V$.

From a) above we see that $|\mathcal{B}_o| \leq \dim_{\mathbb{C}} V_o$ and evidently $|\{Re(\mathcal{B}_\mu*), Im(\mathcal{B}_\mu*)\}| \leq 2 \dim_{\mathbb{C}} V_\mu = \dim_{\mathbb{C}} V_\mu + \dim_{\mathbb{C}} V_{-\mu}$ for all $\mu \in \Lambda^+(V)$. Hence $|\mathcal{B}| \leq \dim_{\mathbb{C}} V_o + \sum_{\mu \in \Lambda^+(V)} \{\dim_{\mathbb{C}} V_\mu + \dim_{\mathbb{C}} V_{-\mu}\} = \dim_{\mathbb{C}} V = \dim_{\mathbb{R}} U$. The inequalities in (*) now become equalities, which completes the proof of Lemma 1.

Finally, we prove a) and b). To prove a) we consider the $\mathbb{C}$-basis $\mathcal{B}_o$* for V_o that is part of our initial $\mathbb{Z}$-basis $\mathcal{B}$* for $\mathfrak{U}(\mathfrak{G})_{\mathbb{Z}}(v)$, and we let I and R be the subsets of $\mathcal{B}_o$* with the properties i) and ii) above. Define $M_o = \mathbb{Z}$ - span ($\mathcal{B}_o$*) and $(V_o)_{\mathbb{Q}}$ = $\mathbb{Q}$ - span ($\mathcal{B}_o$*) $\subseteq V_o$. Define T = (1/2) (Id + J), where J : V $\to$ V is the conjugation map determined by U. Note that T leaves invariant both V_o and $(V_o)_{\mathbb{Q}}$; observe that J $(V_o) \subseteq V_o$ by Proposition 3.2b and J has a $\mathbb{Z}$ - matrix relative to $\mathcal{B}$* and hence also to $\mathcal{B}_o$* by Proposition 3.2e. Note also that $T(\xi)$ = Re (ξ) for all $\xi \in V$.

In the remainder of the proof of a) we regard T as a $\mathbb{Q}$ - linear transformation of $(V_o)_{\mathbb{Q}}$. To prove a) it suffices to prove that $|R| = \text{rank}_{\mathbb{Q}}$ T and $|I| = \dim_{\mathbb{Q}}$ Ker (T) since $\dim_{\mathbb{Q}} (V_o)_{\mathbb{Q}} = |\mathcal{B}_o*| = \dim_{\mathbb{C}} V_o$. More precisely, we show

(**) T $((V_o)_{\mathbb{Q}}) = \mathbb{Q}$ - span (Re (R)) and Ker (T) = $\mathbb{Q}$ - span (i Im (I))

The elements of Re (R) (respectively i Im (I)) are linearly independent over $\mathbb{Q}$ since they are linearly independent over $\mathbb{Z}$ by the definitions of R and I above in i) and ii).

If $\xi \in$ R, then Re (ξ) = T(ξ) $\in$ T $((V_o)_{\mathbb{Q}})$. Hence $\mathbb{Q}$ - span (Re (R)) $\subseteq$ T $((V_o)_{\mathbb{Q}})$. Conversely, if $\xi \in (V_o)_{\mathbb{Q}}$, then $\xi^* = m\xi \in M_o = \mathbb{Z}$ - span ($\mathcal{B}_o$*) for some m $\in \mathbb{Z}$. From the definition of R we see that T(ξ*) = Re (ξ*) $\in \mathbb{Z}$ - span (Re (R)), and we conclude that T(ξ) = (1/m) T(ξ*) $\in \mathbb{Q}$ - span (Re (R)). This proves the first assertion in (**).

If $\xi \in$ I, then T(i Im (ξ)) = (1/2) (Id + J) (i Im (ξ)) = 0, which shows that $\mathbb{Q}$ - span (i Im (I)) $\subseteq$ Ker (T). Conversely, if $\xi \in (V_o)_{\mathbb{Q}}$ is an element of Ker (T), then $\xi^* = m\xi \in M_o \cap$ Ker (T) for some m $\in \mathbb{Z}$. Hence Re (ξ*) = T(ξ*) = 0, which implies that $\xi^* \in$ i Im $(M_o) = \mathbb{Z}$ - span (i Im (I)). We conclude that ξ = (1/m) $\xi^* \in$ $\mathbb{Q}$ - span (i Im (I)), which completes the proof of (**) and a).

We prove b), which will complete the proof of the lemma. We recall that J $(V_\mu) = V_{-\mu}$ for all $\mu \in \Lambda(V)$ by Proposition 3.2b. By Proposition 3.2e J : V $\to$ V has a

$\mathbb{Z}$ - matrix relative to the basis $\mathcal{B}^*$, and by the special nature of $\mathcal{B}^*$ it follows that $J : V_\mu \to V_{-\mu}$ has a $\mathbb{Z}$-matrix relative to the bases $\mathcal{B}_\mu{}^*$ and $\mathcal{B}_{-\mu}{}^*$.

Let $\mu \in \Lambda(V)$ and $\xi \in \mathcal{B}_\mu{}^*$ be given. Then $\mathrm{Re}(\xi) = \mathrm{Re}(J\xi) \in \mathbb{Z}$ - span (Re ($\mathcal{B}_{-\mu}{}^*$)) since $J(\xi) \in \mathbb{Z}$ - span ($\mathcal{B}_{-\mu}{}^*$). Similarly, $\mathrm{Im}(\xi) = -\mathrm{Im}(J\xi) \in \mathbb{Z}$ - span (Im ($\mathcal{B}_{-\mu}{}^*$)). Since $\xi \in \mathcal{B}_\mu{}^*$ was arbitrary we have shown

$$\mathbb{Z}\text{ - span (Re (}\mathcal{B}_\mu{}^*)) \subseteq \mathbb{Z}\text{ - span (Re (}\mathcal{B}_{-\mu}{}^*))$$
$$\mathbb{Z}\text{ - span (Im (}\mathcal{B}_\mu{}^*)) \subseteq \mathbb{Z}\text{ - span (Im (}\mathcal{B}_{-\mu}{}^*))$$

The reverse inclusions follow by reversing the roles of μ and $-\mu$, which completes the proof of b).□

Proof of Lemma 2

By the definitions of the elements in $\mathcal{C}_o$ it follows that $A_\alpha(\mathcal{B}^*) \subseteq \mathbb{Z}$ - span ($\mathcal{B}^*$), $i H_\alpha(\mathcal{B}^*) \subseteq \mathbb{Z}$ - span ($i\,\mathcal{B}^*$) and $B_\alpha(\mathcal{B}^*) \subseteq \mathbb{Z}$ - span ($i\,\mathcal{B}^*$). Hence A_α (Re $\mathcal{B}^*$) = Re $A_\alpha(\mathcal{B}^*) \subseteq \mathbb{Z}$ - span (Re $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span ($\mathcal{B}$). Since Re ($i\,\mathcal{B}^*$) = – Im $\mathcal{B}^*$, similar arguments show that $i H_\alpha$ (Re $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span (Im $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span ($\mathcal{B}$) and B_α (Re $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span (Im $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span ($\mathcal{B}$). We have shown that $\mathcal{C}_o$(Re $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span ($\mathcal{B}$), and a similar argument shows that $\mathcal{C}_o$(Im $\mathcal{B}^*$) $\subseteq \mathbb{Z}$ - span ($\mathcal{B}$).□

Section 5 Rational Approximation

We adopt the following hypotheses and notation in this section. G_o is a compact, connected semisimple Lie group, and $\mathfrak{G}_o$ is the Lie algebra of G_o. U is a finite dimensional real vector space, and $\rho : G_o \to GL(U)$ is a representation with derived representation $d\rho : \mathfrak{G}_o \to \mathrm{End}(U)$. $\mathcal{C}_o$ is a fixed real Chevalley basis of $\mathfrak{G}_o$, and $B(\mathbb{Q}, \mathcal{C}_o)$ is the set of bases $\mathcal{B}$ of U such that $d\rho(\mathcal{C}_o)$ leaves invariant $\mathbb{Q}$ - span ($\mathcal{B}$). For $\mathcal{B} \in B(\mathbb{Q}, \mathcal{C}_o)$, $G_{o,\mathcal{B},\mathbb{Q}} = \{g \in G_o : \rho(g)$ leaves invariant $\mathbb{Q}$-span ($\mathcal{B}$). $G_{o,\mathbb{Q}} = \{g \in G_o : \mathrm{Ad}(g)$ leaves invariant $\mathbb{Q}$ -span ($\mathcal{C}_o$)}.

