Noncommutative Microlocal Analysis
Part I

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Revised Version

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Abstract. Pseudodifferential operators on $\mathbb{R}^n$ in $OPS^m_{\rho,0}$ are built out of smooth families of convolution operators on $\mathbb{R}^n$. Similarly important classes of operators can be built out of smooth families of convolution operators on a noncommutative Lie group G. When the representation theory and harmonic analysis on G are well understood, one can construct a noncommutative symbol calculus. This paper develops some aspects of the resulting noncommutative microlocal analysis. Chapter I treats operators on general Lie groups. The details of the symbol calculus depend on the particular representation theory of the group G, and such a theory is worked out for the Heisenberg group in Chapter II. In Chapter III this theory is applied to a systematic study of operator classes on contact manifolds, including parametrices for naturally occurring subelliptic operators, heat asymptotics, and a study of the Szegö projectors.

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Noncommutative Microlocal Analysis, Part I

Introduction

The theory of pseudodifferential operators has provided a very powerful and flexible tool for treating problems in linear partial differential equations. For example, it provides a framework for "pasting together" parametrices for "frozen" constant coefficient operators (obtained by Fourier analysis on $\mathbb{R}^n$) to produce parametrices for general variable coefficient elliptic equations. Using the theory of Fourier integral operators, one can impose a translation symmetry on operators with simple characteristics and real principal symbol and obtain rich information on such subjects as strictly hyperbolic equations.

In dealing with operators with multiple characteristics, in cases where the geometry of the characteristics can be put in standard form via a homogeneous canonical transformation, one is often led naturally to a problem in noncommutative harmonic analysis. The connection between the Kohn Laplacian and analysis on the Heisenberg group was exploited by Folland and Stein [F4]. More general hypoelliptic operators were studied via analysis on more general nilpotent Lie groups by Rothschild and Stein [R5]. These papers and numerous subsequent ones developed operator calculi for certain classes of pseudodifferential operators. However, they do not exploit harmonic analysis on a nilpotent Lie group in the same way that the most popular approach to pseudodifferential operators on $\mathbb{R}^n$ does, by the group Fourier transform and associated symbol calculus. Some aspects of such a symbol calculus were presented by Dynin in [D2] and [D3], which gave considerable inspiration to this paper.

The sort of operators we deal with arise on a Lie group G from a smooth family of convolution operators on G in the same fashion that classical pseudodifferential operators on $\mathbb{R}^n$ arise from a smooth family of Fourier multipliers (convolution operators) on $\mathbb{R}^n$. In the first section of Chapter I, we develop a general study of such operators and the calculus: adjoints, products, pseudo-locality, etc. This theory proceeds along the lines of what is now a standard treatment of pseudodifferential operators on $\mathbb{R}^n$ with symbols in $S^m_{\rho,0}$ ($\delta = 0$). The symbol of such an operator is defined on $G \times \widehat{G}$, where $\widehat{G}$ is the set of equivalence classes of irreducible representations of G; the symbol takes values as operators on the associated representation spaces. The first chapter presents much of its results in a general framework, though it also focuses on some phenomena special to nilpotent Lie groups with dilations, and occasionally specializes to two-step nilpotent Lie groups.

The second chapter develops the tools to implement symbol calculi on the Heisenberg group $\mathbb{H}^n$. Our goal here is to develop harmonic analysis on the Heisenberg group far enough to construct suitably powerful classes of right invariant pseudodifferential operators on $\mathbb{H}^n$ and understand them through their symbols. We describe the basic representations of the Heisenberg group and analyze the image of a convolution operator under such a representation as an operator in the Weyl functional calculus, which has reached a high degree of development in [H10]. Other approaches to harmonic analysis on $\mathbb{H}^n$ have been pursued, particularly by Geller [G2], [G3], [G5], [C3], and others. A more elementary treatment of harmonic analysis on $\mathbb{H}^n$, having some overlap with the one here, is given in the first chapter of the monograph [T5]. We develop a class of pseudodifferential operators on $\mathbb{H}^n$ containing both the convolutors on $\mathbb{H}^n$ with the dilation properties stressed by Folland and Stein and the algebra OPS^0 of classical pseudodifferential operators. Thus there is available classical microlocalization via elements of the algebra.

The methods of Chapter I allow one to apply the theory of Chapter II to the study of classes of pseudodifferential operators on a contact manifold M, including parametrices for hypoelliptic pseudodifferential operators on M doubly characteristic on a certain symplectic variety $\Lambda \subset T^*M \setminus 0$ (defined by the contact structure), satisfying the condition of subellipticity with loss of one derivative, given originally by Sjöstrand [S6], Boutet de Monvel and Treves [B13], and Boutet de Monvel [B7]. The approach here is in straightforward analogy with the construction, via symbol calculus for OPS^m, of parametrices for elliptic operators. Our noncommutative symbols are operator valued, a property in common with part of Sjöstrand's construction in [S6]. In our case, it is apparent that the parametrices obtained are pseudodifferential operators of type $(1/2, 1/2)$; in fact they are obtained in operator classes that are strict sub-classes of Boutet de Monvel's operator classes in [B7]. Next we construct a parametrix for the "heat" equation determined by such a subelliptic operator, in the case when it is a self-adjoint, semibounded, second-order differential operator, in direct analogy to the construction of a parametrix for the heat equation on a Riemannian manifold. Parametrices have been constructed in greater generality by Menikoff and Sjöstrand [M8], and subsequent papers, using harder work; the approach here is simple enough to proceed from the principal term in the trace of the heat kernel to a complete asymptotic expansion. Finally, we show that the symbol calculus developed in Chapter III gives a very straightforward construction of the Szegö projector, for the boundary of a strictly pseudoconvex domain in $\mathbb{C}^n$. The constuction of Boutet de Monvel and Sjöstrand [B12] lends itself to greater generality, but the treatment via noncommutative symbol calculus seems very natural in this context. The fact that Fourier integral operators are avoided has some advantages, e.g., for L^p estimates.

On the other hand, Fourier integral operators greatly add to the flexibility of pseudodifferential operators as a tool, and will probably play a more crucial role in exploiting harmonic analysis on other groups, in situations modeling other types of symplectic geometry than just a contact line bundle in T^*M. It was originally my intention to include more chapters, discussing applications of harmonic analysis on certain other classes of groups, but the present manuscript has grown to a point where a break is necessary. We will not go into the contents of the projected second part, beyond a few references in the text to a fourth chapter. Let us point out the works of Melin [M4] and of Beals and Greiner [B6], which develop cetain operator calculi on spaces more general than contact manifolds. The operator algebras treated there are not amalgamated with the classical pseudodifferential operators, so further work would be required to microlocalize them.

This paper ends with two appendices. The first gives a brief discussion of some aspects of the Weyl calculus. The second develops a more general sort of Weyl calculus and applies it to a proof of a result of Howe [H11] on an isomorphism between a certain algebra of pseudodifferential operators on $\mathbb{R}^n$, which also arises naturally in Chapter II, and an algebra of Toeplitz operators on the unit ball in $\mathbb{C}^n$.

Remarks on the revised version

The original version of this paper was published in 1984 as an AMS Memoir. Since then I have given away all my free copies and most of the batch I purchased, so it seemed to be a good idea to put out a version in TeX, replacing the product of an old-fashioned typewriter.

I have made some minor changes in the course of retyping this paper in TeX. In particular, the introductions to the various chapters have been amplified, in hopes that the reader can obtain a better outline of the results presented therein by perusing these introductions. To the original bibliography I have appended a handful of references to papers that have appeared since 1984 and bear on the subject treated here. I have also taken the opportunity to reorganize some of the material, splitting up several long, rambling sections into shorter, more focused parts. The following table identifies how old sections were split:

Chapter I

$$\S 1 \mapsto \S\S 1-2$$

Chapter II

$$\S 2 \mapsto \S\S 2-3$$
$$\S 3 \mapsto \S\S 4-5$$

Chapter III

$$\S 1 \mapsto \S\S 1-2$$
$$\S 2 \mapsto \S\S 3-4$$

There have not been many mathematical changes. A little further material on the Neumann operator $\square^+$ for the $\overline{\partial}$-Neumann problem has been added in Chapter III, §4, but not much else.

Chapter I. Noncommutative approach to pseudodifferential operators

Here we produce various classes of pseudodifferential operators on Lie groups in a fashion parallel to the way pseudodifferential operators with symbols in $S^m_{\rho,0}$ are produced from convolution operators on $\mathbb{R}^n$. To a group G and a Frechet space $\widehat{\mathfrak{X}}$ of distributions singular only at the identity element of G, we associate a class of operators $OP\widetilde{\mathfrak{X}}$. We discuss products and adjoints of such operators, as operators in $OP\widetilde{\mathfrak{X}}$, under appropriate hypotheses. A great deal of the resulting operator calculus arises smoothly in close analogy to the development of pseudodifferential operators on $\mathbb{R}^n$, for very general classes of Lie groups.

In §2 we specialize to groups with dilations, and consider some special classes of operators tied to these dilations. Operators that arise in Chapters II and III will be of this sort. We consider in particular operator classes

$$OP\mathfrak{H}(G,\alpha,m) \quad \text{and} \quad OP\mathfrak{H}^{m,\mu}_{\alpha,\delta}, \tag{0.1}$$

of convolution operators homogeneous with respect to the family of dilations $\alpha(t)$, and partaking of homogeneity with respect to both $\alpha(t)$ and the Euclidean dilations $\delta(t)$, respectively, and also the "variable coefficient" extensions, $OP\widetilde{\mathfrak{H}}(G,\alpha,m)$ and $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$.

In §3 we associate a symbol to an element $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}$. This will be a function $\sigma_{\mathfrak{K}}(x,\pi)$, defined for $x \in G$ and an irreducible unitary representation π of G, as an operator on the representation space for π (or at least on a certain dense subspace). The discussion of symbols in this chapter is brief, as this aspect of the study is strongly tied to the particular representation theory of the group at hand.

We will use the following definition of the Euclidean Fourier transform on $\mathbb{R}^n$:

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int u(x) e^{-ix\cdot\xi}\, dx. \tag{0.2}$$

Then the Fourier inversion formula is

$$u(x) = (2\pi)^{-n/2} \int \hat{u}(\xi) e^{-x\cdot\xi}\, d\xi. \tag{0.3}$$

The author will be found guilty of lapses in the text regarding factors of powers of 2π, which may be omitted from many formulas.

Another convenient symbolism we will use is

$$\langle\xi\rangle = \left(1+|\xi|^2\right)^{1/2}, \tag{0.4}$$

for $\xi \in \mathbb{R}^n$.

1. Convolution operators and pseudodifferential operators on Lie groups

In this first chapter we show how to build operators out of convolution operators on a Lie group, in a fashion analogous to constructing pseudodifferential operators out of convolution operators (Fourier multipliers) on Euclidean space. For orientation, let us recall the definition of certain classes of pseudodifferential operators on $\mathbb{R}^n$. We set

$$p(x, D) = (2\pi)^{-n/2} \int p(x, \xi)\hat{u}(\xi)e^{ix\cdot\xi}\, d\xi. \tag{1.1}$$

Suppose the amplitude $p(x, \xi)$ belongs to $S^m_{\rho,\delta}$, which is to say

$$|D^\beta_x D^\alpha_\xi p(x, \xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}. \tag{1.2}$$

We say $p(x, D) \in OPS^m_{\rho,\delta}$. In the particular case $\delta = 0$, we can characterize elements of $OPS^m_{\rho,\delta}$ as follows. The space of Fourier multipliers in $S^m_{\rho,0}$:

$$p(D)u = \hat{p} * u = (2\pi)^{-n/2} \int p(\xi)\hat{u}(\xi)e^{ix\cdot\xi}\, d\xi \tag{1.3}$$

forms a Frechet space $S^m_{\rho\#}$, with seminorms

$$[p]_{\alpha,m,\rho} = \sup_\xi \langle\xi\rangle^{-m+\rho|\alpha|}\, |D^\alpha_\xi p(\xi)|. \tag{1.4}$$

Now we can think of $p(x, \xi) \in S^m_{\rho,0}$ as being a smooth family of elements of $S^m_{\rho\#}$:

$$p_x(\xi) = p(x, \xi), \tag{1.5}$$

and if $P(y)$ is the operator of Fourier multiplication by $p_y(\xi)$, we have

$$p(x, D)u(x) = P(x)u(x). \tag{1.6}$$

The operator calculus for such pseudodifferential operators is given as follows. If $p(x, D) \in OPS^m_{\rho,\delta}$ and $q(x, D) \in OPS^\mu_{\rho,\delta}$, then, as long as $0 \le \delta < \rho \le 1$,

$$p(x, D)q(x, D) = r(x, D) \in OPS^{m+\mu}_{\rho,\delta}, \tag{1.7}$$

with

$$r(x, \xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi p(x, \xi)\, D^\alpha_x q(x, \xi). \tag{1.8}$$

Note that the general term in this asymptotic expansion belongs to $S^{m+\mu-(\rho-\delta)|\alpha|}_{\rho,\delta}$. This result is given in Hörmander [H6]; see also the books [K10], [T2], and [T7] on pseudodifferential operators. The result (1.7) is also valid in case $\rho = \delta < 1$, but in this case (1.8) does not give an asymptotic sum, so we do not have a convenient formula for the principal symbol of a product in this case.

Now let G be a Lie group. Since we intend to work locally, we impose a local coordinate system on a neighborhood of the identity element $e \in G$, such that e is the origin 0; we could use exponential coordinates, identifying a neighborhood of $e \in G$ with a neighborhood of 0 in $\mathfrak{g}$, the Lie algebra of G. The Fourier multipliers (1.3) are replaced by convolution operators on G:

$$\begin{aligned} Ku(x) = k * u(x) &= \int_G k(y)u(y^{-1}x)\,dm(y) \\ &= \int_G k(xy)u(y^{-1})\,dm(y) \qquad (1.9) \\ &= \int_G k(xy^{-1})u(y)\Delta(y^{-1})\,dm(y), \end{aligned}$$

where $dm(y)$ stands for left-invariant Haar measure, $\Delta(y)$ the modular function. We suppose $k \in \mathcal{E}'(G)$ is in some Frechet space $\widehat{\mathfrak{X}}$ of distributions that are singular only at the origin. Consequently the Fourier transform $\hat{k}(\xi)$ belongs to $\mathfrak{X}$, a Frechet space contained in $C^\infty(\mathfrak{g}')$, where $\mathfrak{g}'$ denotes the linear dual of $\mathfrak{g}$. We will suppose that $\mathfrak{X} \subset S^m_{\rho\#}$ for some $m \in \mathbb{R},\ \rho \in (0,1]$. We say K belongs to $OP\mathfrak{X}$.

If $\hat{k}(y,\xi) = \int k(y,x)e^{-ix\cdot\xi}\,dx$ is a smooth function of y with values in $\mathfrak{X}$, for y in a neighborhood of $e \in G$, then $K(y)$, defined by

$$K(y)u(x) = \int_G k(y, xz^{-1})u(z)\Delta(z^{-1})\,dm(z), \tag{1.10}$$

is a smooth function of y, taking values in the Frechet space $OP\mathfrak{X}$. We then associate the operator

$$\mathfrak{K}u(x) = K(x)u(x), \tag{1.11}$$

and we say $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}$.

Oe thing we can say about these operators is that they are pseudodifferential operators in the usual sense. Indeed, we have the following simple result.

Proposition 1.1. *If $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}$ with $\mathfrak{X} \subset S^m_{\rho\#}$, then, modulo a smoothing operator,*

$$\mathfrak{K}u(x) = \iint a(x,y,\xi)e^{i(x-y)\cdot\xi}u(y)\,dy\,d\xi, \tag{1.12}$$

for an amplitude $a \in S^m_{\rho,1-\rho}$*, i.e., such that*

$$|D_y^\gamma D_x^\beta D_\xi^\alpha a(x,y,\xi)| \le C_{\alpha\beta\gamma}\langle\xi\rangle^{m-\rho|\alpha|+(1-\rho)(|\beta|+|\gamma|)}. \tag{1.13}$$

Proof. We can write the group law as

$$xy^{-1} = \Phi(x, x-y), \quad \Phi(x,0) = 0 \quad (e = 0). \tag{1.14}$$

If Haar measure is given by $dm(y) = H(y)\,dy$ and $H_1(y) = \Delta(y^{-1})H(y)$, then

$$\begin{aligned} \mathfrak{K}u(x) &= \int k(x, xy^{-1})u(y)H_1(y)\,dy \\ &= \int u(y)k\big(x, \Phi(x, x-y)\big)H_1(y)\,dy \\ &= \int\int_{\mathbb{R}^n} u(y)e^{i\Phi(x,x-y)\cdot\zeta}\,\hat{k}(x,\zeta)H_1(y)\,d\zeta\,dy. \end{aligned} \tag{1.15}$$

Now by (1.14) we can find a smooth invertible matrix function $\Psi(x,y)$ such that, near the diagonal,

$$\Phi(x, x-y)\cdot\zeta = (x-y)\cdot\Psi(x,y)^t\zeta. \tag{1.16}$$

Hence, modulo a smoothing operator,

$$\mathfrak{K}u(x) = \iint e^{i(x-y)\cdot\Psi(x,y)^t\zeta}\,\hat{k}(x,\zeta)H_1(y)D(x,y)\,dy\,d\zeta, \tag{1.17}$$

where $D(x,y)$ is supported near $x = y$ and equal to 1 on a small neighborhood of $x = y$. If we make the change of variable

$$\xi = \Psi(x,y)^t\zeta,$$

then we can write

$$\zeta = \psi(x,y)\xi,$$

with ψ smooth, and

$$D(x,y)\,d\zeta = \widetilde{D}(x,y)\,d\xi,$$

so

$$\mathfrak{K}u(x) = \iint e^{i(x-y)\cdot\xi}\,\hat{k}(x,\psi(x,y)\xi)H_1(y)\widetilde{D}(x,y)u(y)\,dy\,d\xi, \tag{1.18}$$

so we have the form (1.12) with

$$a(x,y,\xi) = \hat{k}(x,\psi(x,y)\xi)\,H_1(y)\widetilde{D}(x,y). \tag{1.19}$$

It is routine to verify that the hypothesis $\mathfrak{X} \subset S^m_{\rho\#}$ yields (1.13).

In the special case of convolution operators, this result was noted by Strichartz [S10]. If $\rho > 1/2$, then the multiple symbol $a(x, y, \xi)$ can be reduced to a simple symbol, and we have

$$\mathfrak{K}u(x) = \int p(x, \xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \tag{1.20}$$

with

$$p(x, \xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x a(x, y, \xi)\big|_{y=x}. \tag{1.21}$$

See, e.g., [T2], Chapter 2. In the case $\rho = 1/2$, one can still write $\mathfrak{K}$ in the form (1.20), as shown in Beals [B2], but (1.21) is no longer asymptotic, and one does not obtain a neat formula for the principal symbol.

In the case when G is a step 2 ninpotent Lie group, we can obtain the form (1.20) in the general case, rather directly, as follows. In exponential coordinates, Haar measure coincides with Lebesgue measure, and $\Delta(y) \equiv 1$. Also, by the Campbell-Hausdorff formula, we can express the group law as

$$xy^{-1} = L_x(x - y), \tag{1.22}$$

where $L_x = I - (1/2)\mathrm{ad}\, x$ is a smooth family of invertible linear maps on $\mathfrak{g}$. Thus, in this case,

$$\begin{aligned} \mathfrak{K}u(x) &= \int u(y) k\big(x, L_x(x - y)\big)\, dy \\ &= \iint u(y) e^{iL_x(x-y)\cdot\xi} \hat{k}(x, \xi)\, dy\, d\xi \\ &= \int \hat{k}(x, \tilde{L}_x\xi) e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \end{aligned} \tag{1.23}$$

where $\tilde{L}_x = (L^t_x)^{-1}$. Thus we directly obtain (1.20), with

$$p(x, \xi) = \hat{k}(x, \tilde{L}_x\xi), \tag{1.24}$$

which gives $\mathfrak{K} \in OPS^m_{\rho,1-\rho}$. This argument is given in Nagel and Stein [N3] for the case of convolution operators on the Heisenberg group.

From Proposition 1.1 we can conclude, by the Calderon-Vaillancourt theorem, that if $\mathfrak{X} \subset S^0_{1/2\#}$, then $\mathfrak{K} \in OP\mathfrak{X}$ is continuous on L^2, and on any Sobolev space H^s. For $\mathfrak{K} \subset S^0_{\rho\#}$ with $0 < \rho < 1/2$, application of Proposition 1.1 yields only weak continuity results. In fact, it is apparent from first principles that, given any M, if $N > 0$ is large enough and $\mathfrak{X} \subset S^{-N}_{\rho\#}$, $0 < \rho < 1/2$, then any $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}$ maps H^s to H^{s+M}. Sharper continuity results will be considered in the next two sections.

A point we wish to emphasize is that, even though Proposition 1.1 has some uses (it shows $\mathfrak{K}$ is microlocal, for example), if $\rho \le 1/2$ it is not an incisive analysis of $\mathfrak{K}$. In particular, formulas for compositions of such operators are not obtained in a usable fashion from Proposition 1.1, even in case $\rho = 1/2$. We now give an account of compositions of operators of the form (1.11).