Theorem A

Let $\rho : G_o \to GL(U)$ be as above, and for any basis $\mathcal{B}$ of U let $\rho_{\mathcal{B}} : G_o \to GL(n, \mathbb{R})$ denote the corresponding Lie group homomorphism.

1) If $\mathcal{B} \in B(\mathbb{Q}, \mathcal{C}_o)$, then $\rho_{\mathcal{B}}(G_o)$ is an affine algebraic group defined over $\mathbb{Q}$.

2) If $\mathcal{B} \in B(\mathbb{Q}, \mathcal{C}_o)$, then $G_{o,\mathcal{B},\mathbb{Q}}$ is dense in G_o in the Lie group topology.

Remark

In a sequel to this article we shall prove the following

Theorem

Let $\rho : G_o \to GL(U)$ be as in Theorem A above.

1) $G_{o,\mathbb{B},\mathbb{Q}}$ is independent of the basis $\mathbb{B} \in B(\mathbb{Q}, \mathbb{C}_o)$.
2) If ρ has finite kernel, then $G_{o,\mathbb{B},\mathbb{Q}}$ is a normal subgroup of $G_{o,\mathbb{Q}}$. Moreover, the group $G_{o,\mathbb{Q}} / G_{o,\mathbb{B},\mathbb{Q}}$ is abelian and every element of $G_{o,\mathbb{Q}} / G_{o,\mathbb{B},\mathbb{Q}}$ has finite order.

Note that $G_{o,\mathbb{Q}} = G_{o,\mathbb{B},\mathbb{Q}}$ in the case that $\rho = Ad$, $U = \mathfrak{G}_o$ and $\mathbb{B} = \mathbb{C}_o$.

Before stating the second main result of this article we need to recall and introduce some further notation. Let $F = \mathbb{Q}(i) = \mathbb{Q} + i\,\mathbb{Q}$. Let $\mathfrak{G}_o$ be any compact, semisimple Lie algebra. For an arbitrary element $X \in \mathfrak{G}_o$ define $\mathbb{R}_X = \{t \in \mathbb{R}$: for any finite dimensional real representation $\sigma : \mathfrak{G}_o \to End(U)$ the eigenvalues of $\exp(t\,\sigma(X))$ are contained in $F\}$, where $\exp : End(U) \to GL(U)$ is the matrix exponential map. The elements in $\exp(\mathbb{R}\sigma(X)\}$ form a commuting family of semisimple elements in $GL(U)$. Hence, the set $\mathbb{R}_X$ is always an additive subgroup of $\mathbb{R}$, but for a typical element X of $\mathfrak{G}_o$ one expects that $\mathbb{R}_X = \{0\}$.

Theorem B (Universal rational approximation)

Let $\mathfrak{G}_o$ be a compact semisimple Lie algebra. Then there exists a dense subset $\mathfrak{G}_o^{\#}$ of $\mathfrak{G}_o$ with the following properties :

1) If $\sigma : \mathfrak{G}_o \to End(U)$ is any finite dimensional real representation, then there exists a basis $\mathbb{B}$ of U such that the matrix of $\sigma(X)$ relative to $\mathbb{B}$ has entries in $\mathbb{Q}$ for all $X \in \mathfrak{G}_o^{\#}$.
2) If $\sigma : \mathfrak{G}_o \to End(U)$ is any finite dimensional real representation, then the eigenvalues of $\sigma(X)$ are contained in $i\mathbb{Q}$ for all $X \in \mathfrak{G}_o^{\#}$.
3) Let G_o be any compact, connected, semisimple Lie group with Lie algebra $\mathfrak{G}_o$. Then for each $X \in \mathfrak{G}_o^{\#}$ there exists a positive integer m such that $\exp(2\pi mX) = \exp(2\pi X)^m = e$, the identity in G_o.
4) a) $\mathbb{R}_X$ is a dense additive subgroup of $\mathbb{R}$ for all $X \in \mathfrak{G}_o^{\#}$.
   b) Let G_o be any compact, semisimple Lie group with Lie algebra $\mathfrak{G}_o$, and let $G_o^* = \{\exp(tX) : t \in \mathbb{R}_X$ and $X \in \mathfrak{G}_o^{\#}\}$. Then G_o^* is a dense subset of G_o in the Lie topology of G_o, and the eigenvalues of $\rho(g)$ are contained in F for all $g \in G_o^*$ and any finite dimensional real representation $\rho : G_o \to GL(U)$

Remark The subset $\mathfrak{G}_o^{\#}$ of $\mathfrak{G}_o$ is by no means unique. We could replace $\mathfrak{G}_o^{\#}$ by $Ad(g)(\mathfrak{G}_o^{\#})$ for any element g of any compact connected Lie group G_o with Lie algebra $\mathfrak{G}_o$.

Proof of Theorem A

We use the definitions, notation and results stated in the Appendix.

1) Let $\mathfrak{B}$ be any basis in $B(\mathbb{Q}, \mathfrak{C}_o)$, which is nonempty by the theorem in section 4. If we identify GL(U) and End(U) with $GL(n,\mathbb{R})$ and $M(n,\mathbb{R})$ using the basis $\mathfrak{B}$, then by the definition of $B(\mathbb{Q}, \mathfrak{C}_o)$ it follows that $\mathbb{Q}$-span $(d\rho(\mathfrak{C}_o))$ is a semisimple subalgebra of $M(n, \mathbb{Q})$. By the theorem in section 3 of Appendix 2 there exists an algebraic group H in $GL(n,\mathbb{Q})$ with Lie algebra $L(H) = \mathbb{Q}$-span $(d\rho(\mathfrak{C}_o))$. Let $G \subseteq GL(n, \mathbb{R})$ be the affine algebraic variety defined over $\mathbb{Q}$ by the same polynomials that define H. By the discussion in section 1 of Appendix 2, G is an affine algebraic group and $H = G_{\mathbb{Q}} = G \cap GL(n,\mathbb{Q})$. Moreover, $L(G) = L(H) \otimes \mathbb{R} = \mathbb{R}$ - span $(d\rho(\mathfrak{C}_o))$. However, $\mathbb{R}$ - span $(d\rho(\mathfrak{C}_o))$ is also the Lie algebra of $\rho(G_o)$. It follows that $\rho(G_o) = G^o$, the identity component of G, since both $\rho(G_o)$ and G^o are connected Lie groups with the same Lie algebra. The proof of 1) is complete since G^o is also an algebraic group defined over $\mathbb{Q}$ by the discussion in section 1 of Appendix 2.

2) This is an immediate consequence of the next " weak approximation " result of [S, Cor.3.5 (iii)]. See also Theorem 7.7 of [PR, p. 415]. For the convenience of the reader we give a brief proof in section 5 of Appendix 2 in the semisimple case, the one of interest to us here.

Theorem [S, Cor.3.5 (iii)]

Let $H \subseteq GL(n, \mathbb{C})$ be a connected algebraic group defined over $\mathbb{Q}$, and let H^o denote the identity component of H. Then $H^o_{\mathbb{Q}} = H^o \cap GL(n, \mathbb{Q})$ is dense in $H^o_{\mathbb{R}} = H^o \cap GL(n, \mathbb{R})$ in the Lie group topology of $H^o_{\mathbb{R}}$.

Proof of Theorem B

Let $\mathfrak{G} = \mathfrak{G}_o{}^{\mathbb{C}}$. Let $\mathfrak{A}$ be a Cartan subalgebra of $\mathfrak{G}$ and $\mathfrak{C} = \{H_\beta : \beta \in \Delta ; \xi_\alpha : \alpha \in \Phi\}$ a Chevalley basis of $\mathfrak{G}$ such that $\mathfrak{G}_o = \mathbb{R}$-span $(\mathfrak{C}_o)$, where $\mathfrak{C}_o = \{i H_\beta : \beta \in \Delta ; A_\alpha, B_\alpha : \alpha \in \Phi^+\}$ is the corresponding real Chevalley basis. (See section 1.5.) Define $\mathfrak{A}_o = \mathbb{R}$-span $\{i H_\beta : \beta \in \Delta\}$ and $\mathfrak{A}_{o,\mathbb{Q}} = \mathbb{Q}$-span $\{i H_\beta : \beta \in \Delta\} \subseteq \mathbb{Q}$-span $(\mathfrak{C}_o)$. Let $\tilde{G}_o$ be the simply connected Lie group with Lie algebra $\mathfrak{G}_o$. Then $\tilde{G}_o$ is compact by the discussion at the beginning of section 1.5. Let $\tilde{G}_{o,\mathbb{Q}} = \{g \in \tilde{G}_o : Ad(g)$ leaves invariant $\mathbb{Q}$-span $(\mathfrak{C}_o)\}$, and define $\mathfrak{G}_o{}^{\#} = Ad(\tilde{G}_{o,\mathbb{Q}})(\mathfrak{A}_{o,\mathbb{Q}}) \subseteq \mathbb{Q}$-span $(\mathfrak{C}_o)$. We show that $\mathfrak{G}_o{}^{\#}$ has the properties asserted in Theorem C.

Remark

The set $\mathfrak{G}_o{}^{\#} \subseteq \mathbb{Q}$-span $(\mathfrak{C}_o)$ could have been defined, with the same result, by any compact, connected semisimple Lie group G_o with Lie algebra $\mathfrak{G}_o$. If $G_{o,\mathbb{Q}} = \{g \in G_o : Ad(g)$ leaves invariant $\mathbb{Q}$-span $(\mathfrak{C}_o)\}$, then by using a covering homomorphism $\rho_o : \tilde{G}_o \to G_o$ it is routine to prove that $Ad(\tilde{G}_{o,\mathbb{Q}}) = Ad(G_{o,\mathbb{Q}}) \subseteq GL(\mathfrak{G}_o)$.

We shall need the following

Lemma 1

Let A_o be the connected subgroup of $\tilde{G}_o$ with Lie algebra $\mathfrak{A}_o$. Then A_o is a maximal torus of $\tilde{G}_o$.

Proof of the lemma

Note that $\mathfrak{A}_o^{\mathbb{C}} = \mathbb{C}$-span $\{H_\beta : \beta \in \Delta\} = \mathfrak{A}$, a Cartan subalgebra of $\mathfrak{G}_o^{\mathbb{C}}$, which by definition is a maximal abelian subalgebra of $\mathfrak{G}_o^{\mathbb{C}}$ such that ad A is semisimple for all A in $\mathfrak{A}$. Since $\mathfrak{G}_o$ is the Lie algebra of the compact Lie group $\tilde{G}_o$ it follows that ad X : $\mathfrak{G}_o^{\mathbb{C}} \to \mathfrak{G}_o^{\mathbb{C}}$ is semisimple for all $X \in \mathfrak{G}_o$. Therefore, if $\mathfrak{B}$ is any abelian subalgebra of $\mathfrak{G}_o$, then $\mathfrak{B}^{\mathbb{C}}$ is an abelian subalgebra of $\mathfrak{G}_o^{\mathbb{C}}$ such that ad X is semisimple for all $X \in \mathfrak{B}^{\mathbb{C}}$. We conclude that $\mathfrak{A}_o$ is a maximal abelian subalgebra of $\mathfrak{G}_o$. If H is a connected Lie subgroup of $\tilde{G}_o$, then H is a maximal torus of $\tilde{G}_o \Leftrightarrow$ the Lie algebra of H is a maximal abelian subalgebra of $\mathfrak{G}_o$. It follows that A_o is a maximal torus of $\tilde{G}_o$.□

Since $\mathfrak{A}_o$ is the Lie algebra of a maximal torus A_o of $\tilde{G}_o$ by the lemma above we conclude that $\mathfrak{G}_o = \mathrm{Ad}(\tilde{G}_o)(\mathfrak{A}_o)$. Hence $\mathfrak{G}_o^{\#}$ is a dense subset of $\mathfrak{G}_o$ since $\mathfrak{A}_{o,\mathbb{Q}}$ is a dense subset of $\mathfrak{A}_o$ and $\tilde{G}_{o,\mathbb{Q}}$ is a dense subgroup of $\tilde{G}_o$ in the Lie topology by Theorem A.

1) Let $\sigma : \mathfrak{G}_o \to \mathrm{End}(U)$ be a finite dimensional real representation. By the theorem in section 4 there exists a basis $\mathbb{B}$ of U such that $\sigma(\mathbb{C}_o)$ leaves invariant $\mathbb{Z}$-span ($\mathbb{B}$). Hence every element of $\sigma(\mathbb{Q}$-span $(\mathbb{C}_o))$ leaves invariant $\mathbb{Q}$-span ($\mathbb{B}$). Since $\mathfrak{G}_o^{\#} \subseteq \mathbb{Q}$-span $(\mathbb{C}_o)$ the proof of 1) is complete.

2) We shall also need

Lemma 2

Let H be any connected Lie group with Lie algebra $\mathfrak{H}$. Let U be a finite dimensional real vector space, and let $\rho : H \to GL(U)$ be a Lie group homomorphism with derived Lie algebra homomorphism $d\rho : \mathfrak{H} \to \mathrm{End}(U)$. Then $d\rho\,(\mathrm{Ad}(h)\,Z) = \rho(h) \circ d\rho(Z) \circ \rho(h)^{-1}$ for all $h \in H$ and all $Z \in \mathfrak{H}$.

Proof

For any $t \in \mathbb{R}$ we have $e^{t\, d\rho\, \mathrm{Ad}(h) Z} = \rho\,(e^{t\, \mathrm{Ad}(h)\, Z}) = \rho\,(h\, e^{t Z}\, h^{-1}) = \rho(h)\, e^{t\, d\rho(z)}\, \rho(h)^{-1}$. Differentiating at t = 0 proves the lemma.□

Let $\sigma : \mathfrak{G}_o \to \mathrm{End}(U)$ be a finite dimensional real representation. Since $\tilde{G}_o$ is simply connected there exists a (unique) representation $\rho: \tilde{G}_o \to GL(U)$ such that $d\rho = \sigma$. For any $X \in \mathfrak{G}_o$ and any $g \in \tilde{G}_o$ the eigenvalues of $\sigma(X)$ and $\sigma(\mathrm{Ad}(g)X)$ are the same since $\sigma(\mathrm{Ad}(g)X) = d\rho\ (\mathrm{Ad}(g)X) = c_{\rho(g)}\ d\rho(X) = c_{\rho(g)}\ \sigma(X)$ by the Lemma above, where $c_{\rho(g)}$ denotes conjugation by $\rho(g)$. Hence it suffices to prove that the eigenvalues of $\sigma(X)$ are contained in $i\mathbb{Q}$ for any $X \in \mathfrak{A}_{o,\mathbb{Q}} = \mathbb{Q}\text{-span }\{i H_\beta : \beta \in \Delta\}$.

Let $V = U^{\mathbb{C}}$, and let $V = V_o + \sum_{\mu \in \Lambda(V)} V_\mu$ (direct sum) denote the weight space decomposition of V determined by the Cartan subalgebra $\mathfrak{A}$. Let $\sigma : \mathfrak{G} = \mathfrak{G}_o^{\mathbb{C}} \to \mathrm{End}(V)$ also denote the complex representation that extends $\sigma : \mathfrak{G}_o \to \mathrm{End}(U)$. For $\mu \in \Lambda(V)$ and $\beta \in \Delta$ we know that $\sigma(H_\beta) = \mu(H_\beta)$ Id on V_μ, where $\mu(H_\beta) \in \mathbb{Z}$ by the discussion in sections 2.1 and 2.2. If $X = \sum_{\beta\in\Delta} q_\beta\,(i H_\beta)$, then $\sigma(X) = i\,(\sum_{\beta\in\Delta} q_\beta\ \mu(H_\beta))$ Id on V_μ. Hence the eigenvalues of $\sigma(X)$ lie in $i\mathbb{Q}$ for any $X \in \mathbb{Q}\text{-span }\{i H_\beta : \beta \in \Delta\}$.

3) We apply 2) to the adjoint representation $\mathrm{ad} : \mathfrak{G}_o \to \mathrm{End}(\mathfrak{G}_o)$. Let G_o be any compact, connected, semisimple Lie group with Lie algebra $\mathfrak{G}_o$. If X is any element of $\mathfrak{G}_o^{\#}$, then $\mathrm{ad}\,X : \mathfrak{G}_o \to \mathfrak{G}_o$ has eigenvalues in $i\,\mathbb{Q}$ by 2), and hence there exists a positive integer N such that if $X^* = 2\pi N X$, then $\mathrm{ad}\,X^*$ has eigenvalues in $i\,2\pi\mathbb{Z}$. Hence $\mathrm{Ad}\,(\exp\,(X^*)) = e^{\mathrm{ad}X^*}$ has eigenvalues in $e^{i2\pi\mathbb{Z}} = \{1\}$. Since G_o is compact the elements of $\mathrm{Ad}(G_o)$ are semisimple elements of $GL(\mathfrak{G}_o^{\mathbb{C}})$, and hence $\mathrm{Ad}\,(\exp\,(X^*)) = \mathrm{Id}$ on $GL(\mathfrak{G}_o)$. The kernel of $\mathrm{Ad} : G_o \to GL(\mathfrak{G}_o)$ is finite since G_o is compact and semisimple, and we conclude that $e = \exp\,(X^*)^k = \exp\,(kX^*) = \exp\,(2kN\pi X)$ for some positive integer k.