Suppose that for $m \in \mathbb{R}$ (or one might restrict attention to $m \in \mathbb{Z}$), $\mathfrak{X}^m$ is a nested family of Frechet spaces ($\mathfrak{X}^m \supset \mathfrak{X}^\mu$ if $m > \mu$) with the following properties:

$$\mathfrak{X}^m \subset S^m_{\rho\#} \quad \text{for some } \rho \in (0,1],\ m \ge 0, \tag{1.25}$$

$$\mathfrak{X}^m \subset S^{m\sigma}_{\rho\#} \quad \text{if } m < 0, \quad \text{for some } \sigma \in (0,1]. \tag{1.26}$$

$$A \in OP\mathfrak{X}^m,\ B \in OP\mathfrak{X}^\mu \Rightarrow AB \in OP\mathfrak{X}^{m+\mu}, \tag{1.27}$$

the product in (1.27) being continuous. The hypothesis (1.27) is an hypothesis on the composition of two convolution operators. Verifying it for particular classes $\mathfrak{X}^m$ is often a problem in harmonic analysis on G. Some cases of particular importance are considered in §2. Let us note that all these hypotheses are satisfied in the case $\mathfrak{X}^m = S^m_{1/2\#}$ (even easier, in case $\mathfrak{X}^m = S^m_{\rho\#}$ with $\rho \in (1/2,1]$); in all these cases we can take $\sigma = 1$ in (1.26). Now suppose

$$\mathfrak{A} \in OP\widetilde{\mathfrak{X}}^m, \quad \mathfrak{B} \in OP\widetilde{\mathfrak{X}}^\mu. \tag{1.28}$$

Say

$$\mathfrak{A}u(x) = A(x)u(x) = \int a(x, xy^{-1})u(y)H_1(y)\,dy, \tag{1.29}$$

and

$$\mathfrak{B}u(x) = B(x)u(x) = \int b(x, xy^{-1})u(y)H_1(y)\,dy. \tag{1.30}$$

For the composition, we have

$$\begin{aligned}\mathfrak{A}\mathfrak{B}u(x) &= \mathfrak{A}\Big(\int b(x, xy_1^{-1})u(y_1)H_1(y_1)\,dy_1\Big)\\ &= \iint a(x, xy_2^{-1})b(y_2, y_2y_1^{-1})u(y_1)H_1(y_1)H_1(y_2)\,dy_1\,dy_2.\end{aligned} \tag{1.31}$$

Let us compare this with the operator $\mathfrak{C}$, defined by

$$\mathfrak{C}u(x) = A(x)B(x)u(x). \tag{1.32}$$

By hypothesis (1.27), $C(y) = A(y)B(y)$ is a smooth function of y with values in $OP\mathfrak{X}^{m+\mu}$, so $\mathfrak{C} \in OP\widetilde{\mathfrak{X}}^{m+\mu}$. We have the formula

$$\mathfrak{C}u(x) = \iint a(x, xy_2^{-1})b(x, y_2y_1^{-1})u(y_1)H_1(y_1)H_1(y_2)\,dy_1\,dy_2. \tag{1.33}$$

Note that the only difference between (1.31) and (1.33) is in the first argument of b. This suggests making the expansion

$$b(y_2, z) = \sum_{|\gamma|<N} \varphi_\gamma(xy_2^{-1})b_\gamma(x, z) + \sum_{|\gamma|=N} \varphi_\gamma(xy_2^{-1})r_\gamma(x, y_2, z). \tag{1.34}$$

Here, using exponential coordinates, we take

$$\varphi_\gamma(y) = y^\gamma. \tag{1.35}$$

Thus $b_0(x, z) = b(x, z)$, and each $b_\gamma(x, z)$ is a smooth function of x with values in $\widehat{\mathfrak{X}}^\mu$. Furthermore, by Taylor's formula with remainder, $r_\gamma(x, y_2, \cdot)$ is a smooth function of x and y_2 with values in $\widehat{\mathfrak{X}}^\mu$. Thus we have

$$\mathfrak{A}\mathfrak{B}u(x) = \sum_{|\gamma|<N} A^{[\gamma]}(x)B_{[\gamma]}(x)u(x) + \mathfrak{R}_N u(x), \tag{1.36}$$

where

$$A^{[\gamma]}(y)u(x) = \int a(y, xy_2^{-1})\varphi_\gamma(xy_2^{-1})u(y_2)H_1(y_2)\,dy_2, \tag{1.37}$$

$$B_{[\gamma]}(y)u(x) = \int b_\gamma(y, xy_1^{-1})u(y_1)H_1(y_1)\,dy_1, \tag{1.38}$$

and $\mathfrak{R}_N$ is given by

$$\begin{aligned}\mathfrak{R}_N u(x) = \sum_{|\gamma|=N} \iint a(x, xy_2^{-1})\varphi_\gamma(xy_2^{-1})r_\gamma(x, y_2, y_2y_1^{-1})& \\ u(y_1)H_1(y_1)H_1(y_2)\,dy_1\,dy_2.&\end{aligned} \tag{1.39}$$

Suppose $u \in H^s$ is compactly supported. Then crude considerations give $M = M(\mu)$, independent of γ, such that

$$\int r_\gamma(x, y_2, y_2y_1^{-1})u(y_1)H_1(y_1)\,dy_1 = v(x, y_2)$$

is a smooth function of x taking values in the Sobolev space H^{s-M}, as a function of y_2. It will be convenient to switch the order here; pick K large and regard v as an H^{s-M} function of y_2, taking values in the Sobolev space H^K (functions of x). Thus $\mathfrak{R}_N u(x) = w_N(x,x)$, where

$$
\begin{aligned}
w_N(x) &= \sum_{|\gamma|=N} \int a(x, xy_2^{-1})\varphi_\gamma(xy_2^{-1})v(y,y_2)H_1(y_2)\,dy_2 \\
&= \sum_{|\gamma|=N} A^{[\gamma]}(x)v(y,x) \\
&= \sum_{|\gamma|=N} \mathfrak{A}^{[\gamma]}v(y,x).
\end{aligned} \tag{1.40}
$$

Here the operator $\mathfrak{A}^{[\gamma]}$ is applied to the H^K-valued fuction v (of x). For N large, the integral operator $\mathfrak{A}^{[\gamma]}$ has a very weak singularlty, so $w_N(x,y)$ will be in H^K for any given K if N is sufficiently large. Consequently we can interpret (1.36) as an asymptotic expansion

$$\mathfrak{A}\mathfrak{B}u(x) \sim \sum_{\gamma\ge 0} A^{[\gamma]}(x)B_{[\gamma]}(x)u(x). \tag{1.41}$$

As already noted, $B_{[\gamma]}(x)$ is a smooth function of x with values in $OP\mathfrak{X}^\mu$. As for $A^{[\gamma]}(x)$, in view of (1.37) and (1.35), if $\mathfrak{X}^m \subset S^\nu_{\rho\#}$ (with $\nu = m$ for $m \ge 0$, $m\sigma$ for $m < 0$), then $A^{[\gamma]}(x)$ is a smooth function of x with values in $S^{\nu-\rho|\gamma|}_{\rho\#}$. Thus the terms in (1.40) become highly smoothing for large $|\gamma|$. To obtain results within the framework of $OP\widetilde{\mathfrak{X}}^*$, we add the following hypotheses to (1.25)–(1.27):

$$P(\xi) \in \mathfrak{X}^m \Longrightarrow D^\alpha_\xi p(\xi) \in \mathfrak{X}^{m-\tau|\alpha|}, \tag{1.42}$$

for some $\tau \in (0,1]$, and

$$
\begin{aligned}
&K_j \in \mathfrak{X}^{m-\tau j} \Rightarrow \exists\ K \in \mathfrak{X}^m \ \text{ such that, for any } M, \\
&\text{if } N \text{ is sufficiently large, } K - (K_0 + \cdots + K_N) \in S^{-M}_{\rho\#}.
\end{aligned} \tag{1.43}
$$

It follows from these hypotheses and from (1.27) that the γ-term in (1.41) belongs to $OP\widetilde{\mathfrak{X}}^{m+\mu-\tau|\gamma|}$. We summarize the result we have obtained:

Proposition 1.2. *Let $\mathfrak{X}^m$ be a nested family of Frechet spaces satisfying the hypotheses (1.25)–(1.27) and (1.42)–(1.43). If $\mathfrak{A} \in OP\widetilde{\mathfrak{X}}^m$ and $\mathfrak{B} \in OP\widetilde{\mathfrak{X}}^\mu$, then $\mathfrak{A}\mathfrak{B} \in OP\widetilde{\mathfrak{X}}^{m+\mu}$, and, with $\mathfrak{C} \in OP\widetilde{\mathfrak{X}}^{m+\mu}$ given by (1.33), we have $\mathfrak{A}\mathfrak{B} - \mathfrak{C} \in OP\widetilde{\mathfrak{X}}^{m+\mu-\tau}$. More precisely, the asymptotic expansion (1.41) holds.*

We turn to the analysis of adjoints of elements of $OP\widetilde{\mathfrak{X}}^m$. If $\mathfrak{K}$ is given by (1.10)–(1.11), we have

$$\mathfrak{K}^* v(x) = \int k^\#(y, xy^{-1})v(y)H_1(y)\,dy, \tag{1.44}$$

where

$$k^{\#}(y,z)=\overline{k(y,z^{-1})}.$$

We compare this with $\mathfrak{K}^{\#}$, defined by

$$\mathfrak{K}^{\#}v(x)=\int k^{\#}(x,xy^{-1})v(y)H_1(y)\,dy. \tag{1.45}$$

We see that $\mathfrak{K}^{\#}\in OP\widetilde{\mathfrak{X}}^m$ provided $\mathfrak{X}^m$ satisfies the condition

$$p(\xi)\in\mathfrak{X}^m\Longrightarrow\overline{p(\xi)}\in\mathfrak{X}^m. \tag{1.46}$$

The formula (1.45) differs from (1.44) only in the first argument of $k^{\#}$, so, in analogy with (1.34), we make the expansion

$$k^{\#}(y,z)=\sum_{|\gamma|<N}\varphi_\gamma(xy^{-1})k^{\#}_\gamma(x,z)+\sum_{|\gamma|=N}\varphi_\gamma(xy^{-1})r^{\#}_\gamma(x,y,z). \tag{1.47}$$

In this case, $r^{\#}_\gamma(x,y,z)$ is a smooth function of x and y, taking values in $\widehat{\mathfrak{X}}^m$. It follows that

$$\mathfrak{K}^{*}v(x)=\sum_{|\gamma|<N}K_{\{\gamma\}}(x)v(x)+\mathfrak{R}^{\#}_N v(x), \tag{1.48}$$

where

$$K_{\{\gamma\}}(z)v(x)=\int k^{\#}_\gamma(z,xy^{-1})\varphi_\gamma(xy^{-1})v(y)H_1(y)\,dy, \tag{1.49}$$

and $\mathfrak{R}^{\#}_N$ is given by

$$\mathfrak{R}^{\#}_N v(x)=\sum_{|\gamma|=N}\int r^{\#}_\gamma(x,y,xy^{-1})\varphi_\gamma(xy^{-1})v(y)H_1(y)\,dy. \tag{1.50}$$

We see that $k^{\#}(x,z)$ is a smooth function of x taking values in $\widehat{\mathfrak{X}}^m$, and hence, granted hypothesis (1.42), the map $\mathfrak{K}_{\{\gamma\}}$ given by $\mathfrak{K}_{\{\gamma\}}v(x)=K_{\{\gamma\}}(x)v(x)$ belongs to $OP\widetilde{\mathfrak{X}}^{m-\tau|\gamma|}$. Meanwhile, an analysis very like that of $\mathfrak{R}_N$ shows that $\mathfrak{R}^{\#}_N$ is arbitrarily smoothing if N is sufficiently large. We have established the following result.

Proposition 1.3. *Let $\mathfrak{X}^m$ be a nested family of Frechet spaces satisfying the hypotheses (1.25)–(1.27), (1.42)–(1.43), and (1.46). If $\mathfrak{K}\in OP\widetilde{\mathfrak{X}}^m$, then the adjoint $\mathfrak{K}^*$ belongs to $OP\widetilde{\mathfrak{X}}^m$, and one has the asymptotic expansion*

$$\mathfrak{K}^{*}v(x)\sim\sum_{\gamma\ge 0}K_{\{\gamma\}}(x)v(x). \tag{1.51}$$

As we have noted, the hypotheses giving Propositions 1.2 and 1.3 are satisfied by $\mathfrak{X}^m=S^m_{\rho\#}$, if $1/2\le\rho\le 1$. In case $1/2\le\rho<1$, $OP\widetilde{\mathfrak{X}}^m$ is a strict subclass of $OPS^m_{\rho,1-\rho}$. It is useful to note that for $\rho=1$ these two classes coincide.

Proposition 1.4. *If $\mathfrak{X}^m = S^m_{1\#}$, then, locally and modulo smoothing operators, we have*

$$OP\widetilde{\mathfrak{X}}^m = OPS^m_{1,0}. \tag{1.52}$$

Proof. We already have $OP\widetilde{\mathfrak{X}}^m \subset OPS^m_{1,0}$. For the converse, let $p(x,D) \in OPS^m_{1,0}$. Then, guided by (1.19) and (1.21), we define $\hat{k}(x,\xi)$ by

$$p(x,\xi) = \hat{k}\big(x, \psi(x,x)\xi\big) H_1(x) \widetilde{D}(x,x), \tag{1.53}$$

where ψ, H_1, and $\widetilde{D}$ are defined in the proof of Proposition 1.1. It is routine to verify that this does indeed define $\hat{k}(x,\xi)$ as a smooth function of x with values in $S^m_{1\#}$. Now define $\mathfrak{K}_0 \in OP\widetilde{\mathfrak{X}}^m$ by

$$\mathfrak{K}_0 u(x) = \int k(x, xy^{-1}) u(y) H_1(y)\, dy. \tag{1.54}$$

The proof of Proposition 1.1 shows that $\mathfrak{K}_0 \in OPS^m_{1,0}$ and $p(x,D) - \mathfrak{K}_0 = p_2(x,D) \in OPS^{m-1}_{1,0}$. Inductively, we obtain $\mathfrak{K}_j \in OP\widetilde{\mathfrak{X}}^{m-j}$, which asymptotically sum to $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}^m$, and $p(x,D) - \mathfrak{K}$ is a smoothing operator. This finishes the proof.

We next derive a simple abstract hypoellipticity result from the operator calculus developed so far. In the Euclidean case, this is the regularity theorem for elliptic operators.

Proposition 1.5. *Suppose $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}^m$ has the form $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_0$, whth $\mathfrak{K}_0 \in OP\widetilde{\mathfrak{X}}^{m-\tau}$ and $\mathfrak{K}_1 u(x) = K_1(x)u(x)$. Keep the hypotheses on $\mathfrak{X}^m$ used in Proposition 1.2. Assume that*

$$E_1(x) = K_1(x)^{-1} \quad \text{is smooth in } x \quad \text{with values in } OP\mathfrak{X}^{-m}. \tag{1.55}$$

Then $\mathfrak{K}$ is hypoelliptic, with a parametrix in $OP\widetilde{\mathfrak{X}}^{-m}$.

Proof. By (1.55), we can define $\mathfrak{E}_1 \in OP\widetilde{\mathfrak{X}}^{-m}$ by

$$\mathfrak{E}_1 u(x) = E_1(x)u(x). \tag{1.56}$$

Then Proposition 1.2 gives

$$\mathfrak{E}_1 \mathfrak{K} = I - \mathfrak{R}, \quad \mathfrak{R} \in OP\widetilde{\mathfrak{X}}^{-\tau}. \tag{1.57}$$

Hence

$$(I + \mathfrak{R} + \mathfrak{R}^2 + \cdots + \mathfrak{R}^{N-1})\mathfrak{E}_1 \mathfrak{K} = I - \mathfrak{R}^N, \quad \mathfrak{R}^N \in OP\widetilde{\mathfrak{X}}^{-N\tau}. \tag{1.58}$$

Taking N arbitrarily large we deduce $\mathfrak{K}$ is hypoelliptic, with parametrix

$$\mathfrak{E} \sim (1 + \mathfrak{R} + \mathfrak{R}^2 + \cdots)\mathfrak{E}_1.$$

We now take a look at the behavior of such operators as considered above under a change of variable. That is, if $\varphi : G \to G$ is a C^∞ diffeomorphism (not necessarily a group automorphism) with inverse ψ, and $A \in OP\widetilde{\mathfrak{X}}^m$, we want to understand the nature of the conjugated operator $\psi^* A \varphi^*$ (where $\varphi^* u(x) = u(\varphi(x))$. In many cases the class $OP\widetilde{\mathfrak{X}}^m$ will not be invariant under general diffeomorphisms φ. For example, when $G = \mathbb{R}^n,\ \mathfrak{X}^m = S^m_{\rho\#},\ 1/2 \le \rho \le 1$, we have $OP\widetilde{\mathfrak{X}}^m = OPS^m_{\rho,0}$, but the classes invariant under diffeomorphisms are $OPS^m_{\rho,1-\rho}$; unless $\rho = 1,\ OP\widetilde{\mathfrak{X}}^m$ is not invariant under general diffeomorphisms in this case.

Nevertheless, $\mathfrak{X}^m$ in many cases is defined by special properties (e.g., quasi-homogeneity) associated with a certain structure on G (e.g., a contact structure, discussed in Chapter III), and $OP\widetilde{\mathfrak{X}}^m$ may be invariant under those diffeomorphisms preserving such a structure. Results along these lines depend strongly on particular cases, but we will outline some general features of the phenomena here, which will apply to special cases, as in Chapter III.

If an operator A is given by

$$Au(x) = \int a(x, xy^{-1}) u(y) H_1(y)\, dy, \quad x \in G, \tag{1.59}$$

the conjugated operator $B = \psi^* A \varphi^*$ is given by

$$\begin{aligned} Bu(x) &= \int a\big(\psi(x), \psi(x) y^{-1}\big) u\big(\varphi(y)\big) H_1(y)\, dy \\ &= \int a\big(\psi(x), \psi(x)\psi(y)^{-1}\big) u(y) \tilde{H}(y)\, dy \\ &= \int b\big(x, \Psi(x, y, xy^{-1})\big) u(y) \tilde{H}(y)\, dy, \end{aligned} \tag{1.60}$$

where

$$b(x, z) = a(\psi(x), z), \quad \Psi(x, y, xy^{-1}) = \psi(x)\psi(y)^{-1}.$$

There is some freedom in the construction of Ψ, and we can arrange that, in exponential coordinates, $\Psi(x, y, z)$ is linear in z, with

$$\Psi(x, x, z) = D\psi(x) z = \Psi(x) z, \tag{1.61}$$

where $D\psi(x)$ is the derivative of the map ψ, as a linear map on $\mathfrak{g}$. The next natural thing to do is to make a power series expansion of $b\big(x, \Psi(x, y, xy^{-1})\big)$ in the third argument, y, about $y = x$, in analogy with (1.34). We get

$$b\big(x, \Psi(x, y, xy^{-1})\big) \sim \sum_{\gamma \ge 0, |\sigma| \ge |\gamma|} C_{\sigma\gamma}\, (xy^{-1})^{\gamma+\sigma}\, b_{(\gamma)}\big(x, \Psi(x)(xy^{-1})\big), \tag{1.62}$$

where $b_{(\gamma)}(x, z) = D_z^\gamma b(x, z)$. Here, if $w = xy^{-1}$ is given in exponential coordinates by $w = (w_1, \dots, w_n)$, and if $\gamma = (\gamma_1, \dots, \gamma_n)$, we set $w^\gamma = w_1^{\gamma_1} \cdots w_n^{\gamma_n}$. Now we would like the summands in (1.62) to be smoothing of high order if $|\gamma|$ is large. Denote the partial Fourier transform of $b(x,w)$ by $\tilde{b}(x,\xi)$. With $w = xy^{-1}$ in (1.62), the partial Fourier transform of a general term in (1.62) with respect to w is

$$C_{\sigma\gamma}\, D_\xi^{\gamma+\sigma} \xi^\gamma \tilde{b}\big(x, \Psi(x)^t \xi\big). \tag{1.63}$$

If, for example, $\tilde{b}(x,\xi)$ is a smooth function of x with values in $S^m_{1\#}$, then such a term is a smooth function of x with values in $S^{m-|\sigma|}_{1\#} \subset S^{m-|\gamma|}_{1\#}$. Modulo estimating the error term when (1.62) is truncated to a finite sum, this retraces the proof that $OPS^m_{1,0}$ is invariant under coordinate changes. On the other hand, if $\tilde{b}(x,\xi)$ is a smooth function of x with values in $S^m_{\rho\#}$, with $1/2 < \rho < 1$, then the term in (1.63) belongs to $S^{m+(1-\rho)|\gamma|-\rho|\sigma|}_{\rho,1-\rho} \subset S^{m-(2\rho-1)|\gamma|}_{\rho,1-\rho}$, which is not the same as being a smooth function of x with values in $S^{m-(2\rho-1)|\gamma|}_{\rho\#}$.

As mentioned, we will pursue further the question of behavior under changes of variables later on.

2. Operator classes for groups with dilations

Having worked on the level of general Lie groups in §1, we now specialize a bit. Let G be a simply connected Lie group with a one-parameter family $\alpha(t)$ of automorphisms. Denote also by $\alpha(t)$ the automorphisms induced on the Lie algebra $\mathfrak{g}$. We make the assumption that these are dilations. In other words,

$$\mathfrak{g} = \bigoplus_{b\in B} \mathfrak{g}_b, \tag{2.1}$$

where B is a subset of $(0,\infty)$ and

$$\alpha(t)X = e^{tb}X, \quad X \in \mathfrak{g}_b. \tag{2.2}$$

Note that

$$[\mathfrak{g}_b, \mathfrak{g}_c] \subset \mathfrak{g}_{b+c}. \tag{2.3}$$

Hence $\mathfrak{g}$ must be nilpotent. Thus $\exp : \mathfrak{g} \to G$ is a diffeomorphism, taking Lebesgue measure on $\mathfrak{g}$ to Haar measure on G.

Definition. $\widehat{\mathfrak{H}}(G,\alpha,m)$ is the space of distributions u on G such that

$$u \in C^\infty(G \setminus 0), \tag{2.4}$$

$$u = u_1 + u_2,\ u_1 \in \mathcal{E}'(G),\ u_2 \in \mathcal{S}(G), \text{the Schwartz space}, \tag{2.5}$$

$$t \in (0,\infty) \Rightarrow e^{\sigma t}\alpha(t)^* u - e^{-tm}u \in \mathcal{S}(G), \tag{2.6}$$

where $e^{\sigma t}$ is the factor by which $\alpha(t)$ expands volumes in G. If $d_j = \dim \mathfrak{g}_{b_j}$,

$$\sigma = \sum b_j d_j. \tag{2.7}$$

Here, for $v \in \mathcal{S}(G)$, $\alpha(t)^*v(x) = v(\alpha(t)x)$, $x \in G$. Then $\alpha(t)^*$ extends uniquely to a continuous linear map on $\mathcal{S}'(G)$. In words, we say u is approximately homogeneous of degree $-(m+\sigma)$. We can characterize the space $\mathfrak{H}(G,\alpha,m)$ of (Euclidean) Fourier transforms of elements of $\widehat{\mathfrak{H}}(G,\alpha,m)$ as follows.

Proposition 2.1. *The tempered distribution u belongs to $\widehat{\mathfrak{H}}(G,\alpha,m)$ if and only if $\hat{u}(\xi)$ satisfies*

$$\hat{u} \in C^\infty(\mathfrak{g}'), \tag{2.8}$$

where we identify G and $\mathfrak{g}$, and

$$\beta(t)^*\hat{u} - e^{tm}\hat{u} \in \mathcal{S}(\mathfrak{g}'), \quad \forall\ t \in (0,\infty), \tag{2.9}$$

where $\beta(t) = \alpha(t)^t$.

Proof. Since the Fourier transform of $\alpha(t)^*u$ is given by $e^{-\sigma t}\beta(t)^*\hat{u}$, (2.9) and (2.6) are equivalent. Clearly (2.5) implies (2.8). That (2.8) and (2.9) also give (2.4) follows easily from the following observation.

Proposition 2.2. *Suppose* $\hat{u} \in C^\infty(\mathfrak{g}')$. *Then (2.9) is equivalent to the existence of* $v \in C^\infty(\mathfrak{g}' \setminus 0)$, *such that*

$$v(\beta(t)\xi) = e^{tm} v(\xi) \quad \textit{on} \quad \mathfrak{g}' \setminus 0, \quad \forall\, t \in (0, \infty), \tag{2.10}$$

and

$$\chi(\xi)[\hat{u}(\xi) - v(\xi)] \in \mathcal{S}(\mathfrak{g}'), \tag{2.11}$$

where $\chi(\xi)$ *is smooth, equal to* 0 *for* ξ *in a neighborhood of* 0, *and equal to* 1 *for* ξ *outside a compact set.*

Proof. Clearly (2.10) and (2.11) imply (2.9). Conversely, suppose

$$\hat{u}(\beta(1)\xi) - e^m \hat{u}(\xi) = \hat{h}(\xi) \in \mathcal{S}(\mathfrak{g}'). \tag{2.12}$$

Then, replacing ξ by $\beta(j)\xi$, we have

$$\hat{u}(\beta(j+1)\xi) - e^m \hat{u}(\beta(j)\xi) = \hat{h}(\beta(j)\xi).$$

Hence

$$e^{-m(j+1)} \hat{u}(\beta(j+1)\xi) - \hat{u}(\xi) = \sum_{k=0}^{j} e^{-mk} \hat{h}(\beta(k)\xi),$$

so let

$$v(\xi) = \sum_{k=0}^{\infty} e^{-mk} \hat{h}(\beta(k)\xi) + \hat{u}(\xi). \tag{2.13}$$

Since $\hat{h} \in \mathcal{S}(\mathfrak{g}')$, v belongs to $C^\infty(\mathfrak{g}' \setminus 0)$, and (2.12) implies

$$e^{-mj} \hat{u}(\beta(j)\xi) \longrightarrow v(\xi), \quad \text{as} \quad j \to \infty.$$

Using increments of $\beta(1/N)$ instead of $\beta(1)$ and passing to the limit, we have

$$e^{-mt} \hat{u}(\beta(t)\xi) \longrightarrow v(\xi), \quad \text{as} \quad t \to \infty. \tag{2.14}$$

It follows that v satisfies (2.10). Furthermore, (2.13) easily yields (2.11), so the proof is complete.

So we see that, for $u \in \widehat{\mathfrak{H}}(G, \alpha, m)$, $\hat{u}$ differs by a rapidly decreasing function from a homogeneous function. However, it is not always the case that u differs by a smooth function from a homogeneous function. Consider the case $G = \mathbb{R}$, $\alpha(t)x = e^t x$. Then

$$\varphi(x)\, \log|x| \in \widehat{\mathfrak{H}}(\mathbb{R}, e^t, -1), \tag{2.15}$$

where $\varphi \in C_0^\infty(\mathbb{R})$ is equal to 1 near the origin.

Now $OP\mathfrak{H}(G, \alpha, m)$ is the set of convolution operators

$$Ku = k * u, \quad k \in \widehat{\mathfrak{H}}(G, \alpha, m).$$

We have the following result on compositions of such operators.

Proposition 2.3. *If* $K_j \in OP\mathfrak{H}(G,\alpha,m_j)$*, then*

$$K_1K_2 \in OP\mathfrak{H}(G,\alpha,m_1+m_2). \tag{2.16}$$

Proof. We have $K_ju = k_j * u,\ k_j \in \widehat{\mathfrak{H}}(G,\alpha,m_j)$. It suffices to show

$$k_1 * k_2 \in \widehat{\mathfrak{H}}(G,\alpha,m_1+m_2). \tag{2.17}$$

This is very simple. Since $\alpha(t)$ is a group of automorphisms of G,

$$\alpha(t)^*(k_1 * k_2) = e^{\sigma t}\,\alpha(t)^*k_1 * \alpha(t)^*k_2. \tag{2.18}$$

Now, for each $t \in (0,\infty)$,

$$\alpha(t)^*k_j = e^{-t(m_j+\sigma)}k_j + h_j, \quad h_j \in \mathcal{S}(G),$$

and since it is elementary to show

$$k_j * h_\ell,\ h_\ell * k_j \in \mathcal{S}(G),$$

under these circumstances, we have, for each $t \in (0,\infty)$,

$$\alpha(t)^*(k_1 * k_2) = e^{-t(m_1+m_2-\sigma)}\,k_1 * k_2, \quad \mathrm{mod}\ \ \mathcal{S}(G). \tag{2.19}$$

This completes the proof.

As we have said, an element k of $\widehat{\mathfrak{H}}(G,\alpha,m)$ need not be equal to a smooth function plus a homogeneous function. However, for some range of m this turns out to be the case. It is useful to note the following simple results on the behavior of elements of $\widehat{\mathfrak{H}}(G,\alpha,m)$.

Proposition 2.4. *If* $k \in \widehat{\mathfrak{H}}(G,\alpha,m)$*, then*

$$k = k_1 + k_2 \tag{2.20}$$

with $k_2 \in C^\infty(G)$ *and* k_1 *homogeneous, satisfying*

$$\alpha(t)^*k_1 = e^{-t(m+\sigma)}k_1, \tag{2.21}$$

provided

$$m > -\sigma. \tag{2.22}$$

Proof. The point of the hypothesis (2.22) is precisely to guarantee that the homogeneous function $v(\xi)$ of Proposition 2.2 be locally integrable in a neighborhood of the origin. This guarantees that $v(\xi)$ defines a homogeneous distribution in $\mathcal{S}'(\mathfrak{g}')$. We can set k_1 equal to the inverse Fourier transform of $v(\xi)$, and (2.20) follows easily.

We take a look at the mapping properties of convolution operators in $OP\mathfrak{H}(G,\alpha,0)$. As is well known, such operators are bounded on $L^p(G)$, for $1 < p < \infty$. This follows from the general theories of Coifman and Weiss [C4] and Koranyi and Vagi [K8], which were exploited in [F4] and [R5], and which are given a nice exposition in Goodman [G7]. We give a brief description of the situation. We can define a "homogeneous norm" on G (identified with $\mathfrak{g}$ via the exponential map) by

$$|X| = \Big(\sum \|X_i\|^{2/b_i}\Big)^{1/2}, \tag{2.23}$$

where

$$X = \sum X_i, \quad X_i \in \mathfrak{g}_{b_i},$$

and $\|X_i\|$ is a Euclidean norm on $\mathfrak{g}_{b_i}$. Note that

$$|\alpha(t)X| = e^t|X|. \tag{2.24}$$

If $k \in \widehat{\mathfrak{H}}(G,\alpha,0)$, then pick k_1 and k_2 as in Proposition 2.4, so k_1 is homogeneous of degree $-\sigma$ and smooth outsied the origin. We claim k_1 must have mean value zero in each region

$$\mathcal{A}_r = \{X \in \mathfrak{g} : r \le |x| \le 2r\}. \tag{2.25}$$

In fact, we can pick a unique constant C such that

$$k_1(X) = k_1^{\#}(X) + C|X|^{-\sigma} \tag{2.26}$$

for $X \in \mathfrak{g} \setminus 0$, where $k_1^{\#}$ does have mean value zero on each $\mathcal{A}_r$. Then $k_1^{\#}$ defines a principal value distribution, homogeneous of degree $-\sigma$ (cf. [G7]). We can deduce that C in (2.26) is equal to zero if we know that $|X|^{-\sigma}$ cannot be extended to a distribution that is homogeneous of degree $-\sigma$, i.e., whose Fourier transform is homogeneous of degree zero with respect to $\beta(t)$. In fact, not only does $|X|^{-\sigma}$ just fail to be integrable near 0, it also just fails to be integrable near infinity. If a cut-off $\chi(X)$ is used, $\chi \in C^\infty(\mathfrak{g})$, $\chi = 0$ near 0, $\chi = 1$ outside a compact set, and $0 \le \chi(X) \le 1$ for all X, then $\chi(X)|X|^{-\sigma}$ is smooth, ≥ 0, and not quite integrable at infinity. A simple regularization argument shows its Fourier transform must blow up at the origin. However, such Fourier transform must differ by a smooth function from the Fourier transform of a homogeneous distribution extending $|X|^{-\sigma}$, if such exists, and this is not compatible with homogeneity of order zero for its Fourier transform.

Consequently, each $K \in OP\mathfrak{H}(G,\alpha,0)$ is convolution by a k, which differs by a smooth function from a k_1, which defines a principal value distribution, homogeneous of degree $-\sigma$, plus a multiple of the delta function concentrated at 0. As shown in [G7], convolution by k_1 is continuous on $L^p(G)$, $1 < p < \infty$. Thus we have:

Proposition 2.5. *If $K \in OP\mathfrak{H}(G,\alpha,0)$, then $K : L^p \to L^p$, for $1 < p < \infty$.*

From $OP\mathfrak{H}(G,\alpha,m)$, we can pass to the classes of "variable coefficient" operators $OP\widetilde{\mathfrak{H}}(G,\alpha,m)$, where, as usual, $\mathfrak{K} \in OP\widetilde{\mathfrak{H}}(G,\alpha,m)$ is given by $\mathfrak{K}u(x) = K(x)u(x)$, where $K(x)$ is a smooth function of x with values in $OP\mathfrak{H}(G,\alpha,m)$. It follows that if $\mathfrak{K}_j$ belong to $OP\widetilde{\mathfrak{H}}(G,\alpha,m_j)$, then $\mathfrak{K}_1\mathfrak{K}_2$ is in $OP\widetilde{\mathfrak{H}}(G,\alpha,m_1+m_2)$ and $\mathfrak{K}_j^* \in OP\widetilde{\mathfrak{H}}(G,\alpha,m_j)$. We also have L^p continuity for elements of $OP\widetilde{\mathfrak{H}}(G,\alpha,0)$:

Proposition 2.6. *If $\mathfrak{K} \in OP\widetilde{\mathfrak{H}}(G,\alpha,0)$, then*

$$\mathfrak{K} : L^p_{\text{comp}} \longrightarrow L^p_{\text{loc}}, \quad \textit{for} \;\; 1 < p < \infty. \tag{2.27}$$

Proof. Using Proposition 2.4 we can deduce that, modulo a smoothing operator, $\mathfrak{K}$ is given by

$$\mathfrak{K}u(x) = k_x * u(x),$$

where k_y is a smooth function of y taking values in $\widehat{\mathfrak{H}}^0(G,\alpha)$, the space of principal value distributions, homogeneous of degree $-\sigma$, smooth on $\mathfrak{g}\setminus 0$. Working locally, we can suppose $k(y,x) = k_y(x)$ has compact support in y. Now write

$$k(y,x) = \int \ell(\eta,x)e^{iy\cdot\eta}\,d\eta,$$

where

$$\ell(\eta,x) = (2\pi)^{-n/2}\int k(y,x)e^{-iy\cdot\eta}\,dy = \ell_\eta(x).$$

Then ℓ is a rapidly decreasing function of η, taking values in the space $\widehat{\mathfrak{H}}^0(G,\alpha)$ of principal value distributions. It follows from Proposition 2.5 that L_η, defined by $L_\eta u = \ell_\eta * u$, is bounded on $L^p(G)$, with

$$\|L_\eta\|_{\mathcal{L}(L^p)} \le C_N(p)(1+|\eta|)^{-N}.$$

Since the operator of multiplication by $e^{ix\cdot\eta}$ has operator norm 1 on L^p, the identity

$$\mathfrak{K}u(x) = \int e^{ix\cdot\eta}\,L_\eta u\,d\eta$$

shows that (2.27) holds.

We now discuss an amalgamation of a given operator class $OP\widetilde{\mathfrak{H}}(G,\alpha,m)$ with the classical pseudodifferential operators OPS^m. For this, we will assume G is a simply connected, step 2 nilpotent Lie group. In this case the Fourier integral representation for an operator constructed from convolutions takes the simple form (1.23)–(1.24). In particular, if $Ku = k * u$, then

$$Ku = p(x,D)u, \quad p(x,\xi) = \hat{k}(\tilde{L}_x\xi). \tag{2.28}$$

This enables us to establish the following result, following Phong and Stein [P3], who noted it for operators on the Heisenberg group.

Proposition 2.7. *Suppose G is a step 2 nilpotent Lie group. If $Ku = k * u$ defines a convolution operator in $OP\mathfrak{H}(G, \alpha, m)$ and $Lu = \ell * u$ defines a convolution operator in OPS^{μ}, then*

$$LKu = Pu = p * u, \tag{2.29}$$

where $\hat{p}(\xi)$ has an asymptotic expansion of the form

$$\hat{p}(\xi) \sim \hat{k}(\xi)\hat{\ell}(\xi) + \sum_{j \geq 1} a_j(\xi) b_j(\xi). \tag{2.30}$$

Here,

$$a_j(\xi) \in \mathfrak{H}(G, \alpha, m_j), \quad b_j(\xi) \in S^{\mu_j}(\mathfrak{g}'), \tag{2.31}$$

with

$$m_j \leq m, \quad \mu_j \leq \mu, \quad m_j + \mu_j < m + \mu, \quad \text{and } m_j + \mu_j \to -\infty, \text{as } j \to \infty. \tag{2.32}$$

Also

$$KLu = P_1 u = p_1 * u, \tag{2.33}$$

where $\hat{p}_1(\xi)$ has an expansion similar to (2.30), with the same leading term $\hat{k}(\xi)\hat{\ell}(\xi)$.

Proof. By (2.28), convolution operators are characterized by

$$Au = A(x, D)u = \int A(x, \xi)\hat{u}(\xi) e^{ix \cdot \xi} \, d\xi,$$

with

$$A(x, \xi) = \hat{a}(\tilde{L}_x \xi).$$

If we put K in this form, we have

$$Ku = K(x, D)u = \int K(x, \xi)\hat{u}(\xi) e^{ix \cdot \xi} \, d\xi, \tag{2.34}$$

with

$$K(x, \xi) = \hat{k}(\tilde{L}_x \xi).$$

Now if $\hat{k} \in \mathfrak{H}(G, \alpha, m)$, it follows that

$$\hat{k}(\xi) \in S^{M}_{\rho\#}, \tag{2.35}$$

with $\rho = \rho(b_1, \dots, b_i) > 0$ and

$$(2.36)\qquad \begin{aligned} M &= m \max_j b_j, \quad m \ge 0, \\ M &= m \min_j b_j, \quad m < 0. \end{aligned}$$

The pseudodifferential operator calculus as worked out in [H6] allows us to apply

$$(2.37)\qquad L = L(x, D) \in OPS^\mu, \quad L(x, \xi) = \hat{\ell}(\tilde{L}_x \xi)$$

to (2.34), to get

$$(2.38)\qquad LK = P(x, D) \in OPS^{M+\mu}_{\rho, 1-\rho},$$

with

$$(2.39)\qquad P(x, \xi) \sim \sum_{\alpha \ge 0} \frac{1}{\alpha!} L^{(\alpha)}(x, \xi)\, D_x^\alpha K(x, \xi).$$

Since LK is a convolution operator, $P(x, \xi) = \hat{p}(\tilde{L}_x \xi) = P(0, \tilde{L}_x \xi)$, so evaluating (2.39) at $x = 0$ gives (2.30). The analysis of KL is the same.

Proposition 2.7 motivates considering the following classes of symbols.
Definition. We say

$$(2.40)\qquad p(\xi) \in \mathfrak{H}^{m,\mu}_{\alpha,\delta}$$

provided $p(\xi)$ has an asymptotic expansion

$$(2.41)\qquad p(\xi) \sim \sum_{j \ge 0} a_j(\xi) b_j(\xi),$$

where

$$(2.42)\qquad a_j(\xi) \in \mathfrak{H}(G, a, m_j), \quad b_j(\xi) \in \Sigma^{\mu_j}(\mathfrak{g}'),$$

and

$$(2.43)\qquad m_j \le m, \quad \mu_j \le \mu, \quad m_j + \mu_j \to -\infty \ \text{ as } j \to \infty.$$

Here Σ^μ denotes the subspace of $S^\mu_{1\#}$ consisting of $b(\xi) \in S^\mu_{1\#}$ with asymptotic expansion $b(\xi) \sim \sum b_j(\xi)$, b_j homogeneous of degree $\mu - j$ in ξ, with respect to the homogeneous dilations

$$(2.44)\qquad \delta(t)\xi = e^t \xi.$$

From now on, $OP\Sigma^\mu$ will denote the set of convolution operators $Lu = \ell * u$ that belong to OPS^μ.

The pair of subscripts α, δ in (2.40) stands for the pair of dilation groups in effect here. Proposition 2.7 shows that, for G nilpotent of step 2,

$$(2.45)\qquad K \in OP\mathfrak{H}(G, \alpha, m), \ \ L \in OP\Sigma^\mu \Longrightarrow KL \ \text{ and } \ LK \in OP\mathfrak{H}^{m,\mu}_{\alpha,\delta}.$$

We have the following converse result.

Proposition 2.8. *If G is a step 2 nilpotent Lie group and $P \in OP\mathfrak{H}^{m,\mu}_{\alpha,\delta}$, then P has an asymptotic expansion*

$$P \sim \sum_{j \geq 0} K_j L_j, \tag{2.46}$$

in the sense that the difference between P and a sufficiently large partial sum is smoothing to arbitrary order, where

$$L_j \in OP\mathfrak{H}(G, a, m_j), \quad L_j \in OP\Sigma^{\mu_j}, \tag{2.47}$$

with

$$m_j \leq m, \quad \mu_j \leq \mu, \quad \text{and } \ m_j + \mu_j \to -\infty. \tag{2.48}$$

Proof. We can reduce our problem to considering

$$\hat{p}(\xi) = \hat{k}_0(\xi)\hat{\ell}_0(\xi), \quad \hat{k}_0 \in \mathfrak{H}(G, \alpha, m), \quad \hat{\ell}_0 \in \Sigma^{\mu}. \tag{2.49}$$

Then $K_0 u = k_0 * u$ defines an operator $K_0 \in OP\mathfrak{H}(G, \alpha, m)$, and $L_0 u = \ell_0 * u$ defines $L_0 \in OP\Sigma^{\mu}$. If we apply Proposition 1.12 to $K_0 L_0$ we see that $K_0 L_0 \in OP\mathfrak{H}^{m,\mu}_{\alpha,\delta}$, and the principal term in the expansion is given by (2.49). Applying the same reasoning to the remainder terms and iterating repeatedly, we get (2.46).

This result enables us to obtain the following L^p boundedness.

Proposition 2.9. *If G is a step 2 nilpotent Lie group and $P \in OP\mathfrak{H}^{0,0}_{\alpha,\delta}$, then*

$$P : L^p \longrightarrow L^p, \quad 1 < p < \infty.$$

Proof. By (2.46), this follows from Proposition 2.5, which gives

$$K_j : L^p \to L^p, \ \ 1 < p < \infty, \ \text{ if } \ K_j \in OP\mathfrak{H}(G, \alpha, m_j), m_j \leq 0,$$

(the result for $m_j < 0$ being elementary), together with the well known result

$$L_j : L^p \to L^p, \ \ 1 < p < \infty, \ \text{ if } \ L_j \in OPS^{\mu_j}, \mu_j \leq 0.$$

This result was proved by Phong and Stein [P3], in the context of convolution operators on the Heisenberg group. They also treated the harder problem of weak type (1,1) properties, which are not necessarily preserved by taking compositions, as are L^p continuity properties.

The operator classes $OP\mathfrak{H}^{m,\mu}_{\alpha,\delta}$ have the following properties for compositions and adjoints:

Proposition 2.10. *If G is step 2 and $P_j \in OP\mathfrak{H}^{m_j,\mu_j}_{\alpha,\delta}$, then*

$$P_1P_2 \in OP\mathfrak{H}^{m_1+m_2,\mu_1+\mu_2}_{\alpha,\delta}, \tag{2.50}$$

and

$$P_j^* \in OP\mathfrak{H}^{m_j,\mu_j}_{\alpha,\delta}. \tag{2.51}$$

Proof. Apply (2.46) to P_j. Then P_1P_2 is a sum of terms

$$K_kL_kK_k'L_k', \quad K_k \in OP\mathfrak{H}(G,\alpha,m_k),\ K_k' \in OP\mathfrak{H}(G,\alpha,m_k'), \ \text{ etc.}, \tag{2.52}$$

where $m_k \le m_1, m_k' \le m_2$, etc. Now (2.45) implies $L_kK_k' \in OP\mathfrak{H}^{m_k,\mu_k'}_{\alpha,\delta}$, and hence, by Proposition 2.8,

$$L_kK_k' \sim \sum_{\nu\ge 0} K_{k\nu}'L_{k\nu}. \tag{2.53}$$

If we substitute (2.53) into (2.52), we see that (2.50) holds for P_1P_2. The same sort of argument yields (2.51).

We now pass from the classes of convolution operators $OP\mathfrak{H}^{m,\mu}_{\alpha,\delta}$ to their "variable coefficient" versions $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$. Since $\mathfrak{H}^{m,\mu}_{\alpha,\delta}$ does not have a Frechet space structure, we make the special definition that an operator $\mathfrak{K}u(x) = K(x)u(x)$ belongs to $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$ if

$$\hat{k}(y,\xi) \sim \sum_{j\ge 0} a_j(y,\xi)b_j(y,\xi), \tag{2.54}$$

where $a_j(y,\xi)$ is a smooth function of y with values in $\mathfrak{H}(G,\alpha,m_j)$, $b_j(y,\xi)$ is a smooth function of y with values in Σ^{μ_j}, and, as in (2.43),

$$m_j \le m, \quad \mu_j \le \mu, \quad m_j+\mu_j \to -\infty, \text{ as } j \to \infty. \tag{2.55}$$

The meaning of this is that, for any K, if N is large enough,

$$\hat{k}(y,\xi) - \sum_{j\le N} a_j(y,\xi)b_j(y,\xi) \in S^{-K}_{\rho\#}, \quad \text{smoothly in } y, \tag{2.56}$$

where ρ is some number in $(0,1]$. The basic operator calculi developed so far apply. We have the following consequences of Proposition 2.7 and Proposition 2.8.