4)

a) Let $X \in \mathfrak{G}_o^{\#} = \mathrm{Ad}(\tilde{G}_{o,\mathbb{Q}})\,(\mathfrak{A}_{o,\mathbb{Q}})$ be given and write $X = \mathrm{Ad}(g)\,A$ for some $g \in \tilde{G}_{o,\mathbb{Q}}$ and some $A \in \mathfrak{A}_{o,\mathbb{Q}} = \mathbb{Q}\text{-span }\{i H_\beta : \beta \in \Delta\}$). Note that $\mathbb{R}_X = \mathbb{R}_A$ since $\exp(tX) = g\,\exp(tA)\,g^{-1}$ for all $t \in \mathbb{R}$, where $\exp : \mathfrak{G}_o \to \tilde{G}_o$ is the Lie group exponential map. Hence it suffices to show that $\mathbb{R}_A$ is dense in $\mathbb{R}$. Let m be a positive integer such that $mA \in \mathbb{Z}\text{-span }\{i H_\beta : \beta \in \Delta\}$, and let $S = \{t \in \mathbb{R} : e^{it} \in F = \mathbb{Q} + i\,\mathbb{Q}\}$. It is well known that S is a dense additive subgroup of $\mathbb{R}$. We show that $\mathbb{R}_A \supseteq mS$, which will complete the proof of a).

Let $\sigma : \mathfrak{G}_o \to \mathrm{End}(U)$ be a finite dimensional real representation of $\mathfrak{G}_o$. The proof of 2) shows that $\sigma(mA)$ has eigenvalues in $i\mathbb{Z}$ since $mA \in \mathbb{Z}\text{-span }\{i H_\beta : \beta \in \Delta\}$. If $s \in S$ is arbitrary, then $\sigma(msA)$ has eigenvalues in $is\mathbb{Z}$, and hence $\exp(ms\,\sigma(A)) = \exp(\sigma(msA))$ has eigenvalues in $e^{is\mathbb{Z}} \subseteq F$. Therefore $mS \subseteq \mathbb{R}_A$ and the proof of 4a) is complete.

b) The proof is an immediate consequence of a) and the fact that $\mathfrak{G}_o^{\#}$ is a dense subset of $\mathfrak{G}_o$.

Appendix Affine algebraic groups

We describe briefly some definitions, examples and other basic information that we need. For a more thorough treatment see for example [Bo], [C], [CSM] or [Hu 2].

Section 1 Basic definitions

Notation

Let k be any field, and let M(n,k) denote the k-vector space of n x n matrices with entries in k. M(n,k) is a k-algebra under matrix multiplication, but M(n,k) also becomes a Lie algebra by defining [X, Y] = XY – YX for elements X, Y of M(n,k). Let k[M(n,k)] denote the k-algebra of polynomial functions from M(n,k) to k, which is generated by the subspace of linear functions from M(n,k) to k. We let GL(n,k) denote the (general linear) group of invertible elements of M(n,k).

Zariski closed subsets of M(n,k)

A subset X of M(n,k) is <u>Zariski closed</u> if there exists an ideal J of k[M(n,k)] such that $X = \{g \in M(n,k) : p(g) = 0 \text{ for all } p \in J\}$. An ideal of k[M(n,k)] is always finitely generated (Hilbert Basis Theorem) so we may make the equivalent definition that X is a Zariski closed subset of M(n,k) if there exists a finite set of polynomials $\{p_1, p_2, \dots p_N\}$ such that $X = \{g \in M(n,k) : p_i(g) = 0 \text{ for } 1 \le i \le N\}$. The intersection of any family of Zariski closed sets is again a Zariski closed set. Hence the Zariski open sets, which are the complements in M(n,k) of Zariski closed sets, define a topology on M(n,k).

Let X be a Zariski closed subset of M(n,k) defined as the set of common zeros of an ideal J in k[M(n,k)]. One may now associate to X the ideal I of all polynomials in k[M(n,k)] that vanish on X. Clearly $J \subseteq I$, but the inclusion is strict in general. However, it is routine to show that $X = \{g \in M(n,k) : p(g) = 0 \text{ for all } p \in I\}$.

A Zariski closed subset X of M(n,k) is <u>irreducible</u> if X is not the disjoint union of two Zariski closed subsets of M(n,k).

Affine algebraic groups and their Lie algebras

A subgroup G of GL(n,k) is called an <u>affine algebraic group</u> if G is a Zariski closed subset of M(n,k). It follows immediately from the definitions and the discussion above that the intersection of any family of affine algebraic subgroups of GL(n,k) is again an affine algebraic subgroup of GL(n,k).

To every affine algebraic group G one may associate a Lie subalgebra L(G) of M(n,k) called the <u>Lie algebra of G</u> (cf. [C, p. 126]). We give a brief account below in section 4. If $k = \mathbb{R}$ or $\mathbb{C}$, then any affine algebraic group in GL(n,k) is a Lie group since it is closed in the topology induced from the standard Hausdorff topology on $M(n,k) \approx k^{n^2}$.

In this case L(G) is isomorphic as an $\mathbb{R}$-Lie algebra to the standard Lie algebra defined for any Lie group.

If $G \subseteq GL(n,k)$ is an affine algebraic group, then $L(G) \subseteq M(n,k)$ is isomorphic both as a k-algebra and a k-Lie algebra to the set of left invariant derivations of the k-algebra k(G), the polynomial functions from G to k. See section 4 for a brief discussion. For further details of this more intrinsic definition of L(G) see for example [Bo, pp.62-66] or [CMS, pp.161-171].

Identity component of an affine algebraic group

(cf. [Bo, pp. 46 and 65] and [C, pp. 86 and 129])

For every affine algebraic group G of GL(n,k) there exists a unique subgroup G^o with the following properties :

1) G^o is an affine algebraic group in GL(n,k) that is irreducible as a Zariski closed subset.

2) G^o has finite index in G.

3) $L(G^o) = L(G)$.

Moreover, if $k = \mathbb{R}$ or $\mathbb{C}$, then G^o is the connected component of G containing the identity, where G is regarded as a Lie group with the topology induced from the standard Hausdorff topology of M(n,k).

Extension of the base field (cf. [C, pp. 103-105)

Let k and ℓ be fields with $k \subseteq \ell$, and let $G \subseteq GL(n,k)$ be an affine algebraic group. If $I \subseteq k[M(n,k)]$ is the ideal of polynomials that vanish on G, then let I^{ℓ} denote the ideal in $\ell[M(n,\ell)]$ consisting of all ℓ-linear combinations of elements of I. Let $G^{\ell} = \{g \in GL(n,\ell) : p(g) = 0 \text{ for all } p \in I^{\ell}\}$. Then G^{ℓ} has the following properties :

1) G^{ℓ} is an affine algebraic group in $GL(n,\ell)$, the intersection of all affine algebraic groups in $GL(n,\ell)$ that contain G.

2) $G = G^{\ell} \cap M(n,k)$.

3) $L(G^{\ell})$ is the subset of $M(n,\ell)$ consisting of all ℓ-linear combinations of elements of L(G). Equivalently, $L(G^{\ell}) = L(G) \otimes \ell$, the tensor product over k.

Affine algebraic groups defined over subfields

(cf. [Hu 2, pp. 217-218])

Let k and ℓ be fields with $k \subseteq \ell$. Let $G \subseteq GL(n,\ell)$ be an affine algebraic group, and let I be the ideal of all polynomials in $\ell[M(n,\ell)]$ that vanish on G. If I is generated by elements $\{p_1, p_2, \ldots p_N\}$ of k[M(n,k)] , then we say that <u>G is defined over k</u> or that <u>G is a k-group</u>. In this case if J is the ideal in k[M(n,k)] defined by $\{p_1, p_2, \ldots p_N\}$ and if H =

$\{h \in M(n,k) : p(h) = 0$ for all $p \in J\}$, then H is an affine algebraic group in GL(n,k) and H = G $\cap$ M(n,k). Moreover, it is easy to see that $I = J^{\ell}$ and $G = H^{\ell}$ in the terminology above.

The following result is useful.

Proposition

([C, pp. 104-105] and [Bo, p.46]) Let k and ℓ be fields with $k \subseteq \ell$, and let $G \subseteq GL(n,\ell)$ be an affine algebraic group defined over k. Then

1) $G^o \subseteq GL(n,\ell)$ is also an algebraic group defined over k.

2) $[G \cap GL(n,k)]^o = G^o \cap GL(n,k)$.