Proposition 2.11. *If G is step 2 and $\mathfrak{P}_j \in OP\widetilde{\mathfrak{H}}^{m_j,\mu_j}_{\alpha,\delta}$, then*

$$\mathfrak{P}_1\mathfrak{P}_2 \in OP\widetilde{\mathfrak{H}}^{m_1+m_2,\mu_1+\mu_2}_{\alpha,\delta}, \tag{2.57}$$

and

$$\mathfrak{P}^* \in OP\widetilde{\mathfrak{H}}^{m_j,\mu_j}_{\alpha,\delta}. \tag{2.58}$$

We are also able to obtain the following, in parallel with Proposition 2.8:

Proposition 2.12. *If G is step 2 and $\mathfrak{P} \in OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$, then $\mathfrak{P}$ has an asymptotic expansion*

$$\mathfrak{P} \sim \sum_{j \geq 0} \mathfrak{K}_j \mathfrak{L}_j, \tag{2.59}$$

where

$$\mathfrak{K}_j \in OP\widetilde{\mathfrak{H}}(G, \alpha, m_j), \quad \mathfrak{L}_j \in OPS^{\mu_j}, \tag{2.60}$$

with

$$m_j \leq m, \quad \mu_j \leq \mu, \quad m_j + \mu_j \to -\infty. \tag{2.61}$$

Proof. This goes like the proof of Proposition 2.8. We can reduce our problem to considering

$$\hat{p}_y(\xi) = \hat{k}_y(\xi)\hat{\ell}_y(\xi), \quad \hat{k}_y(\xi) \in \mathfrak{H}(G, \alpha, m), \quad \hat{\ell}_y(\xi) \in \Sigma^\mu. \tag{2.62}$$

Then $\mathfrak{K}_0 u = k_x * u(x)$ defines an operator $\mathfrak{K}_0 \in OP\widetilde{\mathfrak{H}}(G, \alpha, m)$ and $\mathfrak{L}_0 u = \ell_x * u(x)$ defines $\mathfrak{L}_0 \in OPS^\mu$. If we apply Proposition 2.11 to $\mathfrak{K}_0\mathfrak{L}_0$ we see that $\mathfrak{K}_0\mathfrak{L}_0$ belongs to $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$ and the principal term in the expansion is given by (2.62). Applying the same reasoning to the remainder term and iterating repeatedly, we get (2.59).

As in Proposition 2.9, we deduce the following L^p boundedness.

Proposition 2.13. *If G is step 2 and $\mathfrak{P} \in OP\widetilde{\mathfrak{H}}^{0,0}_{\alpha,\delta}$, then*

$$\mathfrak{P} : L^p_{\text{comp}} \longrightarrow L^p_{\text{loc}}, \quad 1 < p < \infty. \tag{2.63}$$

Proof. Using the decomposition (2.59), it suffices to invoke Proposition 2.6 and the well known L^p boundedness of $OPS^0_{1,0}$.

3. Symbols

In §§1–2 we have developed what one would call an "operator calculus." We want to elevate this to a "symbol calculus." The symbol of an operator in $OP\widetilde{\mathfrak{X}}^m$ will be described in terms of the irreducible unitary representations of G.

Let $\widehat{G}$ denote the set of (equivalence classes of) such representations. The symbol of a convolution operator $Ku = k * u$ will be defined as

$$\sigma_K(x,\pi) = \pi(k), \quad x \in G,\ \pi \in \widehat{G}, \tag{3.1}$$

where

$$\pi(k) = \int_G k(y)\pi(y)dm(y). \tag{3.2}$$

As before, dm denotes Haar measure on G; in local coordinates $dm(y) = H_1(y)\,dy$. At least if $k \in L^1(G)$, (3.2) is well defined as a bounded operator on the representation space H_π of π. We briefly set up machinery to define $\pi(k)$ for more singular $k \in \mathcal{E}'(G)$.

A vector $v \in H_\pi$ is called a smooth vector (we write $v \in H_\pi^\infty$) if the function $\varphi_v(x) = \pi(x)v$ is C^∞ from G to H_π. Following Rockland [R2], we call v weakly smooth (and write $V \in H_\pi^{\infty w}$) if the function $\varphi_{v,w}(x) = (\pi(x)v, w)$ is C^∞ on G for each $w \in H_\pi$. As is well known, H_π^∞ contains the Gårding space $\{\pi(\psi) : \psi \in C_0^\infty(G)\}$ and consequently is dense in H_π. Clearly $H_\pi^\infty \subset H_\pi^{\infty w}$. As a matter of fact, these two spaces coincide:

$$H_\pi^\infty = H_\pi^{\infty w}. \tag{3.3}$$

To see this, let $\Phi(x)$ be a function from an open set $U \subset G$ (which we identify with a ball in $\mathbb{R}^n$) to H_π that is weakly C^1. Thus we have $\partial_j(\Phi(x), w) = \Psi_j(x, w)$ defined and linear in w, and continuous in x for each w. In particular $(\Phi(x), w)$ is Lipschitz in x for each $w \in H$, i.e.,

$$\Big(\frac{\Phi(x) - \Phi(y)}{|x-y|}, w\Big)$$

is bounded on $U \times U$, for each $w \in H$. The uniform boundedness theorem then implies that there exists $L < \infty$ such that

$$\frac{\|\Phi(x) - \Phi(y)\|}{|x-y|} \le L, \tag{3.4}$$

i.e., Φ is strongly Lipschitz. Since

$$\Psi_j(x,w) = \lim_{h\to 0} h^{-1}\big((\Phi(x+he_j) - \Phi(x)), w\big),$$

(where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$) we deduce that

$$\|\Psi_j(x,w)\| \leq L\|w\|,$$

hence

$$\Psi_j(x,w) = (\Psi_j(x), w), \quad \Psi_j : U \to H.$$

Furthermore if Φ is weakly C^k then each Ψ_j is weakly C^{k-1}, hence strongle Lipschitz, if $k \geq 2$. Continuing this argument we see that Φ is strongly C^{k-1} whenever it is weakly C^k, and this gives (3.3).

Now, given $k \in \mathcal{E}'(G)$, we define

$$\pi(k) : H_\pi^\infty \longrightarrow H_\pi \tag{3.5}$$

by

$$(\pi(k)v, w) = \langle \varphi_{v,w}, k\rangle. \tag{3.6}$$

It is routine to check that this is well defined. Also, for $v \in H_\pi^\infty,\ w \in H_\pi$,

$$(\pi(x)\pi(k)v, w) = \langle \varphi_{v,w}, \rho_x k\rangle,$$

where $\rho_x k(y) = k(x^{-1}y)$ for $k \in C_0^\infty$, extending to $k \in \mathcal{E}'(G)$, and this makes it clear that $\pi(k)v \in H_\pi^{\infty w}$. In view of (3.3), we have

$$\pi(k) : H_\pi^\infty \longrightarrow H_\pi^\infty, \quad \text{for} \quad k \in \mathcal{E}'(G). \tag{3.7}$$

It is routine to verify that, for $k_1, k_2 \in \mathcal{E}'(G),\ k_1 * k_2 \in \mathcal{E}'(G)$ and

$$\pi(k_1 * k_2) = \pi(k_1)\pi(k_2), \tag{3.8}$$

and, if $k^\vee(x) = \overline{k(x^{-1})}$ for $k \in C_0^\infty$, then $k \mapsto k^\vee$ extends to $k \in \mathcal{E}'$, and

$$\pi(k^\vee) = \pi(k)^*. \tag{3.9}$$

With these preliminaries out of the way, we define the symbol of an operator

$$\mathfrak{K}u(x) = \int k(x, xy^{-1})u(y)H_1(y)\,dy \tag{3.10}$$

for $k(x,\cdot)$ a smooth function of x with values in $\mathcal{E}'(G)$, by

$$\sigma_{\mathfrak{K}}(x,\pi) = \pi(k_x), \tag{3.11}$$

with $k_x(y) = k(x,y)$. Thus, loosely speaking,

$$\sigma_{\mathfrak{K}}(x,\pi) = \int\limits_G k(x,y)\pi(y)\,dm(y). \tag{3.12}$$

By (3.8), we have the following formula for the symbol of the term $\mathfrak{C}_\gamma$

$$\mathfrak{C}_\gamma u(x) = A^{[\gamma]}(x)B_{[\gamma]}(x)u(x) \tag{3.13}$$

in the expansion (1.41) for a composite map $\mathfrak{A}\mathfrak{B}$:

$$\sigma_{\mathfrak{C}_\gamma}(x,\pi) = \pi(A^{[\gamma]}(x))\pi(B_{[\gamma]}(x)). \tag{3.14}$$

Thus we can rewrite (1.41) as

$$\sigma_{\mathfrak{A}\mathfrak{B}}(x,\pi) \sim \sum_{\gamma\geq 0} \pi(A^{[\gamma]}(x))\pi(B_{[\gamma]}(x)), \tag{3.15}$$

and in particular

$$\sigma_{\mathfrak{C}}(x,\pi) = \sigma_{\mathfrak{A}}(x,\pi)\sigma_{\mathfrak{B}}(x,\pi), \quad \mathfrak{A}\mathfrak{B} - \mathfrak{C} \in OP\widetilde{\mathfrak{X}}^{m+\mu-\tau}. \tag{3.16}$$

Similarly, on the symbol level, we can write the asymptotic expansion (1.51) for the adjoint $\mathfrak{K}^*$ of $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}^m$ as

$$\sigma_{\mathfrak{K}^*}(x,\pi) \sim \sum_{\gamma\geq 0} \pi(K_{\{\gamma\}}(x)). \tag{3.17}$$

In this case we have

$$\sigma_{\mathfrak{K}^\#}(x,\pi) = \sigma_{\mathfrak{K}}(x,\pi)^*, \quad \mathfrak{K}^* - \mathfrak{K}^\# \in OP\widetilde{\mathfrak{X}}^{m-\tau}. \tag{3.18}$$

The problem that remains in producing symbol calculi is to specify classes $\mathfrak{X}^m$, satisfying (1.25)–(1.27), (1.42)–(1.43), and (1.46), and to describe such classes in terms of their symbols. Important classes of such symbols will arise in the next chapter.

We make some further remarks about the continuity of operators on L^2. If G is a unimodular type I group, there is a Plancherel measure μ on $\widehat{G}$ such that

$$\|u\|^2_{L^2(G)} = \int\limits_{\widehat{G}} \|\pi(u)\|^2_{\mathrm{HS}}\,d\mu(\pi). \tag{3.19}$$

Consequently, a convolution operator $Ku = k * u$ is continuous on $L^2(G)$ if and only if the operator norms of $\pi(k)$ are uniformly bounded, as π ranges over the support of μ in $\widehat{G}$. As for operators of the form (1.10)–(1.11), we have the following result:

Proposition 3.1. *Let $\mathfrak{X}$ be a Frechet space such that for each convolution operator $Ku = k * u$, $K \in OP\mathfrak{X}$, the operator norms $\|\pi(k)\|$ are uniformly bounded as π runs over $\widehat{G}$. Then each $\mathfrak{K} \in OP\widetilde{\mathfrak{X}}$ is continuous on L^2, provided G is a unimodular type I group.*

Proof. We can regard

$$\mathfrak{K}_0 u(x) = K(y)u(x) = v(y,x) \tag{3.20}$$

as a convolution operator on G from scalar functions to functions taking values in the Sobolev space $H^K(G_y)$, with K picked arbitrarily large. Since the operator norms of $\pi(D_y^\alpha k(y,\cdot))$ are uniformly bounded as π runs over $\widehat{G}$, and y runs over a neighborhood of $e \in G$, for each α, it follows that

$$\mathfrak{K}_0 : L^2(G) \longrightarrow L^2(G, H^K). \tag{3.21}$$

If K is picked large enough to apply the Sobolev imbedding theorem, we deduce that for $u \in L^2$, $\mathfrak{K}u(x) = v(x,x)$ also belongs to L^2, which completes the proof.

Let us note that, if G is a type I unimodular group, the Plancherel formula (3.19) polarizes to give

$$(u,v) = \int \operatorname{Tr}\big(\pi(v)^*\pi(u)\big)\, d\mu(\pi). \tag{3.22}$$

If we let v be an approximate identity and pass to the limit, and if we have control over the trace norm of $\pi(u)$, we obtain the inversion formula

$$u(x) = \int_{\widehat{G}} \operatorname{Tr}\big(\pi(x)^*\pi(u)\big)\, d\mu(\pi), \quad x \in G. \tag{3.23}$$

Now, since $\pi(k * u) = \pi(k)\pi(u)$, we get the following formula for the operator $\mathfrak{K}$ given by (3.10)–(3.12) (granted that, for $u \in C_0^\infty(G)$, $\|\pi(u)\|_{\mathrm{Tr}}$ is under control):

$$\mathfrak{K}u(x) = \int_{\widehat{G}} \operatorname{Tr}\big(\pi(x)\sigma_{\mathfrak{K}}(x,\pi)\pi(u)\big)\, d\mu(\pi). \tag{3.24}$$

Note the direct parallel with the formula for a pseudodifferential operator on $\mathbb{R}^n$:

$$p(x,D)u(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi)\hat{u}(\xi)\, d\xi. \tag{3.25}$$

Actually, however neat this parallel is, we will not make much direct use of the formula (3.24).

Chapter II. Harmonic analysis on the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ has the simplest representation theory of all noncommutative Lie groups. We develop several classes of right invariant pseudodifferential operators on $\mathbb{H}^n$. A point worth emphasizing is that we give a unified treatment of convolution operators with the type of homogeneity related to certain automorphisms of $\mathbb{H}^n$ and also of the classical pseudodifferential operators, with the usual Euclidean homogeneity. Using harmonic analysis on $\mathbb{H}^n$ to study this last class is not as direct as studying OPS^m via Fourier analysis on $\mathbb{R}^n$, but once one puts forth the effort to do it this way, one is rewarded with natural amalgamations of classes of operators with different types of homogeneity. This facilitates doing microlocal analysis on the Heisenberg group.

In §1 we define the Heisenberg group, as $\mathbb{R}^{2n+1}$, parametrized by (t, q, p) with $t \in \mathbb{R},\ q, p \in \mathbb{R}^n$, with a particular group law. We describe the irreducible unitary representations of $\mathbb{H}^n$, consisting of the one-parameter families $\pi_{\pm\lambda}$ of infinite dimensional representations and the $2n$-parameter family $\pi_{(y,\eta)}$ of one dimensional representations. We study the images of convolution operators under these representations. In particular $\pi_{\pm\lambda}(k)$ is given in terms of the Weyl calculus. To $Ku = k * u$ we associate the symbols

$$\sigma_K(\pm\lambda)(X, D) = \pi_{\pm\lambda}(k) = \hat{k}(\pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D). \tag{0.1}$$

The Weyl calculus plays a central role in the work in this section. A brief treatment of this subject is given in Appendix A, at the end of this paper.

In §2 we study convolution pseudodifferential operators, homogeneous with respect to the dilations on $\mathbb{H}^n$ of the form $\alpha(s)(t, q, p) = (\sigma^2 t, \sigma q, \sigma p),\ \sigma = e^s$, which are group automorphisms. These operators we denote $OP\Psi_0^m$; they are the same as $OP\mathfrak{H}(\mathbb{H}^n, \alpha, m)$, which made an appearance in Chapter I, §2, in a more general context. Their symbols satisfy

$$\sigma_K(\pm\lambda)(X, D) = \lambda^{m/2}\sigma_K(\pm 1)(X, D), \tag{0.2}$$

and we characterize which operators can be put on the right side of (0.2) to actually define symbols of operators in $OP\Psi_0^m$. This characterization is a crucial tool in the development of our symbol calculus for $OP\Psi_0^m$ and its natural extension, $OP\Psi^m$, involving lower order terms. We develop this symbol calculus and use it in §2 to produce a hypoellipticity criterion for operators in $OP\Psi^m$, including in particular the Heisenberg Laplacian $\mathcal{L}_0$ and variants, $\mathcal{L}_\alpha = \mathcal{L}_0 + i\alpha T$.

In §3 we study the class $OP\Sigma^m$ of convolution operators that are classical pseudodifferential operators, via their symbols. We have $OP\Sigma^m = OP\mathfrak{H}(\mathbb{H}^n, \delta, m)$, where $\delta(s)$ is the family of Euclidean dilations of $\mathbb{H}^n$. Then we study amalgamations of $OP\Psi^*$ and $OP\Sigma^*$. We define the class

$$OP\Omega^{m,k}, \tag{0.3}$$

related to a class of operators studied by Boutet de Monvel. These classes are somewhat larger than the classes $OP\mathfrak{H}^{m',\mu'}_{\alpha,\delta}$, introduced (for more general 2-step nilpotent groups) in Chapter I, §2.

In §4 we study functions of the Heisenberg Laplacian $\mathcal{L}_0$, including fractional powers $(-\mathcal{L}_0)^\gamma$, the "heat" semigroup $e^{s\mathcal{L}_0}$, and the Poisson semigroup $e^{-s(-\mathcal{L}_0)^{1/2}}$. In §5 we produce some results on the Heisenberg wave equation $\partial_s^2 u - \mathcal{L}_0 u = 0$. We conclude this chapter with §6, discussing a hypoellipticity result of Rothschild.

1. Convolution operators on the Heisenberg group

The purpose of this section is to achieve a basic understanding of the symbol $\pi(k)$ of a convolution operator $Ku = k * u$ where k is a compactly supported function (or distribution) on the Heisenberg group. The material in the first half of this section is well known; we collect it for use in the next section. We begin with a brief description of the Heisenberg group $\mathbb{H}^n$ and its irreducible unitary representations. For a more complete discussion, see the first chapter of [T5].

As a C^∞ manifold, $\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$. Let us denote a point in $\mathbb{H}^n$ by (t, q, p), with $t \in \mathbb{R}$, $q, p \in \mathbb{R}^n$. The group law is given by

$$(1.1) \qquad (t_1, q_1, p_1) \cdot (t_2, q_2, p_2) = (t_1 + t_2 + \tfrac{1}{2}(p_1 \cdot q_2 - q_1 \cdot p_2), q_1 + q_2, p_1 + p_2).$$

The Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$ is spanned by the right invariant vector fields

$$(1.2) \qquad T = \frac{\partial}{\partial t}, \quad L_j = \frac{\partial}{\partial q_j} - \frac{p_j}{2}\frac{\partial}{\partial t}, \quad M_j = \frac{\partial}{\partial p_j} + \frac{q_j}{2}\frac{\partial}{\partial t}, \quad 1 \le j \le n.$$

Note that

$$(1.3) \qquad [L_j, M_j] = -[M_j, L_j] = T,$$

all other commutators being zero.

For $\lambda \in (0, \infty)$, irreducible unitary representations of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$ are given by

$$(1.4) \qquad \pi_{\pm\lambda}(t, q, p) = e^{i(\pm\lambda t \pm \lambda^{1/2} q \cdot X + \lambda^{1/2} p \cdot D)},$$

where $Q \cdot X$ is the multiplication operator defined by

$$(1.5) \qquad (Q \cdot X)u(x) = \sum q_j x_j u(x),$$

and $p \cdot D$ is the differential operator given by

$$(1.6) \qquad (P \cdot D)u(x) = \frac{1}{i} \sum p_j \frac{\partial u}{\partial x_j}.$$

An alternative formula, equivalent to (1.4), is

$$(1.7) \qquad \pi_{\pm\lambda}(t, q, p)u(x) = e^{i(\pm\lambda t \pm \lambda^{1/2} q \cdot x \pm \lambda q \cdot p/2)}\, u(x + \lambda^{1/2} p).$$

There are also one-dimensional representations $\pi_{(y,\eta)}$, for $(y, \eta) \in \mathbb{R}^{2n}$, given by

$$(1.8) \qquad \pi_{(y,\eta)}(t, q, p)\, v = e^{i(y \cdot q + \eta \cdot p)}\, v, \quad v \in \mathbb{C}.$$

It is the content of the Stone-von Neumann theorem that any irreducible unitary representation of $\mathbb{H}^n$ is unitarily equivalent to one of those just described.

The Plancherel identity on $\mathbb{H}^n$ is

$$\int_{\mathbb{H}^n} |u(x)|^2\,dx = c_n \int_{\mathbb{R}\setminus 0} \|\pi_\lambda(u)\|^2_{\text{HS}}\,|\lambda|^n\,d\lambda, \tag{1.9}$$

where $\|T\|^2_{\text{HS}} = \text{Tr}\,(T^*T)$ is the squared Hilbert-Schmidt norm of T. (Haar measure on $\mathbb{H}^n$ coincides with Lebesgue measure on $\mathbb{R}^{2n+1}$.) A proof of (1.9) will be given below. In particular, the set of one-dimensional representations (1.8) has Plancherel measure zero.

Given a compactly supported function (or distribution) k on $\mathbb{H}^n$, we want to understand $\pi_{\pm\lambda}(k)$ and $\pi_{(y,\eta)}(k)$. Indeed (1.4) gives

$$\begin{aligned}\pi_{\pm\lambda}(k) &= \int k(t,q,p)e^{i(\pm\lambda t \pm \lambda^{1/2} q\cdot X + \lambda^{1/2} p\cdot D)}\,dt\,dq\,dp\\ &= \hat{k}(\pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D),\end{aligned} \tag{1.10}$$

where $\hat{k}(\tau,y,\eta)$ denotes the Euclidean space (inverse) Fourier transform

$$\hat{k}(\tau,y,\eta) = \int k(t,q,p)e^{i(t\tau + q\cdot y + \eta\cdot p)}\,dt\,dq\,dp, \tag{1.11}$$

and the operator $a(X,D)$ is defined by the Weyl functional calculus:

$$a(X,D) = \int \hat{a}(q,p)e^{i(q\cdot X + p\cdot D)}\,dq\,dp, \tag{1.12}$$

$\hat{a}(q,p)$ denoting the Fourier transform of a. Such operators have been studied by several people, including Grossman, Loupias and Strin [G11], Voros [V3], Hörmander [H10], and Howe [H11]; see also Nelson [N4] and Anderson [A1]. Background material on the Weyl calculus is collected in Appendix A at the end of this paper. Here we mention that a few manipulations of integrals give the following formula for $a(X,D)$:

$$a(X,D)u(x) = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} a\big(\tfrac{1}{2}(x+y),\xi\big)u(y)\,dy\,d\xi. \tag{1.13}$$

To restate (1.10), we have

$$\pi_{\pm\lambda}(k) = \sigma_K(\pm\lambda)(X,D), \tag{1.14}$$

where

$$\sigma_K(\pm\lambda)(x,\xi) = \hat{k}(\pm\lambda, \pm\lambda^{1/2}x, \lambda^{1/2}\xi), \tag{1.15}$$

or equivalently

$$\hat{k}(\pm\tau, y, \eta) = \sigma_K(\pm\tau)(\pm\tau^{-1/2}y, \tau^{-1/2}\eta), \quad \tau > 0. \tag{1.16}$$

The behavior of $\pi_{(y,\eta)}(k)$ is given simply by

$$\begin{aligned} \pi_{(y,\eta)}(k) &= \int k(t,q,p)e^{i(y\cdot q+\eta\cdot p)}\,dt\,dq\,dp \\ &= \hat{k}(0,y,\eta). \end{aligned} \tag{1.17}$$

Regarding the associated representation of the Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$, we have

$$\pi_{\pm\lambda}(T) = \pm i\lambda, \quad \pi_{\pm\lambda}(L_j) = \pm i\lambda^{1/2}x_j, \quad \pi_{\pm\lambda}(M_j) = \lambda^{1/2}\frac{\partial}{\partial x_j}, \tag{1.18}$$

and

$$\pi_{(y,\eta)}(T) = 0, \quad \pi_{(y,\eta)}(L_j) = iy_j, \quad \pi_{(y,\eta)}(M_j) = i\eta_j. \tag{1.19}$$

We now show how formulas (1.10)–(1.13) give a proof of the Plancherel formula (1.9). Note that the squared Hilbert-Schmidt norm of an operator $Au(u) = \int A(x,y)u(y)\,dy$ is $\iint |A(x,y)|^2\,dx\,dy$. Thus (1.10) and (1.13) imply

$$\begin{aligned} c_n\|\pi_{\pm\lambda}(k)\|_{\mathrm{HS}}^2 &= \int\limits_{\mathbb{R}^{2n}} |\hat{k}(\pm\lambda, \pm\lambda^{1/2}x, \lambda^{1/2}\xi)|^2\,dx\,d\xi \\ &= |\lambda|^{-n}\int\limits_{\mathbb{R}^{2n}} |\hat{k}(\pm\lambda, y, \eta)|^2\,dy\,d\eta, \end{aligned} \tag{1.20}$$

so

$$c_n\int_{-\infty}^{\infty} \|\pi_\lambda(k)\|_{\mathrm{HS}}^2\,|\lambda|^n\,d\lambda = \int\limits_{\mathbb{R}^{2n+1}} |\hat{k}(\lambda,y,\eta)|^2\,d\lambda\,dy\,d\eta. \tag{1.21}$$

Now (1.9) follows from this, together with the ordinary Euclidean space Plancherel theorem:

$$\int\limits_{\mathbb{H}^n} |k(z)|^2\,dz = \int\limits_{\mathbb{R}^{2n+1}} |\hat{k}(\lambda,y,\eta)|^2\,d\lambda\,dy\,d\eta. \tag{1.22}$$

This completes the proof of (1.9). Note that polarization of (1.9) gives

$$\int\limits_{\mathbb{H}^n} f(z)\overline{g(z)}\,dz = c_n\int_{-\infty}^{\infty} \mathrm{Tr}\big(\pi_\lambda(g)^*\pi_\lambda(f)\big)|\lambda|^n\,d\lambda. \tag{1.23}$$

If we replace g by a sequence in $C_0^\infty(\mathbb{H}^n)$ tending to the delta function and pass to the limit, we get the inversion formula

$$f(z) = c_n \int_{-\infty}^{\infty} \mathrm{Tr}\big(\pi_\lambda(z)^* \pi_\lambda(f)\big) |\lambda|^n \, d\lambda. \tag{1.24}$$

We can use the Plancherel formula to estimate the L^2-operator norm of a convolution operator $Ku = k * u$. In fact, (1.9) implies

$$\|K\|_{\mathcal{L}(L^2)} = \sup_\lambda \|\sigma_K(\pm\lambda)(X, D)\|, \tag{1.25}$$

where the latter norm is the operator norm on $L^2(\mathbb{R}^n)$. We can estimate this operator norm via the following special case of the Calderon-Vaillancourt theorem. Suppose $a(x,\xi)$ satisfies the estimates

$$|D^\alpha_{x,\xi} a(x,\xi)| \le A, \quad |\alpha| \le K(n), \tag{1.26}$$

where $K(n)$ is sufficiently large. Then

$$\|a(X,D)\| \le C(n)A. \tag{1.27}$$

A proof of this can be found in Appendix A. In light of (1.25) and (1.15), this implies the following estimate on $\|K\|$:

$$\|K\|_{\mathcal{L}(L^2)} \le C(n) \sup_{\pm\lambda, y, \eta} \sup_{|\alpha| \le K(n)} \lambda^{|\alpha|/2} \, |D^\alpha_{y,\eta} \hat{k}(\pm\lambda, y, \eta)|. \tag{1.28}$$

In particular, $K : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ if $\hat{k} \in S^0_{1/2\#}$, where we recall from Chapter I that $\hat{k} \in S^m_{1/2\#}$ means

$$|D^\alpha_{\tau,y,\eta} \hat{k}(\tau, y, \eta)| \le C_\alpha (1 + |\tau| + |y| + |\eta|)^{m - |\alpha|/2}. \tag{1.29}$$

Note that Proposition 1.1 of Chapter I implies $K \in OPS^0_{1/2,1/2}$ in this case, so the L^2 boundedness of K when $\hat{k} \in S^0_{1/2\#}$ also follows directly from the Calderon-Vaillancourt theorem for $OPS^0_{1/2,1/2}$, as already noted in Chapter I.

One significant structure that accompanies the Heisenberg group is the family of dilations

$$\alpha_{\pm\lambda}(t, q, p) = (\pm\lambda t, \pm\lambda^{1/2} q, \lambda^{1/2} p), \quad \lambda > 0. \tag{1.30}$$

These are all automorphisms of $\mathbb{H}^n$. Note that

$$\pi_{\pm\lambda}(w) = \pi_1(\alpha_{\pm\lambda} w), \quad w \in \mathbb{H}^n. \tag{1.31}$$

If we let α_λ^* act on $C_0^\infty(\mathbb{H}^n)$ (or on $\mathcal{E}'(\mathbb{H}^n)$ and other spaces) by

$$\alpha_\lambda^* u(w) = u(\alpha_\lambda w), \tag{1.32}$$

then

$$\pi_\lambda(\alpha_\tau^* u) = |\tau|^{-n-1}\pi_{\lambda/\tau}(u). \tag{1.33}$$

We make a few remarks on the Schwartz space $\mathcal{S}(\mathbb{H}^n)$ of functions that, together with all their derivatives, are rapidly decreasing on $\mathbb{H}^n = \mathbb{R}^{2n+1}$. Note that we could equally specify that $X_1 \cdots X_k u$ be rapidly decreasing for any right invariant vector fields X_j. It is easy to see that $\mathcal{S}(\mathbb{H}^n)$ is a convolution algebra:

$$u, v \in \mathcal{S}(\mathbb{H}^n) \Longrightarrow u * v \in \mathcal{S}(\mathbb{H}^n); \tag{1.34}$$

here we use the Heisenberg group convolution. If $u \in \mathcal{S}(\mathbb{H}^n)$, then $\hat{u}(\tau, y, \eta) \in \mathcal{S}(\mathbb{R}^{2n+1})$, and vice-versa. Note that

$$\pi_{\pm\lambda}(u) = \hat{u}(\pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D)$$

is hence a rapidly decreasing function of λ with values in $OP\mathcal{S}_1^{-\infty}$ as $\lambda \to \infty$, where we define the class $\mathcal{S}_1^m$ to consist of $a(x,\xi)$ such that

$$|D_x^\beta D_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1 + |x| + |\xi|)^{m-|\alpha|-|\beta|}.$$

There is a slightly delicate matter of specifying whether u belongs to $\mathcal{S}(\mathbb{H}^n)$ purely in terms of

$$\pi_{\pm\lambda}(u) = \sigma_u(\pm\lambda)(X, D), \tag{1.35}$$

namely to specify adequately the behavior of $\pi_{\pm\lambda}(u)$ as $\lambda \to 0$. Of course, since

$$\hat{u}(\pm\lambda, y, \eta) = \sigma_u(\pm\lambda)(\pm\lambda^{-1/2}y, \lambda^{-1/2}\eta), \tag{1.36}$$

we could simply specify that the right side of (1.36) define an element of $\mathcal{S}(\mathbb{R}^{2n+1})$, but this is not very explicit. We call the reader's attention to work of Geller [G3], describing $\mathcal{S}(\mathbb{H}^n)$ via $\pi_{\pm\lambda}(u)$.

We now describe a certain transform of $u \in \mathcal{S}(\mathbb{H}^n)$, which struck the author as unexpected and amusing. This transform also helps to establish a certain technical point later in this section. To $u \in \mathcal{S}(\mathbb{H}^n)$, associate $\kappa(u)$ defined by inverting the paramater λ:

$$\sigma_{\kappa(u)}(\pm\lambda)(X, D) = \sigma_u(\pm\lambda^{-1})(X, D). \tag{1.37}$$

This is equivalent to saying

$$\widehat{\kappa(u)}(\pm\tau, y, \eta) = \hat{\ell}(\pm\tau, y, \eta) = \hat{u}(\pm\tau^{-1}, \tau^{-1}y, \tau^{-1}\eta),$$

or

$$\hat{u}(\pm\lambda, y, \eta) = \hat{\ell}(\pm\lambda^{-1}, \lambda^{-1}y, \lambda^{-1}\eta).$$

Thus it is clear that $\ell = \kappa(u)$ is a well defined element of $\mathcal{S}'(\mathbb{R}^{2n+1})$. Indeed, as we will see, it is quite a special element.

If we first restrict attention to $\hat{u}$ on the upper half space $\lambda \geq 0$, we see that $\hat{u}$ coincides with the restriction to $\{\lambda \geq 0\}$ of an element of $\mathcal{S}(\mathbb{R}^{2n+1})$ if and only if $\hat{\ell}(\tau, y, \eta)$, restricted to $\tau \geq 0$, is equal to $\hat{\ell}_+(\tau, y, \eta)$, with the following three properties:

$$\hat{\ell}_+(\tau, y, \eta) \in S^0_{1\#}(\mathbb{R}^{2n+1}), \tag{1.38}$$

$$\hat{\ell}_+(\tau, y, \eta) \sim \sum_{j\geq 0} \hat{\ell}_j(\tau, y, \eta), \tag{1.39}$$

with $\hat{\ell}_j$ homogeneous of degree $-j$ in (τ, y, η), and

$$\hat{\ell}_+(\tau, y, \eta) = 0 \quad \text{for} \;\; \tau < 0. \tag{1.40}$$

Note that (1.38) in particular says $\hat{\ell}_+(\tau, y, \eta)$ is smooth in all of $\mathbb{R}^{2n+1}$, so (1.40) implies $\hat{\ell}(\tau, y, \eta)$ vanishes to infinite order at $\tau = 0$, for $\tau \geq 0$. In addition we see that $\hat{u}$ on $\lambda \leq 0$ coincides with the restriction to $\{\lambda \leq 0\}$ of an element of $\mathcal{S}(\mathbb{R}^{2n+1})$ if and only if $\hat{\ell} = \hat{\ell}_-$ on $\tau < 0$, where $\hat{\ell}_-$ satisfies (1.38), (1.39), and, instead of (1.40), we have

$$\hat{\ell}_-(\tau, y, \eta) = 0 \quad \text{for} \;\; \tau > 0. \tag{1.41}$$

To see whether $\hat{u}$ satisfying both these conditions is actually in $\mathcal{S}(\mathbb{R}^{2n+1})$ we need to know whether all the appropriate compatibility conditions hold at $\lambda = 0$. It is simplest to state these if we break up $\hat{u}$ into its even and odd parts:

$$\hat{u} = \hat{u}_e + \hat{u}_o, \tag{1.42}$$

$$\hat{u}_e(\lambda, y, \eta) = \hat{u}_e(-\lambda, y, \eta), \quad \hat{u}_o(\lambda, y, \eta) = -\hat{u}_o(-\lambda, y, \eta). \tag{1.43}$$

Clearly the compatibility conditions are equivalent to

$$\begin{aligned} D^j_\lambda \hat{u}_e(0, y, \eta) &= 0 \quad \text{for} \;\; j \text{ odd}, \\ D^j_\lambda \hat{u}_o(0, y, \eta) &= 0 \quad \text{for} \;\; j \text{ even}, \end{aligned} \tag{1.44}$$

the latter case including $j = 0$. The associated decomposition

$$\hat{\ell}(\tau, y, \eta) = \hat{\ell}_e(\tau, y, \eta) + \hat{\ell}_o(\tau, y, \eta) \tag{1.45}$$

stores up these conditions in the form

$$\begin{aligned} \hat{\ell}_e(\tau, y, \eta) &\sim \sum_{j\geq 0} \hat{\ell}_{2j}(\tau, y, \eta), \\ \hat{\ell}_o(\tau, y, \eta) &\sim \sum_{j\geq 0} \hat{\ell}_{2j+1}(\tau, y, \eta), \end{aligned} \tag{1.46}$$

where $\hat{\ell}_k(\tau, y, \eta)$ is homogeneous of degree $-k$ in (τ, y, η). If we let $\kappa_1(u)$ denote the operator of left convolution by $\kappa(u)$, these observations produce the following:

Inversion Trick. *The transformation $u \mapsto \kappa_1(u)$ is an isomorphism of $\mathcal{S}(\mathbb{H}^n)$ onto the class of right invariant pseudodifferential operators on $\mathbb{H}^n$, in OPS^0, of the form $Lf = \ell * f$ ($\ell = \kappa(u)$) with $\hat{\ell}$ characterized by (1.38)–(1.41) and (1.45)–(1.46). If only (1.38)–(1.41) are considered, we get the isomorphic image of the "piecewise" elements $\hat{u}$ of $\mathcal{S}(\overline{\mathbb{R}^{2n+1}_+}) + \mathcal{S}(\overline{\mathbb{R}^{2n+1}_-})$. Furthermore, these isomorphisms are isomorphisms of convolution algebras:*

$$\kappa(u * v) = \kappa(u) * \kappa(v). \tag{1.47}$$

Note that (1.47) follows directly from (1.37). Note also that $\kappa(u) \in \mathcal{S}'(\mathbb{H}^n)$ is singular only at the origin (the identity element of $\mathbb{H}^n$) and is equal to an element of $\mathcal{S}(\mathbb{H}^n)$ outside any neighborhood of the origin. Let us denote by $OP\mathfrak{C}$ the set of convolution operators $Lu = \ell * u$ with $\hat{\ell}$ satisfying (1.38)–(1.41) and by $OP\mathfrak{C}_0$ those that also satisfy (1.45)–(1.46). Thus $OP\mathfrak{C}_0$ is the image of the convolution algebra $\mathcal{S}(\mathbb{H}^n)$. It follows that $OP\mathfrak{C}_0$ is a convolution algebra. It is clear that $OP\mathfrak{C}$ is a convolution algebra; hence the set of u with u a piecewise element of $\mathcal{S}(\mathbb{R}^{2n+1})$, with simple jump across $\lambda = 0$, is also a convolution algebra; let us call this convolution algebra $\hat{\mathcal{S}}_P(\mathbb{H}^n)$.

Let us note that $\hat{u}(\tau, y, \eta) \in \mathcal{S}(\mathbb{R}^{2n+1})$ vanishes to infinite order at $(\lambda, y, \eta) = 0$ if and only if $\hat{\ell}(\tau, y, \eta) \in S^0_{1\#}$ vanishes to infinite order along the rays $(\pm\tau, 0, 0)$, $\tau \to \infty$. We can restate this:

Corollary to inversion trick. *The map $u \mapsto \kappa_1(u)$ sets up an isomorphism between the set of $u \in \mathcal{S}(\mathbb{H}^n)$ such that $\hat{u}(\lambda, y, \eta)$ vanishes to infinite order at $(\lambda, y, \eta) = 0$ and the subalgebra of $OP\mathfrak{C}_0$ consisting of pseudodifferential operators whose full symbols vanish to infinite order on the conic subset of $T^*\mathbb{H}^n \setminus 0$ which is the right invariant set whose fiber over the origin is generated by dt.*

This line bundle Λ, which is being intersected with $T^*\mathbb{H}^n \setminus 0$, furnishes a contact structure on $\mathbb{H}^n$, and will arise in other contexts later. We remark that the property of a pseudodifferential operator in $OPS^m_{1,0}(\Omega)$ (or even more general classes) of having its complete symbol vanish to infinite order on a closed conic subset of $T^*(\Omega) \setminus 0$ is invariant.

Let us record the following definition:

$$u \in \mathcal{S}_{00}(\mathbb{H}^n) \Leftrightarrow \hat{u}(\tau, y, \eta) \text{ vanishes to infinite order at the origin.} \tag{1.48}$$

We will use the corollary, characterizing the image under κ of $\mathcal{S}_{00}(\mathbb{H}^n)$, to study the action on $\mathcal{S}_{00}(\mathbb{H}^n)$ of convolution operators $Lu = \ell * u$, where $\hat{\ell}$ is singular at the origin, so L does not map $\mathcal{S}(\mathbb{H}^n)$ into itself. This will be useful in streamlining some results to be developed in the next section. We begin by observing that $u \in \mathcal{S}_{00}(\mathbb{H}^n)$ if and only if $u \in \mathcal{S}(\mathbb{H}^n)$ and $\langle u, p \rangle = 0$ for every polynomial function on $\mathbb{H}^n = \mathbb{R}^{2n+1}$. Since the translate $p_y(x) = p(y^{-1}x)$ is a polynomial in x for each $y \in \mathbb{H}^n$ whenever $p(x)$ is a polynomial, we see that if $u \in \mathcal{S}_{00}(\mathbb{H}^n)$, then so is $u_y(x) = u(y^{-1}x)$. This easily gives:

$$u \in \mathcal{S}_{00}(\mathbb{H}^n),\ \ell \in \mathcal{E}'(\mathbb{H}^n) \Longrightarrow \ell * u \in \mathcal{S}_{00}(\mathbb{H}^n), \tag{1.49}$$

and

$$u \in \mathcal{S}_{00}(\mathbb{H}^n),\ \ell \in \mathcal{S}(\mathbb{H}^n) \Longrightarrow \ell * u \in \mathcal{S}_{00}(\mathbb{H}^n). \tag{1.50}$$

Of course, (1.50) can also be obtained from the corollary to the inversion trick, since $L_j \in OP\mathfrak{C}_0$ implies that $L_1 L_2 \in OP\mathfrak{C}_0$ has full symbol vanishing to infinite order at Λ provided either factor does.

We now want to study $Ku = k*u$ for $u \in \mathcal{S}_{00}(\mathbb{H}^n)$, where $\hat{k}(\tau, y, \eta)$ satisfies the following conditions:

$$\hat{k} \in C^\infty(\mathbb{R}^{2n+1} \setminus 0), \tag{1.51}$$

$$\hat{k}(\pm s\lambda, s^{1/2}y, s^{1/2}\eta) = s^{m/2}\hat{k}(\pm\lambda, y, \eta), \quad s > 0. \tag{1.52}$$

Note that (1.52) implies

$$\sigma_K(\pm\lambda)(X, D) = \lambda^{m/2}\sigma_K(\pm 1)(X, D). \tag{1.53}$$

Now if $\chi(y, \eta)$ is a smooth cut-off, equal to 1 for $|y| + |\eta| \le \sigma,\ 0$ for $|y| + |\eta| \ge 2\sigma$, and if $\psi(\lambda)$ is a smooth cut-off, equal to 1 for $|\lambda| \le \sigma,\ 0$ for $|\lambda| \ge 2\sigma$, write

$$\hat{k} = \hat{k}_1 + \hat{k}_2 = \chi\psi\hat{k} + (1 - \chi\psi)\hat{k}.$$

Thus $\hat{k}_1$ is supported on $|\lambda| \le 2\sigma, |y| + |\eta| \le 2\sigma$, while $\hat{k}_2 \in S^m_{1/2\#}$ vanishes near 0. It follows that k_2 is the sum of an element of $\mathcal{S}(\mathbb{H}^n)$ and an element of $\mathcal{E}'(\mathbb{H}^n)$, so, by (1.49)–(1.50), $k_2 u \in \mathcal{S}_{00}(\mathbb{H}^n)$ if $u \in \mathcal{S}_{00}(\mathbb{H}^n)$. Now we look at $k_1 * u$, for $u \in \mathcal{S}_{00}(\mathbb{H}^n)$. Note that, if

$$\ell = \kappa(k_1 * u),$$

we have, for $Lf = \ell * f$,

$$L \in OPS^0(\mathbb{H}^n),$$

since the full symbol of $\kappa_1(u)$ vanishes to infinite order at Λ. Hence,

$$L \in OP\mathfrak{C}.$$

This implies that $k_1 * u$ has Euclidean Fourier transform piecewise in $\mathcal{S}$ on $\overline{\mathbb{R}^{2n+1}_+}$ and $\overline{\mathbb{R}^{2n+1}_-}$. In fact, the full symbol of L vanishes outside a small conic neighborhood of Λ, if σ is chosen small, and this implies the Fourier transform of $k_1 * u$ vanishes to infinite order as $\pm\tau \to 0$, for $|y| + |\eta| \ge \sigma'$, where σ' is small if σ is. It follows that $Ku = k_1 * u + k_2 * u$ has Euclidean Fourier transform piecewise in $\mathcal{S}$ in $\overline{\mathbb{R}^{2n+1}_+}$ and $\overline{\mathbb{R}^{2n+1}_-}$, and these functions in the upper and lower half spaces match up at $\tau = 0$ to be smooth, provided $(y, \eta) \ne (0, 0)$. But the origin cannot carry any singularity, so in fact $Ku \in \mathcal{S}(\mathbb{H}^n)$. Furthermore, since the full symbol of $L = \kappa_1(k_1 * u)$ vanishes to infinite order at Λ, it follows that $k_1 * u \in \mathcal{S}_{00}(\mathbb{H}^n)$, so finally we have

$$u \in \mathcal{S}_{00}(\mathbb{H}^n) \Longrightarrow k * u \in \mathcal{S}_{00}(\mathbb{H}^n), \quad \text{if } \ k \ \text{ satisfies (1.51)–(1.52).} \tag{1.53}$$

Let us renotate the decomposition $k = k_1 + k_2$ by

$$k = k^o + k^b. \tag{1.54}$$

Here, k^o is singular only at the origin, and equal to an element of $\mathcal{S}(\mathbb{H}^n)$ outside a neighborhood of the origin, while k^b is C^∞, with $\hat{k}^b(\tau, y, \eta)$ supported in $|\tau| \le 2\sigma, |y|+|\eta| \le 2\sigma$. If k_j satisfy (1.51)–(1.52) with decompositions $k_j = k_j^o + k_j^b$ as in (1.54), set $K_j f = K_j^o f + K_j^b f = k_j^o * f + k_j^b * f$. Note that $K_j^o : \mathcal{S}(\mathbb{H}^n) \to \mathcal{S}(\mathbb{H}^n)$ and $K^b : \mathcal{E}'(\mathbb{H}^n) \to C^\infty(\mathbb{H}^n)$. Now

$$\begin{aligned} K_3(u - \chi u) = {} & k_1^o * k_2^o * (u - \chi u) + k_1^o * k_1^b * (u - \chi u) \\ & + k_1^b * k_2^o * (u - \chi u) + k_1^b * k_2^b * (u - \chi u), \end{aligned} \tag{1.55}$$

where K_3 is defined by

$$\sigma_{K_3}(\pm\lambda)(X, D) = \sigma_{K_1}(\pm\lambda)(X, D)\sigma_{K_2}(\pm\lambda)(X, D), \tag{1.56}$$

and χu is given by

$$\chi u = \psi * u \tag{1.57}$$

where $\hat{\psi}(\tau, y, \eta)$ is a smooth function, with compact support, equal to 1 for $|\tau| \le 2\sigma, |y| + |\eta| \le 2\sigma$. Note that the second and third terms on the right side of (1.55) represent convolution of $u - \chi u$ by C^∞ functions. This identity generalizes to more general distributions, in particular to $u = \delta$. We have

$$K_3(\delta - \chi\delta) = K_1^o K_2^o \delta + k_1^b * k_2^b * (\delta - \chi\delta) + \varphi, \quad \varphi \in C^\infty. \tag{1.58}$$

Now, under the transformation κ_1, we get from $k_1^b * k_2^b * (\delta - \chi\delta)$ the operator product of a pair of right invariant pseudodifferential operators with symbol supported in a certain conic neighborhood of Λ, with a right invariant pseudodifferential operator whose full symbol vanishes on a conic neighborhood of such support. Such a product is a smoothing operator, and hence belongs to the image under κ_1 of an element of $\mathcal{S}(\mathbb{R}^{2n+1})$. In other words,

$$K_3\delta = K_1^o K_2^o \delta + \tilde{\varphi}, \quad \tilde{\varphi} \in C^\infty(\mathbb{H}^n). \tag{1.59}$$

This enables one to manipulate some classes of pseudodifferential operators symbolically, in a clean fashion, without worrying too much about the necessity to smooth out the singularity of their symbols at the origin. Compare the remarks at the end of the proof of Proposition 2.6, in the next section.