Warning

Let J be an ideal in ℓ[M(n,ℓ)] that is generated by polynomials $\{p_1, p_2, ... p_N\}$ in k[M(n,k)], and let $X = \{g \in M(n,\ell) : p(g) = 0$ for all $p \in J\}$. If I is the ideal of all polynomials in ℓ[M(n,ℓ)] that vanish on the Zariski closed set X, then in general I is strictly larger than J and may not be generated by finitely many polynomials in k[M(n,k)].

However, if k has characteristic zero, then I is generated by finitely many polynomials in k[M(n,k)] whenever J has this property (cf. [Hu 2, p. 217]). Hence we obtain the following

Proposition

Let k and ℓ be fields with $k \subseteq \ell$, and suppose that k has characteristic zero. Let $G \subseteq GL(n,\ell)$ be an affine algebraic group such that $G = \{g \in GL(n,\ell) : p_i(g) = 0$ for $1 \le i \le N\}$, where $\{p_1, p_2, ... p_N\}$ are polynomials in k[M(n,k)]. Then G is defined over k.

Algebraic Lie algebras

(cf. [C, pp. 171-185] and [Bo, pp. pp. 105-110])

Following Chevalley, a Lie subalgebra $\mathfrak{G}$ of M(n,k) will be called <u>algebraic</u> if there exists an affine algebraic group G in GL(n,k) such that $L(G) = \mathfrak{G}$. Not every Lie subalgebra of M(n,k) is algebraic, but if k has characteristic zero (e.g. $k \subseteq \mathbb{C}$), then Chevalley has developed a theory to identify many algebraic subalgebras of M(n,k). In particular one has the following

Theorem ([C, p. 177] and [Bo, p. 109]) Let k be a field of characteristic zero, and let $\mathfrak{G}$ be any Lie subalgebra of M(n,k). Then the commutator subalgebra $[\mathfrak{G},\mathfrak{G}]$ generated by $\{[X, Y] : X, Y \in \mathfrak{G}\}$ is an algebraic subalgebra of M(n,k).

We shall prove this result later in the appendix, except for one lemma that we quote without proof (see example 6 below). Observe that if $\mathfrak{G}$ is a simple subalgebra of M(n,k), then $\mathfrak{G}$ must equal $[\mathfrak{G},\mathfrak{G}]$ since $\mathfrak{G}$ is nonabelian and contains no nontrivial ideals. Hence, if $\mathfrak{G}$ is semisimple (the direct sum of simple subalgebras $\mathfrak{G}_i$), then $[\mathfrak{G},\mathfrak{G}]$ is the direct sum of the subalgebras $[\mathfrak{G}_i, \mathfrak{G}_i] = \mathfrak{G}_i$. Hence $\mathfrak{G} = [\mathfrak{G}, \mathfrak{G}]$, and by the theorem above we obtain the following

Corollary Let k be a field of characteristic zero, and let $\mathfrak{G}$ be any semisimple Lie subalgebra of M(n,k). Then $\mathfrak{G}$ is an algebraic subalgebra of M(n,k).

Section 2 Examples

Let k be a field. We present some basic examples of affine algebraic groups G in GL(n,k) and their Lie algebras L(G) in M(n,k) that will be useful to us. For the convenience of the reader we include some short proofs but omit calculations of the Lie algebras.

In the remainder of this appendix we shall drop the word affine and use only the expression algebraic group, with the understanding that we always mean affine algebraic group.

Example 1

Let G = SL(n,k), the n x n matrices with entries in k whose determinant is 1. The determinant function is a polynomial function on k^n so SL(n,k) is the locus of zeros of p = det −1. L(G) = $s\ell$(n,k), the n x n matrices with entries in k whose trace is zero.

Example 2

Let $\mathfrak{G}$ be a semisimple subalgebra of M(n,$\mathbb{C}$), and let G = Aut ($\mathfrak{G}$), the group of Lie algebra automorphisms of $\mathfrak{G}$. Then G is an algebraic subgroup of GL(n,$\mathbb{C}$) defined over $\mathbb{Q}$. L(G) = Der ($\mathfrak{G}$), the derivations of $\mathfrak{G}$.

Proof

Let $\{X_1, X_2, \ldots X_n\}$ be a basis of $\mathfrak{G}$ with structure constants in $\mathbb{Q}$; that is $[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k$, where $\{C_{ij}^k\} \subseteq \mathbb{Q}$. For example, we may take a Chevalley basis whose structure constants actually lie in $\mathbb{Z}$. Given $g \in$ GL(n,$\mathbb{C}$) we consider the matrix A = (A_{ij}) defined by $g(X_i) = \sum_{j=1}^{n} A_{ji} X_j$. A routine computation shows that g lies in Aut ($\mathfrak{G}$) $\Leftrightarrow P_t(A) = 0$ for $1 \le t \le n$, where $P_t(z_{ij})$ is a quadratic polynomial with rational coefficients in the coordinates $\{z_{ij}\}$ of $\mathbb{Q}^{n^2}$ = M(n,$\mathbb{Q}$) given by $P_t(z_{ij}) = \sum_{r\ s=1}^{n} C_{rs}^t z_{ri} z_{sj} - \sum_{k=1}^{n} C_{ij}^k z_{tk}$. Since $\mathbb{Q}$ has characteristic zero and G is defined by the vanishing of the polynomials $\{p_1, p_2, \ldots p_n\}$ in $\mathbb{Q}$[M(n,$\mathbb{Q}$)] it follows from the discussion in section 1 that G is defined over $\mathbb{Q}$. □

Example 3

Let G_o be a compact, connected, semisimple Lie group, and let Ad : $G_o \to GL(\mathfrak{G}_o)$ denote the adjoint representation, where $\mathfrak{G}_o$ is the Lie algebra of G. Then G = $Ad(G_o)$ is an algebraic subgroup of $GL(n,\mathbb{R})$ defined over $\mathbb{Q}$. L(G) = Der $(\mathfrak{G}_o)$, the derivations of $\mathfrak{G}_o$.

Proof

Identify $GL(\mathfrak{G}_o)$ with $GL(n,\mathbb{R})$, n = dim G, by choosing a basis $\{X_1, X_2, \ldots X_n\}$ of $\mathfrak{G}_o$ with structure constants in $\mathbb{Q}$, for example a real Chevalley basis $\mathfrak{C}_o$ as defined in section 1.5. The argument used in example 2 shows that H = Aut $(\mathfrak{G}_o)$ is an algebraic subgroup of $GL(n,\mathbb{R})$ defined over $\mathbb{Q}$. Since G_o is semisimple, $Ad(G_o)$ is the identity component H^o of the Lie group H (cf. [He, p.122]). By the discussion in section 1, H^o is also the unique irreducible algebraic subgroup of H that contains the identity. Since H is defined over $\mathbb{Q}$ this is also true for H^o by the discussion in section 1. □

Example 4

Let k be any field and let $X \in M(n,k)$ be any nonzero vector. Let $G_X = \{g \in GL(n,k) : Ad(g)X = X\}$, where $Ad(g)Y = gYg^{-1}$ for $Y \in M(n,k)$ and $g \in GL(n,k)$. Then G_X is an algebraic group and $L(G_X) = \{Y \in M(n,k) : [Y,X] = 0\}$.

Proof

By definition $G_X = \{g \in G : gX = Xg\} = G \cap \{g \in GL(n,k) : p_{ij}(g) = 0$ for all $1 \le i,j \le n\}$, where $p_{ij}(z) = \sum_{k=1}^{n} \{X_{kj}\, z_{ik} - X_{ik}\, z_{kj}\}$, a polynomial of degree 1 in the variables $\{z_{\alpha\beta}\}$. □

Example 5 (cf. [C, pp. 172-173] and [Bo, p.107]).

Let V and W be subspaces of M(n,k) with $W \subseteq V$. Let $G = \{g \in GL(n,k) : Ad(g)V = V, Ad(g)W = W$ and $(Ad(g) - Id)(V) \subseteq W\}$. Then G is an algebraic group and $L(G) = \{Y \in M(n,k) : ad(Y)(V) \subseteq W\}$, where $ad(Y)X = [Y,X]$ for $X,Y \in M(n,k)$.

In particular, if V = W, then $G_V = \{g \in GL(n,k) : Ad(g)V = V\}$ is an algebraic group with $L(G_V) = \{Y \in M(n,k) : ad(Y)(V) \subseteq V\}$.