2. Right invariant pseudodifferential operators on $\mathbb{H}^n$, I: α-homogeneous operators

Having made a basic study of convolution operators on $\mathbb{H}^n$, we now determine some classes of operators that deserve the exalted title of pseudodifferential operators. One important class is generated by the study of the "Heisenberg Laplacian"

$$\mathcal{L}_0 = \sum_{j=1}^{n} (L_j^2 + M_j^2), \tag{2.1}$$

a second-order nonelliptic (but hypoelliptic) operator. Note that, by (1.18)–(1.19),

$$\pi_{\pm\lambda}(\mathcal{L}_0) = \lambda \sum_{j=1}^{n} \Big(\frac{\partial^2}{\partial x_j^2} - x_j^2 \Big), \tag{2.2}$$

and

$$\pi_{(y,\eta)}(\mathcal{L}_0) = \sum_{j=1}^{n} (y_j^2 + \eta_j^2). \tag{2.3}$$

We define the following classes:

Definition 2.1. *The class Ψ_0^m consists of functions $\hat{k}(\tau, y, \eta)$, smooth except at 0, such that, for $\tau > 0$,*

$$\hat{k}(\pm\tau, y, \eta) = \tau^{m/2} \hat{k}(\pm 1, \tau^{-1/2} y, \tau^{-1/2} \eta). \tag{2.4}$$

Neglecting the singularity of the symbol at the origin, elements of Ψ_0^m belong to $S^m_{1/2\#}$ if $m \geq 0$ and to $S^{m/2}_{1/2\#}$ if $m < 0$. If $\hat{k} \in \Psi_0^m$, we say $K \in OP\Psi_0^m$, with $Ku = k * u$. Proposition 1.1 of Chapter I implies $OP\Psi_0^m \subset OPS^m_{1/2,1/2}$ for $m \geq 0$, $OPS^{m/2}_{1/2,1,2}$ for $m < 0$. In fact, using the dilations $\alpha(s)(t, q, p) = (\sigma^2 t, \sigma q, \sigma p)$, $\sigma = e^s$, we see that, modulo smoothing operators,

$$OP\Psi_0^m = OP\mathfrak{H}(\mathbb{H}^n, \alpha, m),$$

a special case of a class studied in Chapter I, §2. From (1.15), we have $K \in OP\Psi_0^m$ provided $\hat{k}(\tau, y, \eta)$ is smooth away from $(\tau, y, \eta) = 0$ and, for $\lambda > 0$,

$$\sigma_K(\pm\lambda)(x, \xi) = \lambda^{m/2} \sigma_K(\pm 1)(x, \xi). \tag{2.5}$$

Thus (2.2) gives

$$\mathcal{L}_0 \in OP\Psi_0^2. \tag{2.6}$$

Note also that

$$\pi_{\pm\lambda}(T) = \pm i\lambda, \tag{2.7}$$

and hence

$$T \in OP\Psi_0^2. \tag{2.8}$$

The following is a convenient characterization of Ψ_0^m. For a related result, see Geller [G2].

Proposition 2.2. *Let $a_\pm(x,\xi) \in C^\infty(\mathbb{R}^{2n})$. Then (2.5) defines an element of Ψ_0^m, with*

$$\sigma_K(\pm 1)(x,\xi) = a_\pm(x,\xi),$$

if and only if $a_\pm(x,\xi)$ have the compatible asymptotic expansions

$$a_\pm(ry, r\eta) \sim \sum_{j\geq 0} r^{m-2j}(\pm 1)^j \varphi_j(\pm y, \eta), \quad r \to +\infty, \tag{2.9}$$

for $|y|^2 + |\eta|^2 = 1$, $\varphi_j \in C^\infty(S^{2n-1})$.

Proof. We have

$$\hat{k}(\pm\tau, y, \eta) = \tau^{m/2} a_\pm(\pm\tau^{-1/2} y, \tau^{-1/2}\eta),$$

for $\tau \neq 0$. The only problem is to specify when this extends to a function smooth at $\tau = 0$, $(y,\eta) \neq (0,0)$. Writing

$$\begin{aligned}\hat{k}(\pm\tau, y, \eta) &\sim \sum_{j\geq 0} (\pm\tau)^j \varphi_j(y,\eta) \\ &= \tau^{m/2} a_\pm(\pm\tau^{-1/2} y, \tau^{-1/2}\eta),\end{aligned}$$

and changing variables, setting $r = \tau^{-1/2}$, makes the characterization (2.9) apparent.

A characterization equivalent to (2.9) is

$$\sigma_K(\pm 1)(x,\xi) \sim \sum_{j\geq 0} (\pm 1)^j \varphi_j(\pm x, \xi), \quad |x|^2 + |\xi|^2 \to \infty, \tag{2.10}$$

with $\varphi_j(x,\xi)$ homogeneous of degree $m - 2j$ in (x,ξ). Note from (1.17) that

$$\pi_{(y,\eta)}(k) = \varphi_0(y,\eta). \tag{2.11}$$

In other words, $\pi_{(y,\eta)}(k)$ is the principal symbol of the operator $\sigma_K(+1)(X,D)$, evaluated at (y,η).

Note in particular that if $\sigma_K(\pm 1)(x,\xi) \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions, then (2.5) defines an element of $OP\Psi_0^m$.

Since $\pi_{\pm 1}(\mathcal{L}_0 + i\alpha T) = \sum_1^n (\partial^2/\partial x_j^2 - x_j^2) \mp \alpha$, we have

$$\sigma_{\mathcal{L}_0 + i\alpha T}(\pm 1)(x,\xi) = -|\xi|^2 - |x|^2 \mp \alpha, \tag{2.12}$$

which is consistent with (2.10).

Proposition 2.2 motivates us to make the following definitions.

Definition 2.3. *We say $a(x,\xi) \in \mathcal{H}_b^m$ if $a(x,\xi)$ is smooth and has the asymptotic expansion*

$$a(x,\xi) \sim \sum_{j\ge 0} \varphi_j(x,\xi), \quad |x|^2 + |\xi|^2 \to \infty, \tag{2.13}$$

where $\varphi_j(x,\xi)$ is homogeneous of degree $m-2j$ in (x,ξ).

Definition 2.4. *We say the pair $a_\pm(x,\xi)$ belongs to $\mathcal{H}^m$ if both $a_+(x,\xi)$ and $a_-(x,\xi)$ belong to $\mathcal{H}_b^m$, and if furthermore their expansions are compatible, in the sense that*

$$a_\pm(x,\xi) \sim \sum_{j\ge 0} (\pm 1)^j \varphi_j(\pm x,\xi). \tag{2.14}$$

The content of Proposition 2.2 is hence that $K \in OP\Psi_0^m$ precisely when (2.5) holds and $\sigma_K(\pm 1)(x,\xi) \in \mathcal{H}^m$. Note that if $L \in OP\Psi_0^\mu$, then

$$\sigma_{KL}(\pm\lambda)(X,D) = \lambda^{(m+\mu)/2}\sigma_K(\pm 1)(X,D)\sigma_L(\pm 1)(X,D), \quad \lambda > 0. \tag{2.15}$$

Thus, to deduce from Proposition 2.2 that the composition KL belongs to $OP\Psi_0^{m+\mu}$, we need to discuss compositions of operators in the Weyl calculus. This theory has been worked out, in enough generality for our needs, in [G11], and in much more generality in [H10]. we state some results here, referring to these sources for proofs. See also Appendix A of this paper for further discussion of the Weyl calculus.

The class $\mathcal{H}_b^m$ of symbols is a subset of the class $\mathcal{S}_1^m$, which we define to consist of $a(x,\xi)$, smooth, such that

$$|D_x^\beta D_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|}. \tag{2.16}$$

If $a(x,\xi) \in \mathcal{S}_1^m$, we say $a(X,D) \in OPS_1^m$; similarly if $a(x,\xi) \in \mathcal{H}_b^m$, we say $a(X,D) \in OP\mathcal{H}_b^m$; finally, if $a_\pm(x,\xi) \in \mathcal{H}^m$, we say $a_\pm(X,D) \in OP\mathcal{H}^m$. Now if $a(X,D) \in OP\mathcal{S}_1^m$ and $b(X,D) \in OPS_1^\mu$, then

$$a(X,D)b(X,D) = c(X,D) \in OP\mathcal{S}_1^{m+\mu}, \tag{2.17}$$

and $c(x,\xi)$ has the asymptotic expansion

$$c(x,\xi) \sim \sum_{j\ge 0} \frac{1}{j!}\{a,b\}_j(x,\xi), \tag{2.18}$$

where $\{a,b\}_j(x,\xi)$ is defined by

$$\{a,b\}_0(x,\xi) = a(x,\xi)b(x,\xi), \tag{2.19}$$

and, for $j \ge 1$,

$$\{a,b\}_j(x,\xi) = \Big(\frac{1}{2i}\Big)^j \sum_{k=1}^n \Big(\frac{\partial^2}{\partial y_k \partial \xi_k} - \frac{\partial^2}{\partial x_k \partial \eta_k}\Big)^j a(x,\xi)b(y,\eta)\Big|_{y=x,\eta=\xi}. \tag{2.20}$$

Note that if $a_k(x,\xi)$ and $b_\ell(x,\xi)$ are homogeneous of degree k and ℓ, respectively (not necessarily integers), then $\{a_k,b_\ell\}_j(x,\xi)$ is homogeneous of degree $k+\ell-2j$ in (x,ξ). This shows that if $a(X,D) \in OP\mathcal{H}_b^m$ and $b(X,D) \in OP\mathcal{H}_b^\mu$, then $a(X,D)b(X,D) \in OP\mathcal{H}_b^{m+\mu}$. Furthermore, we have:

Proposition 2.5. *If $a_\pm(X,D) \in OP\mathcal{H}^m$ and $b_\pm(X,D) \in OP\mathcal{H}^\mu$, then*

$$a_\pm(X,D)b_\pm(X,D) = c_\pm(X,D) \in OP\mathcal{H}^{m+\mu}. \tag{2.21}$$

Proof. We know that $c_+(X,D)$ and $c_-(X,D)$ belong to $OP\mathcal{H}_b^{m+\mu}$ and

$$c_\pm(x,\xi) \sim \sum_{j\geq 0} \frac{1}{j!}\{a_\pm, b_\pm\}_j(x,\xi). \tag{2.22}$$

Let us say

$$f_\pm(x,\xi) \in \mathcal{H}^{m,k} \Leftrightarrow f_\pm(x,\xi) \sim \sum_{j\geq k} (\pm 1)^j \psi_j(\pm x, \xi), \tag{2.23}$$

with $\psi_j(x,\xi)$ homogeneous of degree $m-2j$ in (x,ξ). Note that $\mathcal{H}^{m,k} \subset \mathcal{H}^m \cap \mathcal{S}_1^{m-2k}$, and formal series of elements of $\mathcal{H}^{m,k}$, $k = 1,2,\ldots$, asymptotically sum to elements of $\mathcal{H}^m$. Now the jth term of (2.22) is seen to belong to $\mathcal{H}^{m+\mu,j}$, so the proof is complete.

An immediate consequence of Proposition 2.5 is:

Proposition 2.6. *If $K_1 \in OP\Psi_0^m$, $K_2 \in OP\Psi_0^\mu$, then $K_1K_2 \in OP\Psi_0^{m+\mu}$, and*

$$\sigma_{K_1K_2}(\pm\lambda)(X,D) = \sigma_{K_1}(\pm\lambda)(X,D)\sigma_{K_2}(\pm\lambda)(X,D). \tag{2.24}$$

In view of the discussion at the end of §1, leading up to (1.59), we have a similar result for the operators obtained by regularizing $\hat{k}_j$ near the origin.

Similarly we can discuss adjoints. It follows from the Weyl calculus that if $a(X,D) \in OP\mathcal{S}_1^m$, then $a(X,D)^* = a^*(X,D) \in OP\mathcal{S}_1^m$, and

$$a^*(x,\xi) = \overline{a(x,\xi)}. \tag{2.25}$$

This formula shows that if $a(X,D) \in OP\mathcal{H}_b^m$ then $a(X,D)^* \in OP\mathcal{H}_b^m$ and furthermore if $a_\pm(X,D) \in OP\mathcal{H}^m$ then $a_\pm(X,D)^* \in OP\mathcal{H}^m$. Consequently we have:

Proposition 2.7. *If $K \in OP\Psi_0^m$, then $K^* \in OP\Psi_0^m$, and*

$$\sigma_{K^*}(\pm\lambda)(X,D) = \sigma_K(\pm\lambda)(X,D)^*. \tag{2.26}$$

Our next task is to examine $K \in OP\Psi_0^m$ in the case when the operators $\sigma_K(\pm 1)(X,D)$ are elliptic. In general, whenever $a(X,D) \in OP\mathcal{S}_1^m$ is elliptic, there exists a parametrix $b(X,D) \in OP\mathcal{S}_1^{-m}$.

Proposition 2.8. *If $a_\pm(X,D) \in OP\mathcal{H}^m$, with both operators elliptic, having parametrices $b_\pm(X,D)$, belonging a priori to $OP\mathcal{S}_1^{-m}$, then $b_\pm(X,D) \in OP\mathcal{H}^{-m}$.*

Proof. The content of this proposition is that

$$b_\pm(x,\xi) \sim \sum_{j\ge 0} (\pm 1)^j \psi_j(\pm x,\xi), \tag{2.27}$$

with $\psi_j(\pm x,\xi)$ homogeneous of degree $-m-2j$ in (x,ξ). If we assume (2.14), pick $\beta_\pm(x,\xi)$ smooth, with

$$\beta_\pm(x,\xi) = \varphi_0(\pm x,\xi)^{-1}, \quad |x|^2 + |\xi|^2 \ \text{ large.} \tag{2.28}$$

Then clearly $\beta_\pm(x,\xi) \in \mathcal{H}^{-m}$. By (2.17)–(2.18), we have

$$\beta_\pm(X,D)a_\pm(X,D) = I + r_\pm(X,D), \tag{2.29}$$

with

$$r_\pm(x,\xi) \in \mathcal{H}^{0,1}. \tag{2.30}$$

A simple extension of Proposition 2.5 is that

$$r_j^\pm(X,D) \in OP\mathcal{H}^{m_j,k_j} \Longrightarrow r_1^\pm(X,D) r_2^\pm(X,D) \in OP\mathcal{H}^{m_1+m_2,k_1+k_2}. \tag{2.31}$$

In particular,

$$r_\pm(X,D)^k \in OP\mathcal{H}^{0,k} \subset OP\mathcal{S}_1^{-2k}. \tag{2.32}$$

Consequently

$$b_\pm \sim \big(I - r_\pm(X,D) + r_\pm(X,D)^2 - \cdots\big)\beta_\pm(X,D) \tag{2.33}$$

belongs to $OP\mathcal{H}^{-m}$, as asserted.

Suppose that $K \in OP\Psi_0^m$ and that $\sigma_K(\pm 1)(X,D)$ are elliptic. Then denoting parametrices by $\sigma_L(\pm 1)(X,D)$, we obtain an operator $L \in OP\Psi_0^{-m}$. It does not follow that $KL - I$ or $LK - I$ is smoothing. Rather, one has

$$\sigma_{KL}(\pm\lambda)(X,D) = I + r_\pm(X,D), \tag{2.34}$$

with $r_\pm(x,\xi) \in \mathcal{S}_1^{-\infty}$. An operator $R \in OP\Psi_0^0$ with

$$\sigma_R(\pm\lambda)(x,\xi) \in \mathcal{S}_1^{-\infty} \tag{2.35}$$

will be said to belong to $OP\Psi_0^{0,\infty}$. More generally, we have the following notion:

Definition 2.9. *We say* $R \in OP\Psi_0^{m,k}$ *if* $R \in OP\Psi_0^m$ *with*

$$\sigma_R(\pm 1)(X,D) \in OP\mathcal{H}^{m,k}. \tag{2.36}$$

We denote $\cap_{k\geq 0} OP\Psi_0^{m,k}$ *by* $OP\Psi_0^{m,\infty}$.

In order to construct a (left or right) parametrix for $K \in OP\Psi_0^m$ with $\sigma_K(\pm 1)(X,D)$ elliptic, it is necessary to assume $\sigma_K(\pm 1)(X,D)$ has a (left or right) inverse. As shown, e.g., in [B4] or [G12], the following are equivalent if $a(X,D) \in OP\mathcal{S}_1^m$ is elliptic:

$$a(X,D) \text{ has a left inverse in } OP\mathcal{S}_1^{-m}, \tag{2.37}$$

$$a(X,D) \text{ is injective on the Schwartz space } \mathcal{S}(\mathbb{R}^n), \tag{2.38}$$

$$a(X,D) \text{ is injective on the space } \mathcal{S}'(\mathbb{R}^n). \tag{2.39}$$

Similarly, the following three conditions are equivalent:

$$a(X,D) \text{ has a right inverse in } OP\mathcal{S}_1^{-m}, \tag{2.40}$$

$$a(X,D) \text{ is surjective on } \mathcal{S}(\mathbb{R}^n), \tag{2.41}$$

$$a(X,D)^* \text{ is injective on } \mathcal{S}(\mathbb{R}^n). \tag{2.42}$$

Our regularity result is the following.

Theorem 2.10. *If* $K \in OP\Psi_0^m$ *has the property that* $\sigma_K(\pm 1)(X,D)$ *are elliptic, then* K *has a left inverse* $L \in OP\Psi_0^{-m}$ *if and only if* $\sigma_K(\pm 1)(X,D)^*$ *are injective on* $\mathcal{S}(\mathbb{R}^n)$*, and such a right inverse if and only if* $\sigma_K(\pm 1)(X,D)$ *are injective on* $\mathcal{S}(\mathbb{R}^n)$*.*

Proof. In light of the discussion above, this is an immediate consequence of Proposition 2.8.

Corollary 2.11. *If* $K \in OP\Psi_0^m$ *has the property that* $\sigma_K(\pm 1)(X,D)$ *are elliptic and injective on* $\mathcal{S}(\mathbb{R}^n)$*, then* K *is hypoelliptic.*

In the case of scalar differential operators, the hypoellipticity is proved by Miller [M10], using somewhat different arguments; see also Rockland [R2]. A more general result has been proved by Helffer and Nourrigat [H3], by different means. This result was also announced by Dynin [D2]; his announcement did not include proofs, but it is fairly likely that his argument was similar to that given here. If K is scalar, with $\sigma_K(\pm 1)(X,D)$ elliptic, then, as observed by Grusin [G12], the operators $\sigma_K(\pm 1)(X,D)$ are Fredholm of index zero, so one-sided and two-sided invertibility coincide. The propositions above also work for $k\times k$ matrices of operators, in which case $\sigma_K(\pm 1)(X,D)$ may have nonzero index. As a counterpoint to Corollary 2.11, we have:

Proposition 2.12. *If* $K \in OP\Psi_0^m$ *has the property that* $\sigma_K(\pm 1)(X,D)$ *are elliptic but not both injective on* $\mathcal{S}(\mathbb{R}^n)$*, then* K *is not hypoelliptic.*

Proof. Say $\sigma_K(+1)(X,D)$ has a nontrivial kernel. Such a null space must be a finite-dimensional subspace of $\mathcal{S}(\mathbb{R}^n)$, so the orthogonal projection of $L^2(\mathbb{R}^n)$ onto this null space is $r(X,D)$ with $r(x,\xi)\in\mathcal{S}_1^{-\infty}$. Then

$$\sigma_S(+\lambda)(X,D)=r(X,D),\quad \sigma_S(-\lambda)(X,D)=0, \tag{2.43}$$

for $\lambda>0$, defines an element $S\in OP\Psi_0^{0,\infty}$, and $KS=0$. Since S is right invariant and not smoothing, $v=S\delta_0$ is not smooth; it is smooth except at the origin. Since $Kv=0$, K is not hypoelliptic on any neighborhood of the origin. By right invariance, K is not hypoelliptic on any open set.