Proof

Let $\{\xi_1, \ldots, \xi_{n^2}\}$ be a basis for M(n,k) such that $\{\xi_1, \ldots, \xi_p\}$ is a basis for W and $\{\xi_1, \ldots, \xi_{p+q}\}$ is a basis for V. Let $\{\psi_1, \ldots, \psi_{n^2}\}$ be the dual basis in Hom (M(n,k),k). Let $\{E_{\alpha\beta} : 1 \le \alpha, \beta \le n^2\}$ be the standard basis of M(n,k) whose dual basis is given by the coordinate functions $\{z_{\alpha\beta} : 1 \le \alpha, \beta \le n^2\}$ in $M(n,k) = k^{n^2}$. Since $\{\xi_1, \ldots, \xi_{n^2}\} \subseteq M(n,k)$ we may write

(1) $\xi_j = \sum_{\alpha\, \beta=1}^{n} B^j_{\alpha\beta}\, E_{\alpha\beta}$, $\{B^j_{\alpha\beta}\} \subseteq k$

$$\psi_j = \sum_{\alpha\,\beta=1}^{n} A^j_{\alpha\beta}\, z_{\alpha\beta}\ , \qquad \{A^j_{\alpha\beta}\} \subseteq k$$

The second equation says that each ψ_j lies in k[M(n,k)], the polynomials in $\{z_{\alpha\beta}\}$ with coefficients in k.

If det $\in$ k[z] denotes the determinant function, then SL(n,k) = {g $\in$ GL(n,k) ; d(g) = 0}, where d = det − 1 $\in$ k[z]. For g $\in$ GL(n,k) let $g^{\alpha\beta}$ denote $z_{\alpha\beta}(g^{-1})$. Then $g^{\alpha\beta} = P_{\alpha\beta}(g)$ for some polynomial $P_{\alpha\beta} \in k[z]$. From (1) we obtain

(2) $(g\,\xi_r\,g^{-1})_{\alpha\beta} = z_{\alpha\beta}(g\,\xi_r\,g^{-1}) = F^r_{\alpha\beta}(g)$

where g $\in$ GL(n,k) and $F^r_{\alpha\beta} \in k[z]$ for $1 \le r \le n^2$, $1 \le \alpha, \beta \le n^2$.

We obtain from (2), since the composition of two polynomial functions is a polynomial function,

(3) $\psi_i(g\,\xi_r\,g^{-1}) = P_{ir}(g)$

where $P_{ir} \in k[z]$ for $1 \le i \le n^2$, $1 \le r \le n^2$.

For g $\in$ GL(n,k), Ad(g)W = W $\Leftrightarrow$ $\psi_i(g\,\xi_r\,g^{-1}) = 0$ for $p+1 \le i \le n^2$, $1 \le r \le p$. $\Leftrightarrow$ $P_{ir}(g) = 0$ for $p+1 \le i \le n^2$, $1 \le r \le p$. Arguing in similar fashion we obtain

Ad(g)W = W $\Leftrightarrow$ $P_{ir}(g) = 0$ for $p+1 \le i \le n^2$, $1 \le r \le p$.

(4) Ad(g)V = V $\Leftrightarrow$ $P_{ir}(g) = 0$ for $p+q+1 \le i \le n^2$, $1 \le r \le p+q$.

(Ad(g) − Id)V $\subseteq$ W $\Leftrightarrow$ $P_{ir}(g) - \psi_i(\xi_r) = 0$ for $p+1 \le i \le n^2$, $1 \le r \le p+q$.

Since $\psi_i(\xi_r) \in k$ it follows that G = {g $\in$ GL(n,k) : Ad(g)V = V, Ad(g)W = W and (Ad(g) − Id)(V) $\subseteq$ W} is an algebraic group.□

Example 6

Let k be a field of characteristic zero, and let G $\subseteq$ GL(n,k) be a connected algebraic group. Then [G,G], the subgroup of G generated by $\{[g,h] = ghg^{-1}h^{-1} : g,h \in G\}$ is an algebraic group, and L([G,G]) = [L(G), L(G)], the subalgebra of L(G) generated by {[X,Y] : X,Y $\in$ L(G)}.

Proof

The proof is less elementary than the proofs above, but may be found, for example, in Proposition 7.8 of [Bo, p. 108].□

Section 3 More on algebraic Lie algebras

In this section we prove the following theorem, which was previously stated in the discussion in section 1 of algebraic Lie algebras.

Theorem Let k be a field of characteristic zero, and let $\mathfrak{G}$ be any Lie subalgebra of M(n,k). Then the commutator subalgebra $[\mathfrak{G},\mathfrak{G}]$ generated by $\{[X, Y] : X, Y \in \mathfrak{G}\}$ is an algebraic subalgebra of M(n,k).

Proof

Let k be a field of characteristic zero, and let M be any subset of M(n,k). Let $\mathcal{A}$(M) denote the intersection of all algebraic Lie algebras that contain M. It is known that $\mathcal{A}$(M) is an algebraic Lie algebra L(G), where G is the intersection of all algebraic subgroups H of GL(n,k) such that M $\subseteq$ L(H). See [Bo, p. 105].

Let V and W be subspaces of M(n,k) with W $\subseteq$ V, and let tr (V,W) = {X $\in$ M(n,k) : ad X(V) $\subseteq$ W}. Then tr (V,W) is an algebraic Lie algebra by example 5 above. If $\mathfrak{G}$ is any Lie subalgebra of M(n,k), then by definition $\mathfrak{G} \subseteq$ tr ($\mathfrak{G}$, [$\mathfrak{G}$,$\mathfrak{G}$]). Since tr ($\mathfrak{G}$, [$\mathfrak{G}$,$\mathfrak{G}$]) is an algebraic Lie algebra it follows from the definition of $\mathcal{A}(\mathfrak{G})$ above that $\mathcal{A}(\mathfrak{G}) \subseteq$ tr ($\mathfrak{G}$, [$\mathfrak{G}$,$\mathfrak{G}$]) or equivalently that [$\mathcal{A}(\mathfrak{G})$, $\mathfrak{G}$] $\subseteq$ [$\mathfrak{G}$,$\mathfrak{G}$]. Since $\mathfrak{G} \subseteq$ tr ($\mathcal{A}(\mathfrak{G})$, [$\mathfrak{G}$,$\mathfrak{G}$]) the same argument implies that $\mathcal{A}(\mathfrak{G}) \subseteq$ tr ($\mathcal{A}(\mathfrak{G})$, [$\mathfrak{G}$,$\mathfrak{G}$]) or equivalently that [$\mathcal{A}(\mathfrak{G})$, $\mathcal{A}(\mathfrak{G})$] $\subseteq$ [$\mathfrak{G}$,$\mathfrak{G}$]. It follows that [$\mathfrak{G}$,$\mathfrak{G}$] = [$\mathcal{A}(\mathfrak{G})$, $\mathcal{A}(\mathfrak{G})$] since $\mathfrak{G} \subseteq \mathcal{A}(\mathfrak{G})$, and [$\mathcal{A}(\mathfrak{G})$, $\mathcal{A}(\mathfrak{G})$] is algebraic by example 6 since $\mathcal{A}(\mathfrak{G})$ is algebraic. □

Section 4 The Lie algebra of an algebraic group

Left and right invariant derivations of k[M(n,k)]

Let k be any field and let k[M(n,k)] denote the k-algebra of polynomial functions from M(n,k) to k. An element x of M(n,k) defines a left multiplication λ(x) : k[M(n,k)] $\rightarrow$ k[M(n,k)] and a right multiplication ρ(x) : k[M(n,k)] $\rightarrow$ k[M(n,k)] by [λ(x)f](y) = f(x^{-1}y) and [ρ(x)f](y) = f(yx) for any f $\in$ k[M(n,k)] and any y $\in$ M(n,k). It is routine to check that the maps {λ(x), ρ(x) : x $\in$ M(n,k)} are k-algebra isomorphisms of k[M(n,k)] for all x $\in$ GL(n,k) and λ(xy) = λ(x) o λ(y) , ρ(xy) = ρ(x) o ρ(y) for all x,y in M(n,k).

A k-linear map D : k[M(n,k)] $\rightarrow$ k[M(n,k)] is a <u>derivation</u> if D(fg) = D(f) g + f D(g) for all f,g in k[M(n,k)]. A derivation D is <u>left invariant</u> (respectively <u>right invariant</u>) if D o λ(x) = λ(x) o D (respectively D o ρ(x) = ρ(x) o D) for all x $\in$ M(n,k). Each element x of M(n,k) defines a left invariant derivation L_x : k[M(n,k)] $\rightarrow$ k[M(n,k)], which is uniquely determined by its value on linear functions u : M(n,k) $\rightarrow$ k given by (L_xu)(y) = u(yx) for all y $\in$ M(n,k). Similarly one defines a right invariant derivation R_x : k[M(n,k)] $\rightarrow$ k[M(n,k)], which is uniquely determined by its value on linear functions u by defining (R_xu)(y) = $-$u(xy) for all y $\in$ M(n,k).