Let us apply these results on hypoellipticity to the well known important example $\mathcal{L}_0+i\alpha T$, first treated by Folland and Stein [F4], by a different method. By (2.12), we have

$$\sigma_{\mathcal{L}_0+i\alpha T}(\pm1)(X,D)=\Delta-|x|^2\mp\alpha, \tag{2.44}$$

which clearly gives an elliptic pair in $OP\mathcal{H}^2$. We determine the spectrum of $\Delta-|x|^2$. In fact, as is well known, $-d^2/dx^2+x^2$, acting on functions on $\mathbb{R}$, has discrete spectrum consisting of the eigenvalues $2j+1,\ j=0,1,2,\dots$, all simple. Indeed the complete set of eigenfunctions of d^2/dx^2-x^2 is given by the set of Hermite functions

$$h_j(x)=[\pi^{1/2}2^jj!]^{-1/2}\Big(\frac{d}{dx}-x\Big)^je^{-x^2/2},\quad \Big(\frac{d^2}{dx^2}-x^2\Big)h_j(x)=-(2j+1)h_j(x). \tag{2.45}$$

It follows that, on $L^2(\mathbb{R}^n)$, $\Delta-|x|^2$ has spectrum consisting of all the negative integers of the form $-n-2k,\ k=0,1,2,\dots$. Thus Theorem 2.10 and Proposition 2.12 yield the result of Folland and Stein:

Proposition 2.13. *For $\alpha\in\mathbb{C}$, $\mathcal{L}_0+i\alpha T$ is hypoelliptic on $\mathbb{H}^n$ if and only if*

$$-(n\pm\alpha)\notin\{0,1,2,\dots\}. \tag{2.46}$$

If (2.46) holds, $\mathcal{L}_0+i\alpha T$ has a two-sided inverse in $OP\Psi_0^{-2}$. If (2.46) fails, there exists a non-smoothing element S of $OP\Psi_0^{0,\infty}$ such that $(\mathcal{L}_0+i\alpha T)S=0$.

For a large class of homogeneous second-order polynomials $Q(x,\xi)$, the spectrum of $Q(X,D)$ is completely specified by the following result; see Sjöstrand [S6], Hörmander [H8], Grigis [G10].

Proposition 2.14. *Let $Q(x,\xi)$ be a second-order homogeneous polynomial in (x,ξ). Suppose that Q takes values in a cone of the form*

$$\Gamma=\{z\in\mathbb{C}:|\mathrm{Im}\,z|\le K\,\mathrm{Re}\,z\}. \tag{2.47}$$

Denote by $i\mu_j$ the eigenvalues in $\Gamma\setminus 0$ of the Hamilton map F of Q, defined by

$$Q(v,v')=\sigma(v,Fv'), \tag{2.48}$$

where σ is the symplectic form on $\mathbb{R}^{2n}$:

$$\sigma\big((x,\xi),(x',\xi')\big) = x'\cdot\xi - x\cdot\xi', \tag{2.49}$$

and $Q(v,v')$ is the bilinear form polarizing Q. Let $N_0 \subset \mathbb{C}^{2n}$ be the space of generalized eigenvectors of F associated to the eigenvalue 0. Then α is not in the spectrum of $Q(X,D)$ if and only if

$$-\alpha + Q(v,\overline{v}) + \sum_j (2k_j+1)\mu_j \neq 0, \quad \forall\ v\in N_0,\ k_j\in\{0,1,2,3,\dots\} = \mathbb{Z}^+. \tag{2.50}$$

In particular, if $Q(x,\xi)$ is positive-definite, the spectrum of $Q(X,D)$ consists of

$$\Big\{\sum_j (2k_j+1)\mu_j : k_j\in\mathbb{Z}^+\Big\}. \tag{2.51}$$

The result on the spectrum of $Q(X,D)$ in the positive-definite case is older; using the metaplectic representation, one can reduce it to a simple result in linear algebra, plus the analysis of the harmonic oscillator $d^2/dx^2 - x^2$ described above. Compare the calculations in (4.55)–(4.64) of this chapter, and those in (3.56)–(3.65) of Chapter III. See also Chapter I of [T5] for an exposition of this case. For certain homogeneous polynomials of degree 3, the spectrum of $Q(X,D)$ has been analyzed by Helffer [H1]. For general $Q(X,D)\in OP\mathcal{H}^m$, it would be out of the question to explicitly describe the spectrum, though qualitative studies of the spectrum have been made; see [H2] and references therein.

Here we mean the operator $Q(X,D)$ to be defined by the Weyl calculus, of course. In particular,

$$Q(x,\xi) = x_j\xi_k \Longrightarrow Q(X,D)u(x) = \frac{i}{2}\Big(x_j\frac{\partial u}{\partial x_k} + \frac{\partial}{\partial x_k}(x_j u)\Big).$$

The fact that hypoellipticity of an element K of $OP\Psi_0^m$ depends not just on the ellipticity of $\sigma_K(\pm 1)(X,D)$ but also on the invertibility of these operators, is closely related to the essential non-commutativity of $OP\Psi_0^0$. In fact, if $L\in OP\Psi_0^\mu$, then the commutator $[K,L]=KL-LK$ has symbol

$$\sigma_{[K,L]}(\pm\lambda)(X,D) = \lambda^{(m+\mu)/2}[\sigma_K(\pm 1)(X,D),\sigma_L(\pm 1)(X,D)]. \tag{2.52}$$

Clearly this is not the symbol of an operator belonging to $OP\Psi_0^{m+\mu-1}$. In fact, the result one has is the following; recall Definition 2.9:

Proposition 2.15. *If $K\in OP\Psi_0^m$ and $L\in OP\Psi_0^\mu$, then*

$$[K,L]\in OP\Psi_0^{m+\mu,1}. \tag{2.53}$$

More generally, if $K\in OP\Psi_0^{m,k}$, $L\in OP\Psi_0^{\mu,\ell}$, then

$$[K,L]\in OP\Psi_0^{m+\mu,k+\ell+1}. \tag{2.54}$$

Proof. It suffices to show that, if $a_\pm(X,D) \in OP\mathcal{H}^{m,k}$ and $b_\pm(X,D) \in OP\mathcal{H}^{\mu,\ell}$, then

$$[a_\pm(X,D), b_\pm(X,D)] \in OP\mathcal{H}^{m+\mu,k+\ell+1}. \tag{2.55}$$

This follows from the formula (2.18) for the symbol of a product, along the same lines as the proof of Proposition 2.5.

We now want to pass from operators in $OP\Psi_0^m$ with homogeneous symbols to nonhomogeneous operators given as asymptotic sums. We define the class $OP\Psi^m$ as follows.

Definition 2.16. *We say a convolution operator K belongs to $OP\Psi^m$ if*

$$K \sim \sum_{j\ge 0} K_j, \quad K_j \in OP\Psi_0^{m-j}, \tag{2.56}$$

where (2.56) means $K - \sum_{j\le N} K_j$ is arbitrarily smoothing for N picked sufficiently large.

In order to fit $OP\Psi^m$ into the framework of Chapter I, we make Ψ^m into a Frechet space as follows. We say $\hat{k}(\tau,y,\eta) \in \Psi^m$ if $\hat{k}$ is C^∞ and

$$\hat{k} \sim \sum_{j\ge 0} k_j(\tau,y,\eta), \quad \text{as} \quad |\tau|^2 + |y|^2 + |\eta|^2 \to \infty, \tag{2.57}$$

where $\hat{k}_j$ satisfy the homogeneity conditions

$$\hat{k}_j(r\tau, r^{1/2}y, r^{1/2}\eta) = r^{(m-j)/2}\hat{k}_j(\tau,y,\eta). \tag{2.58}$$

Here, (2.57) holds in the sense that, if $\varphi(\tau,y,\eta)$ vanishes near 0 and is 1 near ∞,

$$\hat{k} - \varphi\sum_{j=1}^{N} \hat{k}_j \in \mathcal{S}_1^{m-N/2}.$$

Since we have required $\hat{k}$ to be C^∞, even at the origin, strictly speaking Ψ_0^m is not contained in Ψ^m, but altering $\hat{k}$ on a compact set in (τ,y,η) space changes a convolution operator by a smoothing operator, so, modulo smoothing operators $OP\Psi_0^m$ is contained in $OP\Psi^m$.

Now to describe the Frechet space structure on Ψ^m, we first pick a large compact neighborhood of the origin and use the C^j norms of $\hat{k}$ on this neighborhood. For the rest, take $(\tau,y,\eta) \in S^{2n}$, and define

$$\kappa(s,\tau,y,\eta) = s^m\,\hat{k}(s^{-2}\tau, s^{-1}y, s^{-1}\eta), \quad 0 < s \le 1. \tag{2.59}$$

Then $\hat{k} \in \Psi^m$ if and only if $\hat{k} \in C^\infty$ and $\kappa \in C^\infty([0,1]\times S^{2n})$, so we transfer the seminorms of this Frechet space to obtain a complementary set of seminorms on Ψ^m.

It is a simple consequence of Proposition 2.6 that

$$K_j \in OP\Psi^{m_j} \Longrightarrow K_1K_2 \in OP\Psi^{m_1+m_2}, \quad K_j^* \in OP\Psi^{m_j}. \tag{2.60}$$

Note the remark following the proof of Proposition 2.6. We also easily obtain the following result.

Theorem 2.17. *Let $K \in OP\Psi^m$. Suppose K_0, in the expansion (2.56), satisfies the conditions that $\sigma_{K_0}(\pm 1)(X, D)$ are elliptic and both are injective (resp., surjective; resp., invertible) on $\mathcal{S}(\mathbb{R}^n)$. Then K has a left (resp., right; resp., two-sided) parametrix $L \in OP\Psi^{-m}$.*

Proof. Take the case that $\sigma_{K_0}(\pm 1)(X, D)$ are injective. Then Theorem 2.10 gives $L_0 \in OP\Psi_0^{-m}$ such that $L_0 K_0 = I$. It follows that

$$L_0 K = I + S_1, \quad S_1 \in OP\Psi^{-1}. \tag{2.61}$$

Then we can construct $L \in OP\Psi^{-m}$ such that

$$L \sim (I - S_1 + S_1^2 - \cdots)L_0, \tag{2.62}$$

and then $LK = I$ modulo a smoothing operator. The other assertions of this theorem follow similarly.

For example, $\mathcal{L}_0 + i\alpha T + P$, with $P \in OP\Psi^1$, has a parametrix in $OP\Psi^{-2}$ provided α satisfies the condition (2.46). In the case of scalar differential operators, the regularity result contained in Theorem 2.17 was obtained by Miller [M10], via energy estimates, rather than a symbolic construction of a parametrix.

Unlike the situation in Proposition 2.12, if $K \in OP\Psi^m$ as in Theorem 2.17, with $\sigma_{K_0}(\pm)(X, D)$ elliptic but not injective, it is still possible for K to be hypoelliptic. We consider the following example due to Stein [S7]; see also Rothschild [R4]. Let

$$K = \mathcal{L}_0 + i\alpha T + \beta \in OP\Psi^2, \tag{2.63}$$

where α is chosen to be an integer such that the condition (2.46) for hypoellipticity of $K_0 = \mathcal{L}_0 + i\alpha T$ is violated, and we take $\beta \in \mathbb{C}$. Let $S \in OP\Psi_0^{0,\infty}$ be the projection produced by the proof of Proposition 2.12. Then, as in (2.34), we have $E_0 \in OP\Psi^{-2}$ such that

$$E_0(\mathcal{L}_0 + i\alpha T) = I - S, \tag{2.64}$$

while

$$S(\mathcal{L}_0 + i\alpha T) = 0. \tag{2.65}$$

Then, if $\beta \neq 0$,

$$(E_0 + \beta^{-1}S)K = (E_0 + \beta^{-1}S)(K_0 + \beta) = I + \beta E_0. \tag{2.66}$$

Note that $R = \beta E_0 \in OP\Psi_0^{-2}$. It follows that

$$E \sim (I - R + R^2 - \cdots)(E_0 + \beta^{-1}S) \in OP\Psi^0 \tag{2.67}$$

gives a left parametrix for K, so the operator (2.63) is hypoelliptic, with loss of two derivatives, whenever $\beta \neq 0$. A generalization, in the case of right invariant differential operators on $\mathbb{H}^n$, was given by Rothschild, and will be discussed in §6 of this chapter. We will look at further generalizations of this phenomenon in Chapter III.

The machinery summarized in Theorem 2.17 and Proposition 2.14 is particularly effective in examining hypoellipticity and constructing parametrices for a second-order differential operator P (right invariant) on $\mathbb{H}^n$, which is doubly characteristic on the set $\Lambda \subset T^*\mathbb{H}^n$ of characteristics for $\mathcal{L}_0$. Note that Λ is characterized as a line bundle which is translation invariant and its fiber over the identity $0 \in \mathbb{H}^n$ is the linear span of dt. Suppose the principal symbol of P is positive and vanishes to exactly second order on Λ. It follows that the principal symbol of P must agree with that of

$$\sum_{j,k=1}^{2n} a_{jk} X_j X_k = P_2, \tag{2.68}$$

where we have set

$$X_j = L_j, \quad X_{j+n} = M_j, \quad 1 \leq j \leq n. \tag{2.69}$$

The matrix (a_{jk}) is symmetric and positive definite. Since $L_j, M_j \in OP\Psi_0^1$, by (2.5), it follows that

$$P = \sum_{j,k} a_{jk} X_j X_k + i\alpha T + B, \quad B \in OP\Psi^1, \tag{2.70}$$

for some $\alpha \in \mathbb{C}$. The sum of the first two terms belongs to $OP\Psi_0^2$, and we have

$$\sigma_{P_2 + i\alpha T}(\pm 1)(x, \xi) = -\sum a_{jk} \chi_j \chi_k \mp \alpha, \tag{2.71}$$

where we have set

$$\chi_j = x_j, \quad \chi_{j+n} = \xi_j, \quad 1 \leq j \leq n. \tag{2.72}$$

In particular, the ellipticity hypothesis of Theorem 2.17 holds, and Proposition 2.14 applies in a straightforward fashion to the question of invertibility. Non-translation-invariant generalizations form a very important class of operators, which will be investigated in Chapter III.

3. Right invariant pseudodifferential operators on $\mathbb{H}^n$, II: δ-homogeneous operators and amalgamations

In order to achieve a genuine microlocal analysis, we want to amalgamate the operator classes $OP\Psi^m$ and their non-right-invariant extensions, with the classical operator classes OPS^m. We begin by considering right invariant operators in OPS^m from the point of view of the representations of $\mathbb{H}^n$. We shall discuss connections with Propositions 2.7–2.13 of Chapter I in §1 of Chapter III.

Definition 3.1. *We say $\hat{k} \in \Sigma_0^m$ provided $\hat{k}(\tau, y, \eta)$ is C^∞ away from $(0,0,0)$ and*

$$\hat{k}(\lambda\tau, \lambda y, \lambda\eta) = \lambda^m \hat{k}(\tau, y, \eta), \quad \lambda > 0. \tag{3.1}$$

In such a case, we say $Ku = k * u$ defines $K \in OP\Sigma_0^m$. If

$$\hat{k} \sim \sum_{j\geq 0} \hat{k}_j, \quad \hat{k}_j \in \Sigma_0^{m-j}, \tag{3.2}$$

we say $\hat{k} \in \Sigma^m$ and $K \in OP\Sigma^m$.

The proof of Proposition 1.4 from Chapter I immediately implies that, modulo smoothing operators, $OP\Sigma^m$ is identical with the class of right invariant operators in OPS^m. Note that (3.1) is equivalent to

$$\sigma_K(\pm\lambda)(x, \xi) = \lambda^m \sigma_K(\pm 1)(\lambda^{-1/2}x, \lambda^{-1/2}\xi). \tag{3.3}$$

Parallel to Proposition 2.2, we have the following simple result, whose proof we omit.

Proposition 3.2. *The formula (3.3) defines an element of Σ_0^m if and only if $\sigma_K(\pm 1)(x, \xi)$ are smooth with asymptotic behavior*

$$\sigma_K(\pm 1)(x, \xi) \sim \sum_{j\geq 0} (\pm 1)^j \varphi_{m-j}(\pm x, \xi), \tag{3.4}$$

where $\varphi_{m-j}(x, \xi)$ is homogeneous of degree $m - j$ in (x, ξ).

Of course the operator calculus for OPS^m gives us that if $P_j \in OP\Sigma^{m_j}$, then $P_1P_2 \in OP\Sigma^{m_1+m_2}$. We want to perceive this via the representation theory of $\mathbb{H}^n$. The argument proving Proposition 2.6, on products of elements of $OP\Psi^{m_j}$, will have to be modified, since, in (3.3), the argument of $\sigma_K(\pm 1)$ is $(\lambda^{-1/2}x, \lambda^{-1/2}\xi)$. In fact, in this case, letting $\lambda \to 0$ causes technical problems we wish to avoid. One way to avoid this problem is to restrict attention to operators defined by (3.3) with $\sigma_K(\pm 1)(x, \xi) \in C_0^\infty(\mathbb{R}^{2n})$. It is clear that this subset of OPS^m contains operators that agree with any right invariant operator in OPS^m, microlocally on a conic neighborhood of the line bundle Λ (characteristic set of $\mathcal{L}_0$) discussed in §2. That this is a sufficiently rich class for our purposes is guaranteed by the following result.

Proposition 3.3. *If $K \in OP\Psi^m$, then, microlocally on the complement of any conic neighborhood of $\Lambda \subset T^*\mathbb{H}^n \setminus 0$, K belongs to OPS^m.*

Proof. Given the formula (1.23) from Chapter I, analyzing K as a pseudodifferential operator, we see this is a simple consequence of the following observation. If $\hat{k}(\tau, y, \eta) \in \Psi_0^m$, then, on the complement of any conic neighborhood of $\{y = \eta = 0\}$ in $\mathbb{R}^{2n+1} \setminus 0$, $\hat{k}$ agrees with an element of Σ^m.

Let us now consider the symbolic calculus applied to a product of $A \in OP\Sigma_0^m$ and $B \in OP\Sigma_0^\mu$, with

$$\sigma_A(\pm 1)(x,\xi) = a_\pm(x,\xi), \ \sigma_B(\pm 1)(x,\xi) = b_\pm(x,\xi) \in C_0^\infty(\mathbb{R}^{2n}). \tag{3.5}$$

Let us denote

$$a_{\pm,\lambda}(x,\xi) = a_\pm(\lambda^{-1/2}x, \lambda^{-1/2}\xi), \tag{3.6}$$

with $b_{\pm,\lambda}(x,\xi)$ similarly defined. Consequently,

$$\sigma_{AB}(\pm\lambda)(X,D) = \lambda^{m+\mu} e_\pm(\lambda, X, D), \tag{3.7}$$

with

$$e_\pm(\lambda, x, \xi) = \sum_{j=0}^{N} \frac{1}{j!}\{a_{\pm,\lambda}, b_{\pm,\lambda}\}_j(x,\xi) + \lambda^{-N} R_N^\pm(\lambda, x, \xi). \tag{3.8}$$

Recall that $\{a,b\}_j$ is given by (2.20). To see that we can avoid considering $\lambda \to 0$, note that, if A' is defined by

$$\sigma_{A'}(\pm\lambda)(X,D) = \psi(\lambda)\sigma_A(\pm\lambda)(X,D), \tag{3.9}$$

where $\psi(\lambda)$ is smooth, equal to 1 for $|\lambda| \geq 1$, and to 0 for $|\lambda| \leq 1/2$, then A and A' differ by an operation of convolution by w, where $\hat{w}(\tau, y, \eta)$ has compact support. Hence $w \in C^\infty(\mathbb{H}^n)$, so A and A' differ by a smoothing operator. Hence, in (3.7)–(3.8), we can restrict attention to $|\lambda| \geq 1$. We can rewrite (3.8) as

$$e_\pm(\lambda, X, D) = \sum_{j=0}^{N} \lambda^{-j} e_j^\pm(\lambda^{-1/2}X, \lambda^{-1/2}D) + \lambda^{-N} R_N^\pm(\lambda, X, D). \tag{3.10}$$

Results of [H10] imply that $R_N^\pm(\lambda, X, D)$ is bounded in $OP\mathcal{S}_1^0$ (defined by (2.16)), for $|\lambda| \geq 1$. To see that the remainder in (3.10) makes a contribution that is smoothing to a high degree when N is large, consider the following. The operator $T \in OP\Sigma_0^1$ has the symbol

$$\sigma_T(\pm\lambda)(X,D) = \pm i\lambda. \tag{3.11}$$

Thus $A(BT^k)$ has a form similar to (3.7), (3.10), with $\lambda^{m+\mu}$ replaced by $(\pm i)^k \lambda^{m+\mu+k}$; in fact,

$$\begin{aligned}\sigma_{ABT^k}(\pm\lambda)(X,D) = (\pm i)^k \lambda^{m+\mu+k} \sum_{j=0}^{N} \lambda^{-j} e_j^{\pm}(\lambda^{-1/2}X, \lambda^{-1/2}D) \\ + (\pm i)^k \lambda^{m+\mu+k-N} R_N^{\pm}(\lambda, X, D).\end{aligned} \tag{3.12}$$

Consequently, (3.7) and (3.10) give AB as a sum

$$AB = \sum_{j=1}^{N} E_j + S_N, \tag{3.13}$$

where $E_j \in OP\Sigma_0^{m+\mu-j}$ and S_N has the following property:

$$T^{k_1} S_N T^{k_2} : L^2(\mathbb{H}^n) \longrightarrow L^2(\mathbb{H}^n), \quad \text{for} \quad N \geq m + \mu + k_1 + k_2. \tag{3.14}$$

Since S_N is certainly an operator in $OPS^{m+\mu}$ whose symbol is essentially supported near Λ, and since $T \in OPS^1$ is microlocally elliptic near Λ, this implies that S_N is arbitrarily smoothing for N large. Consequently (3.13) is an asymptotic relation, so, for $A \in OP\Sigma_0^m,\ B \in OP\Sigma_0^\mu$, we have $AB \in OP\Sigma^{m+\mu}$, and

$$\sigma_{AB}(\pm\lambda)(X,D) \sim \lambda^{m+\mu} \sum_{j \geq 0} \lambda^{-j} e_j^{\pm}(\lambda^{-1/2}X, \lambda^{-1/2}D). \tag{3.15}$$

Note that

$$e_0^{\pm}(x,\xi) = a_{\pm}(x,\xi) b_{\pm}(x,\xi), \tag{3.16}$$

which verifies that (for scalar operators)

$$[A,B] = AB - BA \in OP\Sigma^{m+\mu-1}. \tag{3.17}$$

Of course, examining products of operators in $OP\Sigma^{m_j}$ via the Heisenberg group harmonic analysis involves a bit more work than just appealing to the OPS^m operator calculus. The point is that it does work to produce a symbol calculus for such operators, whereas commutative harmonic analysis does not work to produce a symbol calculus for $OP\Psi^m$. The calculations just done can be viewed as a warm-up for what is to come.