It is straightforward to verify that $L_x \circ L_y - L_y \circ L_x = L_{[x,y]}$ and $R_x \circ R_y - R_y \circ R_x = R_{[x,y]}$ for all x,y in M(n,k).

Left and right invariant derivations of k[G]

Let G $\subseteq$ GL(n,k) be an algebraic group, and let I $\subseteq$ k[M(n,k)] be the ideal of polynomials that vanish on G. A <u>polynomial on G</u> is the restriction to G of an element of k[M(n,k)]. The set k[G] of polynomials on G is a k-algebra and the restriction map r : k[M(n,k)] $\to$ k[G] is a surjective k-algebra homomorphism whose kernel is I. Note that if $f \in I$ and $x \in G$, then $\lambda(x)f \in I$ and $\rho(x)f \in I$.

If x is an element of G, then the left and right multiplications $\lambda(x)$ and $\rho(x)$: k[M(n,k)] $\to$ k[M(n,k)] induce corresponding left and right multiplications $\bar{\lambda}(x)$ and $\bar{\rho}(x)$: k[G] $\to$ k[G] such that $r \circ \lambda(x) = \bar{\lambda}(x) \circ r$ and $r \circ \rho(x) = \bar{\rho}(x) \circ r$. A derivation of k[G] is a k-linear map $\bar{D}$: k[G] $\to$ k[G] such that $\bar{D}(fg) = \bar{D}(f)\, g + f\, \bar{D}(g)$ for all f, g in k[G]. A derivation $\bar{D}$ of k[G[is left invariant (respectively right invariant) if $\bar{D} \circ \bar{\lambda}(x) = \bar{\lambda}(x) \circ \bar{D}$ (respectively $\bar{D} \circ \bar{\rho}(x) = \bar{\rho}(x) \circ \bar{D}$) for all x in G. The left invariant (respectively right invariant) derivations of k[G] form a k-Lie algebra.

Definition of the Lie algebra of an algebraic group

Let $G \subseteq GL(n,k)$ be an algebraic group, and let $I \subseteq k[M(n,k)]$ be the ideal of polynomials that vanish on G. Let $L(G) = \{x \in M(n,k) : L_x(I) \subseteq I\}$, where L_x is the left invariant derivation of k[M(n,k)] determined by x as defined above.

L(G) is called the Lie algebra of the algebraic group G.

Remarks

1) From the discussion above it is easy to check that L(G) is a Lie subalgebra of M(n,k).

2) If k is a subfield of ℓ and $I^{\ell} \subseteq \ell[M(n,\ell)]$ and $G^{\ell} \subseteq GL(n,\ell)$ are defined as in section 1, then it is routine to verify from the definitions that $L(G^{\ell})$ consists of all ℓ-linear combinations of elements of L(G), or equivalently that $L(G^{\ell}) = L(G)\otimes\ell$, the tensor product over k.

3) One may also give an equivalent intrinsic definition of L(G). If x is an element of L(G), then the left invariant derivation L_x of k[M(n,k)] defined above induces a left invariant derivation $\bar{L}_x$ of k[G] such that $r \circ L_x = \bar{L}_x \circ r$, where r : k[M(n,k)] $\to$ k[G] is the restriction homomorphism. Conversely, if $\bar{L}$: k[G] $\to$ k[G] is a left invariant derivation of k[G], then it is not difficult to show that $\bar{L} = \bar{L}_x$ for a unique element x of L(G). The map $x \to \bar{L}_x$ is a k-Lie algebra isomorphism of L(G) onto the k-Lie algebra of left invariant derivations of k[G].

Section 5 A density theorem

Preliminaries

Before beginning, we review some notation and basic facts from section 1 that we need. Let $H \subseteq GL(n,\mathbb{C})$ be a semisimple algebraic group defined over $\mathbb{Q}$, and let H^o be the irreducible algebraic subgroup of H that contains the identity. Let $H_{\mathbb{Q}} = H \cap GL(n,\mathbb{Q})$ and $H_{\mathbb{R}} = H \cap GL(n,\mathbb{R})$. The groups $H_{\mathbb{Q}}$ and $H_{\mathbb{R}}$ are algebraic groups in $GL(n,\mathbb{Q})$ and

$GL(n,\mathbb{R})$ respectively, and we let $H^o_{\mathbb{Q}}$ and $H^o_{\mathbb{R}}$ denote the corresponding irreducible algebraic subgroups that contain the identity. By the discussion in section 1, $H^o_{\mathbb{Q}} = H^o \cap GL(n,\mathbb{Q})$ and $H^o_{\mathbb{R}} = H^o \cap GL(n,\mathbb{R})$.

Theorem [S, Cor.3.5 (iii)]

Let $H \subseteq GL(n, \mathbb{C})$ be a connected algebraic group defined over $\mathbb{Q}$. Then $H^o_{\mathbb{Q}}$ is dense in $H^o_{\mathbb{R}}$ in the Lie group topology of $H^o_{\mathbb{R}}$.

An elementary proof of this result seems to be difficult to find in the literature. We provide an elementary proof below in the case that H is semisimple, which is sufficient for our purposes. The theorem in the semisimple case is a consequence of the Proposition below, whose proof will be carried out in several steps. We wish to thank S. Kumar and G. Prasad, who explained and outlined the proof of Lemma 1 in the next result.

Proposition

Let $H \subseteq GL(n,\mathbb{C})$ be a semisimple algebraic group defined over $\mathbb{Q}$ with identity component H^o. If $K = \overline{H_{\mathbb{Q}}}$, the closure of $H_{\mathbb{Q}}$ in the Lie topology of $H_{\mathbb{R}}$, then $H^o_{\mathbb{R}} = K^o$, the identity component of K regarded as a Lie group.

We assume this result for the moment and use it to prove that $H^o_{\mathbb{Q}}$ is dense in the Lie topology of $H^o_{\mathbb{R}}$, thereby completing the proof of the Theorem stated above in the case that H is semisimple. Let $\xi \in H^o_{\mathbb{R}}$ be given. Since $(\overline{H_{\mathbb{Q}}})^o = K^o = H^o_{\mathbb{R}}$ we can find a sequence $\{\xi_k\} \subseteq H_{\mathbb{Q}}$ such that $\xi_k \to \xi \in H^o_{\mathbb{R}} \subseteq H_{\mathbb{R}}$. Hence $\varepsilon_k = \xi^{-1}\xi_k \to 1$ in $H_{\mathbb{R}}$, and we conclude that $\varepsilon_k \in H^o_{\mathbb{R}}$ for all $k \geq k_o$, where k_o is some positive integer. Hence $\xi_k = \xi\varepsilon_k \in H^o_{\mathbb{R}} \cap H_{\mathbb{Q}}$ for $k \geq k_o$, which proves that $H^o_{\mathbb{R}} \subseteq \overline{H^o_{\mathbb{R}} \cap H_{\mathbb{Q}}} = \overline{H^o_{\mathbb{Q}}}$ since $\xi \in H^o_{\mathbb{R}}$ was arbitrary. The reverse inclusion is obvious.□

Proof of the Proposition

Lemma 1

Let $K = \overline{H_{\mathbb{Q}}}$, the closure of $H_{\mathbb{Q}}$ in the Lie topology of $H_{\mathbb{R}}$. Then K^o is normalized by $H^o_{\mathbb{R}}$.

Proof

We break the proof into steps. We adopt the conventions that Cl (X) and $\overline{X}$ denote the closure of a set X in the Zariski and Lie topologies of $GL(n,\mathbb{R})$ respectively.

Step 1

The group $H_{\mathbb{Q}}$ is dense in $H_{\mathbb{R}} \subseteq GL(n,\mathbb{R})$ in the Zariski topology of $GL(n,\mathbb{R})$.

Proof

See Corollary 18.3 on [Bo, p.220].□

Step 2

$H_{\mathbb{Q}} \cap SL(n,\mathbb{R})$ is dense in $H \cap SL(n,\mathbb{R}) = H_{\mathbb{R}} \cap SL(n,\mathbb{R})$ in the Zariski topology of $GL(n,\mathbb{R})$.

Proof

If $A \in H_{\mathbb{Q}} \subseteq H_{\mathbb{R}}$, then $A^n \in H_{\mathbb{R}}^o$ for some positive integer n since $H_{\mathbb{R}}^o$ has finite index in $H_{\mathbb{R}}$. Note that $H_{\mathbb{R}}^o \subseteq SL(n,\mathbb{R})$ since H is semisimple. Hence $1 = \det(A^n) = (\det A)^n$, which implies that $\det A = \pm 1$ since $\det A \in \mathbb{Q}$. Hence we may write

(1) $H_{\mathbb{Q}} = H_{\mathbb{Q}}^+ \cup H_{\mathbb{Q}}^-$, where $H_{\mathbb{Q}}^+ = H_{\mathbb{Q}} \cap SL(n,\mathbb{R})$, $H_{\mathbb{Q}}^- = H_{\mathbb{Q}} \cap SL^-(n,\mathbb{R})$ and $SL^-(n,\mathbb{R}) = \{g \in GL(n,\mathbb{R}) : \det(g) = -1\}$.

We obtain

(2) $H_{\mathbb{R}} = Cl(H_{\mathbb{Q}}) = Cl(H_{\mathbb{Q}}^+) \cup Cl(H_{\mathbb{Q}}^-) \subseteq \{H_{\mathbb{R}} \cap SL(n,\mathbb{R})\} \cup \{H_{\mathbb{R}} \cap SL^-(n,\mathbb{R})\}$

However, the argument used to prove (1) also proves

(3) $H_{\mathbb{R}} = \{H_{\mathbb{R}} \cap SL(n,\mathbb{R})\} \cup \{H_{\mathbb{R}} \cap SL^-(n,\mathbb{R})\}$

Combining (2) and (3), all the inclusions in (2) become equalities, and in particular, $Cl(H_{\mathbb{Q}}^+) = \{H_{\mathbb{R}} \cap SL(n,\mathbb{R})\}$, which is the statement of Step 2.

Step 3 (completion of the proof)

Let $\mathfrak{K}$ denote the Lie algebra of $K = \overline{H_{\mathbb{Q}}} \subseteq H_{\mathbb{R}} \subseteq GL(n,\mathbb{R})$. Then from (1) we obtain $\overline{H_{\mathbb{Q}}^+} \subseteq \overline{H_{\mathbb{Q}}} \cap SL(n,\mathbb{R}) = K \cap SL(n,\mathbb{R}) \subseteq N = \{g \in SL(n,\mathbb{R}) : g K^o g^{-1} \subseteq K^o\} = \{g \in SL(n,\mathbb{R}) : g\, \mathfrak{K}\, g^{-1} \subseteq \mathfrak{K}\}$. Since $\mathfrak{K}$ is a subspace of $M(n,\mathbb{R})$ it follows from example 5 in section 2 that N is an algebraic subgroup of $GL(n,\mathbb{R})$. The group $H_{\mathbb{R}}$ is clearly an algebraic subgroup of $GL(n,\mathbb{R})$, and hence $H_{\mathbb{R}} \cap N$ is an algebraic subgroup of $GL(n,\mathbb{R})$.

The discussion above shows that $H_{\mathbb{Q}}^+ \subseteq H_{\mathbb{R}} \cap N \subseteq H_{\mathbb{R}} \cap SL(n,\mathbb{R})$, and $H_{\mathbb{R}} \cap N$ is Zariski closed in $GL(n,\mathbb{R})$. By Step 2 $H_{\mathbb{Q}}^+ = H_{\mathbb{Q}} \cap SL(n,\mathbb{R})$ is dense in $H_{\mathbb{R}} \cap SL(n,\mathbb{R})$ in the Zariski topology of $GL(n,\mathbb{R})$, and we conclude that $H_{\mathbb{R}} \cap N = H_{\mathbb{R}} \cap SL(n,\mathbb{R})$. Finally, $H_{\mathbb{R}}^o \subseteq H_{\mathbb{R}} \cap SL(n,\mathbb{R}) = H_{\mathbb{R}} \cap N \subseteq N$. It follows that $H_{\mathbb{R}}^o$ normalizes K^o by the definition of N. The proof of Lemma 1 is complete. □

We shall need a further intermediate result.

Lemma 2

$Cl(H_{\mathbb{Q}} \cap K^o) = H_{\mathbb{R}}^o$, where Cl denotes closure in the Zariski topology of $GL(n,\mathbb{R})$.

Proof

Since K^o has finite index in K it follows that $H_{\mathbb{Q}} \cap K^o$ has finite index in $H_{\mathbb{Q}} \cap K = H_{\mathbb{Q}}$, and hence $Cl(H_{\mathbb{Q}} \cap K^o)$ has finite index in $Cl(H_{\mathbb{Q}}) = H_{\mathbb{R}}$ by Step 1 of Lemma

1. Let $\{h_1, h_2, \ldots h_p\}$ be elements of $H_{\mathbb{R}}$ such that $H_{\mathbb{R}} = \bigcup_{i=1}^{p} h_i A$, where $A = Cl(H_{\mathbb{Q}} \cap K^o)$. Hence $H^o_{\mathbb{R}}$ is the disjoint union of the Zariski closed subsets $(h_i A) \cap H^o_{\mathbb{R}}$, $1 \le i \le p\}$. Since $H^o_{\mathbb{R}}$ is connected and the identity lies in $A \cap H^o_{\mathbb{R}}$ it follows that $H^o_{\mathbb{R}} = A \cap H^o_{\mathbb{R}}$ or $H^o_{\mathbb{R}} \subseteq A = Cl(H_{\mathbb{Q}} \cap K^o)$. The reverse inclusion follows since $H_{\mathbb{Q}} \cap K^o \subseteq K^o \subseteq H^o_{\mathbb{R}}$, and $H^o_{\mathbb{R}}$ is a Zariski closed subset of $GL(n,\mathbb{R})$. □

Proof of the Proposition

We now complete the proof of the proposition stated at the beginning of this section. By Lemma 1 we know that K^o is a normal subgroup of $H^o_{\mathbb{R}}$. If $\mathfrak{H}$ and $\mathfrak{K}$ denote the Lie algebras of $H^o_{\mathbb{R}}$ and K^o, then by the semisimplicity of $\mathfrak{H}$ it follows that $\mathfrak{H}$ is a direct sum $\mathfrak{K} \oplus \mathfrak{K}^{\perp}$ of commuting ideals (cf. Corollary 6.3 of [He,p.121]).

It suffices to show that $\mathfrak{K}^{\perp}$ lies in the center of $\mathfrak{H}$, which will show that $\mathfrak{K}^{\perp} = \{0\}$ and $K^o = H^o_{\mathbb{R}}$ since $\mathfrak{H}$ is semisimple. Given an arbitrary element X of $\mathfrak{K}^{\perp}$ it follows that $Ad(e^Y)(X) = e^{ad(Y)}(X) = X$ for all Y in $\mathfrak{K}$. Hence $Ad(g)(X) = X$ for all g in K^o. By the discussion in section 1 and example 4 in section 2 it follows that $H_X = \{g \in H_{\mathbb{R}} : Ad(g)(X) = X\}$ and $H_X{}^o$ are algebraic subgroups of $GL(n,\mathbb{R})$. Furthermore $H_{\mathbb{Q}} \cap K^o \subseteq K^o \subseteq H_X{}^o \subseteq H^o_{\mathbb{R}}$. Since $H_{\mathbb{Q}} \cap K^o$ is Zariski dense in $H^o_{\mathbb{R}}$ by Lemma 2, it follows that $H_X{}^o = H^o_{\mathbb{R}}$. We conclude that X lies in the center of $\mathfrak{H}$ for all X in $\mathfrak{K}^{\perp}$. □

References

[Bo] A. Borel. Linear Algebraic Groups, Graduate Texts in Mathematics, Number 126, Springer, New York, 1991.

[B-tD] T. Bröcker and T. tom Dieck, Representations of Compact Lie Groups, Graduate Texts in Mathematics, Number 98, Springer, Heidelberg, 1985.

[C] C. Chevalley, Théorie des Groupes de Lie : Groupes algébriques, théoremes généraux sur les algebres de Lie, Hermann, Paris, 1968

[CSM] R. Carter, I. Macdonald and G. Segal, Lectures on Lie groups and Lie Algebras, London Mathematical Society Student Texts, vol. 32, Cambridge University Press, Cambridge, 1995.

[E] P. Eberlein " Rational approximation in compact Lie groups and their Lie algebras ", preprint 1999.

[He] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, New York, 1978.

[Hu 1] J. Humphreys, Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics, Number 9, Springer, Heidelberg, 1972.

[Hu 2] -------, Linear Algebraic Groups , Graduate Texts in Mathematics, Number 21, Springer, New York, 1981.

[J] N. Jacobson, Lie algebras, Interscience , 1962, and Dover, 1979.

[PR] V. Platonov and A. Rapinchuk, Algebraic groups and Number Theory, Pure and Applied Mathematics, vol. 139, Academic Press, Boston, 1991.

[S] J. J. Sansuc, " Groupe de Brauer et arithmétique des groupes algébriques linéaires sur un corps de nombres ", J. Reine Angew. Math. 327 (1981), 12-80.