Now let us consider products PK and KP, given $K \in OP\Psi^\mu,\ P \in OP\Sigma_0^m$, the symbol of P being supported near Λ. Thus

$$\sigma_{PK}(\pm\lambda)(X,D) = \lambda^{m+\mu/2} p_{\pm}(\lambda^{-1/2}X, \lambda^{-1/2}D) a_{\pm}(X,D). \tag{3.18}$$

As above, we can restrict attention to $|\lambda| \geq 1$. As in the analysis of (3.7), we have

$$\sigma_{PK}(\pm\lambda)(X, D) = e_{\pm}(\lambda, X, D), \tag{3.19}$$

where, with $p_{\pm,\lambda}(x, \xi) = p_{\pm}(\lambda^{-1/2}x, \lambda^{-1/2}\xi)$,

$$e_{\pm}(\lambda, x, \xi) = \sum_{j=0}^{N} \frac{1}{j!}\{p_{\pm,\lambda}, a_{\pm}\}_j(x, \xi) + \lambda^{-N/2}R_N^{\pm}(\lambda, x, \xi). \tag{3.20}$$

This time, $R_N^{\pm}(\lambda, x, \xi)$ is bounded in $\mathcal{S}_1^{\mu}$ for $|\lambda| \geq 1$.

In this case, the formula (2.20) for $\{a, b\}_j$ gives

$$e_{\pm}(\lambda, x, \xi) = \sum_{j=1}^{N}\sum_{k=1}^{K(j)} \lambda^{-j/2} p_{kj}^{\pm}(\lambda^{-1/2}x, \lambda^{-1/2}\xi) a_{kj}(x, \xi) + \lambda^{-N/2}R_N^{\pm}(\lambda, x, \xi), \tag{3.21}$$

where

$$p_{kj}^{\pm}(x, \xi) \in C_0^{\infty}(\mathbb{R}^{2n}), \quad a_{kj}^{\pm}(x, \xi) \in \mathcal{H}_b^{\mu-j}. \tag{3.22}$$

The same argument as before shows the remainder in (3.21) contributes a term smoothing to high order. To summarize, we have:

Proposition 3.4. *If $K \in OP\Psi_0^{\mu}$ and $P \in OP\Sigma_0^{m}$, the symbol of P being supported near Λ, then*

$$\sigma_{PK}(\pm\lambda)(X, D) \sim \lambda^{m+\mu/2} \sum_{j\geq 0} \lambda^{-j/2} e_j^{\pm}(\lambda, X, D), \tag{3.23}$$

where

$$e_0^{\pm}(\lambda, x, \xi) = p_{\pm}(\lambda^{-1/2}x, \lambda^{-1/2}\xi) a_{\pm}(x, \xi), \tag{3.24}$$

and generally

$$e_j^{\pm}(\lambda, x, \xi) = \sum_{k=1}^{K(j)} p_{kj}^{\pm}(\lambda^{-1/2}x, \lambda^{-1/2}\xi) a_{kj}^{\pm}(x, \xi), \tag{3.25}$$

with (3.22) holding. In a similar fashion we have

$$\sigma_{KP}(\pm\lambda)(X, D) \sim \lambda^{m+\mu/2} \sum_{j\geq 0} \lambda^{-j/2} f_j^{\pm}(\lambda, X, D), \tag{3.26}$$

where

$$f_0^{\pm} = e_0^{\pm}, \tag{3.27}$$

and $f_j^{\pm}(\lambda, x, \xi)$ has an expression similar in form to (3.25).

Note that this analysis gives for the commutator $[P, K]$:

$$\sigma_{[P,K]}(\pm\lambda)(X, D) \sim \lambda^{m+\mu/2} \sum_{j\geq 1} \lambda^{-j/2} [e_j^{\pm}(\lambda, X, X) - f_j^{\pm}(\lambda, X, D)]. \tag{3.28}$$

Inductively, from the representation (3.23)–(3.25), we obtain:

Corollary 3.5. *If $K \in OP\Psi^{\mu}$, $P \in OP\Sigma_0^m$ are as above, then*

$$[P,K] \sim \sum_{j\geq 1} \sum_{k=1}^{K(j)} P_{kj} A_{kj}, \quad P_{kj} \in OP\Sigma^m, \ A_{kj} \in OP\Psi^{\mu-j}. \tag{3.29}$$

It is worth noting that more precise statements can be made about PK when $K \in OP\Psi^{\mu,\infty}$. Of course, elements of $OP\Psi^{\mu,\infty}$ have special properties, one of which is the following.

Proposition 3.6. *If $K \in OP\Psi_0^{\mu,\infty}$, then, microlocally on the complement of any conic neighborhood of Λ, K belongs to $OPS^{-\infty}$.*

Proof. In view of the analysis of K as a pseudodifferential operator given in Chapter I, Proposition 1.1, this follows from the observation that, if $\hat{k}(\tau,y,\eta) \in \Psi_0^{\mu,\infty}$, then $\hat{k}(\tau,y,\eta)$ is rapidly decreasing outside any conic neighborhood of $\{y=\eta=0\}$ in $\mathbb{R}^{2n+1}\setminus 0$.

Now, if $P \in OP\Sigma^m$ is as above and $K \in OP\Psi_0^{\mu,\infty}$, the formulas (3.18)–(3.22) hold, and in addition we have

$$a_{kj}^{\pm}(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.30}$$

We can expand $p_{kj}^{\pm}(\lambda^{-1/2}x, \lambda^{-1/2}\xi)$ is a power series, in powers of $\lambda^{-1/2}$. Indeed,

$$p_{kj}^{\pm}(\lambda^{-1/2}x, \lambda^{-1/2}\xi) = \sum_{\ell=0}^{N} \frac{1}{\ell!} \lambda^{-\ell/2}\, \Omega^{\ell} p_{kj}^{\pm}(x,\xi,0,0) + \lambda^{-N/2} r_{kjN}^{\pm}(\lambda,x,\xi), \tag{3.31}$$

where

$$\Omega p(x,\xi,y,\eta) = (x,\xi)\cdot \nabla_{y,\eta} p(y,\eta), \tag{3.32}$$

and

$$r_{kjN}^{\pm}(\lambda,x,\xi) \ \text{ is bounded in } \ \mathcal{S}_1^N, \quad \text{for } \lambda \geq 1. \tag{3.33}$$

Since the factors $a_{kj}^{\pm}(x,\xi)$ in (3.22) belong to $\mathcal{S}(\mathbb{R}^{2n}) = \mathcal{S}_1^{-\infty}$ in this case, we have

$$\sigma_{PK}(\pm\lambda)(X,D) = \lambda^{m+\mu/2} e_{\pm}(\lambda,X,D), \tag{3.34}$$

with

$$e_{\pm}(\lambda,X,D) \sim \sum_{j\geq 0} \lambda^{-j/2} e_j^{\pm}(X,D), \quad e_j^{\pm}(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n}). \tag{3.35}$$

Note that

$$e_0^{\pm}(x,\xi) = p_{\pm}(0,0) a_{\pm}(0,0). \tag{3.36}$$

We have proven the following result.

Proposition 3.7. *If $P \in OP\Sigma^m$ and $K \in OP\Psi_0^{\mu,\infty}$, then $PK \in OP\Psi^{2m+\mu,\infty}$. Moreover, modulo a smoothing operator, $PK = L$, with*

$$\sigma_L(\pm\lambda)(x,\xi) \sim \sum_{j\geq 0} \lambda^{m+\mu/2-j/2} e_j^{\pm}(x,\xi), \tag{3.37}$$

with $e_j^{\pm}(x,\xi) \in \mathcal{S}(\mathbb{R}^{2n})$ and

$$e_0^{\pm}(x,\xi) = p_{\pm}^0 \, \sigma_K(\pm 1)(x,\xi). \tag{3.38}$$

The factor $p_{\pm}^0$ is euqal to the principal symbol of P, evaluated at the point $\pm dt$ lying in the fiber of $T^*\mathbb{H}^n$ over the origin. The product $KP \in OP\Psi^{2m+\mu,\infty}$ has a similar behavior.

Note that, in light of Proposition 3.6, we need not make the assumption that the symbol of P is essentially supported near Λ. From Proposition 3.7 we obtain the following simple corollary.

Corollary 3.8. *If $K \in OP\Psi^{\mu,\infty}$, then there exists $A \in OP\Sigma^{\mu/2}$ such that*

$$K = AK_0, \quad K_0 \in OP\Psi^{0,\infty}. \tag{3.39}$$

It follows that elements of $OP\Psi^{\mu,\infty}$ enjoy stronger continuity properties on Sobolev spaces than general elements of $OP\Psi^{\mu}$, if $\mu > 0$.

We now consider a certain synthesis of the classical (right invariant) pseudodifferential operator classes $OP\Sigma^*$ and the operator classes $OP\Psi^*$. We will first consider a class of operators that are microlocally supported in a small conic neighborhood of the line bundle Λ in $T^*\mathbb{H}^n \setminus 0$. In light of our calculations involving one-parameter families of pseudodifferential operators on $\mathbb{R}^n$, the following class is quite natural.

Definition 3.9. *We say a right invariant operator A belongs to $OP\Omega_b^{m,k}$ provided the symbol $b_{\pm}(\lambda,x,\xi) = \sigma_A(\pm\lambda)(x,\xi)$ satisfies the following two conditions:*

$$b_{\pm}(\lambda,x,\xi) \ \text{ is smooth and supported on } \ |x| + |\xi| \leq C\lambda^{1/2}, \ \lambda \geq C', \tag{3.40}$$

and

$$\lambda^{j-m} \, D_\lambda^j b_{\pm}(\lambda,\cdot,\cdot) \ \text{ is bounded in } \ \mathcal{S}_1^k, \ \text{for } \lambda \in \mathbb{R}^+. \tag{3.41}$$

Recall that the symbol class $\mathcal{S}_1^k$ is defined by (2.16). Hence (3.41) is equivalent to

$$|D_\lambda^j D_x^\beta D_\xi^\alpha b_{\pm}(\lambda,x,\xi)| \leq C_{j\alpha\beta} \, \lambda^{m-j} \, (1+|x|+|\xi|)^{k-|\alpha|-|\beta|}. \tag{3.42}$$

If $Au = a * u$, then, according to (1.16), we have

$$\hat{a}(\pm\tau,y,\eta) = b_{\pm}(\tau, \pm\tau^{-1/2}y, \tau^{-1/2}\eta). \tag{3.43}$$

Hence we see that the support condition in (3.40) is equivalent to:

$$\hat{a}(\tau,y,\eta) \ \text{ is supported on } \ |y| + |\eta| \leq C|\tau|, \ |\tau| \geq C'. \tag{3.44}$$

As for the content of (3.41), we have the following.

Proposition 3.10. *The hypothesis (3.41) is equivalent to the following estimate on $\hat{a}$:*

$$\begin{aligned}|D_\tau^j D_y^\beta D_\eta^\alpha \hat{a}(\tau,y,\eta)| &\le C_{j\alpha\beta}\tau^{m-j-k/2}\ (\tau^{1/2}+|y|+|\eta|)^{k-|\alpha|-|\beta|} \\ &= C_{j\alpha\beta}\,\tau^{m-j-(|\alpha|+|\beta|)/2}\left(1+|\tau^{-1/2}y|+|\tau^{-1/2}\eta|\right)^{k-|\alpha|-|\beta|}.\end{aligned} \tag{3.45}$$

Proof. It is straightforward to differentiate the right side of (3.43) with respect to y and η. We get

$$\begin{aligned}|D_y^\beta D_\eta^\alpha \hat{a}(\tau,y,\eta)| &= \tau^{-(|\alpha|+|\beta|)/2}\,|D_x^\beta D_\xi^\alpha b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta)| \\ &\le C\tau^{m-(|\alpha|+|\beta|)/2}\left(1+|\tau^{-1/2}y|+|\tau^{-1/2}\eta|\right)^{k-|\alpha|-|\beta|},\end{aligned} \tag{3.46}$$

thereby verifying (3.45) in case $j=0$. Since it is not so easy to apply D_τ^j to the right side of (3.43), we proceed more carefully. Let us say that $b_\pm(\lambda,x,\xi)$ belongs to $\Omega(m,k)$ if (3.40) and (3.41) hold. now applying one τ derivative to (3.43) gives

$$\begin{aligned}D_\tau b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta) &= D_\lambda b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta) \\ &\quad\mp \tfrac{1}{2}\tau^{-3/2}y\,D_x b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta) \\ &\quad- \tfrac{1}{2}\tau^{-3/2}\eta\,D_\xi b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta).\end{aligned} \tag{3.47}$$

Now, note that

$$c_\pm(\lambda,x,\xi)\in\Omega(m,k)\Longrightarrow D_\lambda c_\pm,\ \lambda^{-1}x\,D_x c_\pm,\ \lambda^{-1}\xi\,D_\xi c_\pm\in\Omega(m-1,k).$$

We deduce that, if $b_\pm(\lambda,x,\xi)\in\Omega(m,k)$, then

$$D_\tau b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta)=\sum_\ell c_{\ell\pm}(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta),$$

with $c_{\ell\pm}(\lambda,x,\xi)\in\Omega(m-1,k)$. It follows by induction that

$$D_\tau^j b_\pm(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta)=\sum_\ell c_{j\ell\pm}(\tau,\pm\tau^{-1/2}y,\tau^{-1/2}\eta),$$

with

$$c_{j\ell\pm}(\lambda,x,\xi)\in\Omega(m-j,k).$$

In view of this, the estimate (3.46) shows that (3.45) follows from (3.41). The converse result is proven by a similar argument.

The estimates (3.45) show that

$$\hat{a}(\tau,y,\eta)\in S^{m+\kappa/2}_{1/2\#},\quad \kappa=\max(k,0), \tag{3.48}$$

and, outside any conic neighborhood of $\{y=\eta=0\}$, $\hat{a}$ satisfies the estimates for membership in $S^{m+k/2}_{1\#}$. Recalling the support condition (3.44), we deduce the following from Proposition 1.1 and (1.23) of Chapter I.

Proposition 3.11. *If $A \in OP\Omega_b^{m,k}$, then*

$$A \in OPS^{m+\kappa/2}_{1/2,1/2}, \quad \kappa = \max(k,0). \tag{3.49}$$

Furthermore, A belongs to $OPS^{m+k/2}_{1,0}$ microlocally outside any conic neighborhood of the line bundle $\Lambda \subset T^*\mathbb{H}^n \setminus 0$, and, outside a certain conic neighborhood $\mathcal{V}$ of Λ, A is microlocally in $OPS^{-\infty}$.

The following result on the symbol calculus for $OP\Omega_b^{m,k}$ follows easily from the properties of $OP\mathcal{S}_1^*$.

Proposition 3.12. *If $A \in OP\Omega_b^{m,k}$ and $B \in OP\Omega_b^{m',k'}$, then, modulo a smoothing operator, $AB \in OP\Omega_b^{m+m',k+k'}$. Also $A^* \in OP\Omega_b^{m,k}$.*

Proof. We have $AB \in OPS^{m+m'+(\kappa+\kappa')/2}_{1/2,1/2}$, and

$$\sigma_{AB}(\pm\lambda)(X,D) = \sigma_A(\pm\lambda)(X,D)\sigma_B(\pm\lambda)(X,D). \tag{3.50}$$

It is clear that $\sigma_{AB}(\pm\lambda)(x,\xi)$ satisfies the condition (3.41), with m replaced by $m+m'$ and k replaced by $k+k'$. The support condition (3.40) is not quite verified, but since both A and B are microlocally in $OPS^{-\infty}$ outside $\mathcal{V}$, by Proposition 3.11, so is their product, so one can subtract a smoothing operator from AB to recover the support property (3.40). The proof for A^* is immediate.

In view of the material developed above, it is reasonable to introduce the following classes of operators.

Definition 3.13. *We say a right invariant operator A belongs to $OP\Omega^{m,k}$ provided that A coincides with an element of $OP\Omega_b^{m,k}$ microlocally on some conic neighborhood of Λ and A belongs to $OPS^{m+k/2}_{1,0}$ outside any conic neighborhood of Λ.*

A straightforward amalgamation of the symbol calculus (3.50) and the usual operator calculus for $OPS^m_{1,0}$ handles adjoints and products of operators in $OP\Omega^{m,k}$. We should point out that, in view of the characterization (3.45), $OP\Omega^{m,k}$ is contained in Boutet de Monvel's class

$$OPS^{m+k/2,k}(\mathbb{H}^n, \Lambda). \tag{3.51}$$

See Boutet de Monvel [B7] for a development of these operator classes. In fact, $OP\Omega^{m,k}$ coincides (modulo smoothing operators) with the set of right invariant operators in Boutet de Monvel's class. However, when the machinery of Chapter I is implemented to produce $OP\widetilde{\Omega}^{m,k}$ (see Chapter III), this class will be a strict subclass of $OPS^{m+k/2,k}(\mathbb{H}^n,\Lambda)$, and will have the advantage of possessing the symbol calculus we have developed.

Let us note the following inclusions of operator classes (which are all evident):

(3.52) $$OP\Psi^m \subset OP\Omega^{m/2,m},$$

(3.53) $$OP\Psi^{m,k} \subset OP\Omega^{m/2,m-k}, \quad k \geq 0,$$

(3.54) $$OP\Sigma^m \subset OP\Omega^{m,\mu}, \quad \mu = \max(m,0).$$

The classes $OP\Omega^{m,k}$ have associated hypoellipticity results, extending Theorem 2.17. We will postpone discussing such results until Chapter III.

4. Functional calculus for the Heisenberg Laplacian and for the harmonic oscillator

In §2 we considered inverses of the operators

$$\mathcal{L}_\alpha = \mathcal{L}_0 + i\alpha T, \tag{4.1}$$

where

$$\mathcal{L}_0 = \sum (L_j^2 + M_j^2) \tag{4.2}$$

is the "Heisenberg Laplacian." We saw that $\mathcal{L}_\alpha$ is hypoelliptic, with inverse in $OP\Psi_0^{-2}$, provided α avoids the set

$$\{\dots, -n-2, -n-1, -n\} \cup \{n, n+1, n+2, \dots\}. \tag{4.3}$$

Here we would like to understand the behavior of more general functions $f(\mathcal{L}_\alpha)$. There is some overlap between the material of this section and the material presented in Chapter 1, §7, of [T5], but here we will concentrate more on the technical aspects having to do with the theory of pseudodifferential operators. Recall that

$$\sigma_{\mathcal{L}_\alpha}(\pm\lambda)(X, D) = -\lambda(-\Delta + |x|^2 \pm \alpha). \tag{4.4}$$

It follows that

$$\sigma_{f(\mathcal{L}_\alpha)}(\pm\lambda)(X, D) = f\big(-\lambda(\Delta + |x|^2 \pm \alpha)\big). \tag{4.5}$$

Thus we need to understand

$$f(H), \quad H = -\Delta + |x|^2 \tag{4.6}$$

for a general class of functions of the harmonic oscillator $H = -\Delta + |x|^2$.

We begin by computing the Weyl symbol of the operator semigroup

$$e^{-tH} = h_t(X, D). \tag{4.7}$$

Formula (4.8) below was given by Peetre [P1]; see also Unterberger [U1].

Proposition 4.1. *We have*

$$\tilde{h}_t(q,p) = c_n'(sinh\, t)^{-n}\ e^{-(|q|^2+|p|^2)\coth t} \tag{4.8}$$

and

$$h_t(x,\xi) = (cosh\, t)^{-n}\ e^{-(|x|^2+|\xi|^2)\tanh t}. \tag{4.9}$$

Proof. As a first step, note that, by commutativity,

$$e^{-tH} = e^{-tH_1}\cdots e^{-tH_n},$$

where

$$H_j = -\frac{\partial^2}{\partial x_j^2} + x_j^2.$$

Thus $h_t(x,\xi) = h_t(x_1,\xi_1)\cdots h_t(x_n,\xi_n)$, and $\tilde{h}_t(q,p)$ satisfies the analogous multiplicative condition. Since the right sides of (4.8) and (4.9) are also multiplicative, it suffices to prove the proposition for $H = -d^2/dx^2 + x^2$, acting on functions of one variable. Now the Weyl symbol $h_t(x,\xi)$ is related to the integral kernel of the operator e^{-tH}, defined by

$$e^{-tH}u(x) = \int K_t(x,y)u(y)\,dy, \tag{4.10}$$

by

$$K_t(x,y) = \int e^{i(x-y)\xi}\, h_t\big(\tfrac{1}{2}(x+y),\xi\big)\,d\xi. \tag{4.11}$$

Consequently, the identity (4.9) (for $n=1$) is equivalent to

$$K_t(x,y) = (2\pi)^{-1/2}(\sinh 2t)^{-1/2}\ \exp\big\{[-\tfrac{1}{2}(\coth 2t)(x^2+y^2) - xy]/\sinh 2t\big\}, \tag{4.12}$$

which in turn is equivalent to Mehler's generating function identity

$$\sum_{j=0}^{\infty} h_j(x)h_j(y)t^j = \pi^{-1/2}(1-t^2)^{-1/2}\ \exp\big\{[2xyt+(x^2+y^2)t^2]/(1-t^2)\big\}\ e^{-x^2/2-y^2/2} \tag{4.13}$$

for hermite functions; see Lebedev [L4], pp. 61–63. This proves (4.9), and (4.8) follows by taking the Fourier transform.

A different proof of Proposition 4.1, making use of the Weyl calculus, is given in Appendix A of this paper; see (A.18). Yet another proof, closer to that of Peetre, utilizing the Bargmann-Fok representation of $\mathbb{H}^n$, is given in Chapter 1 of [T5].

Now we can make an analytic continuation of (4.7)–(4.9) to Re $t > 0$, and pass to the limit as t becomes purely imaginary. In that way we get formulas for the Weyl symbol of $e^{itH},\ t \in \mathbb{R}$. We have

$$e^{itH} = E_t(X, D) \tag{4.14}$$

with

$$E_t(x,\xi) = (\cos t)^{-n}\, e^{-i(\tan t)(|x|^2+|\xi|^2)}, \tag{4.15}$$

at least when t is different from a half-integral multiple of π. As such singular values, $E_t(x,\xi)$ achieves a limiting value in $\mathcal{S}'(\mathbb{R}^{2n})$.

One simple consequence of (4.15) is the following analysis of a general class of operators as pseudodifferential operators.

Proposition 4.2. *Let $f(\lambda) \in S^m_{\rho\#}$, i.e., suppose $|f^{(j)}(\lambda)| \le C_j(1+|\lambda|)^{m-\rho j}$. If $1/2 \le \rho \le 1$, then*

$$f(H) = F(X,D) \in OP\mathcal{S}^{2m}_{\rho'}, \quad \rho' = 2\rho - 1, \tag{4.16}$$

i.e.,

$$|D_x^\beta D_\xi^\alpha F(x,\xi)| \le C_{\alpha\beta}(1+|x|+|\xi|)^{2m-\rho'(|\alpha|+|\beta|)}.$$

Furthermore,

$$F(x,\xi) = f(|x|^2+|\xi|^2) + r(|x|^2+|\xi|^2), \tag{4.17}$$

with

$$r(|x|^2+|\xi|^2) \in \mathcal{S}^{2m-2(2\rho-1)}_{\rho'}. \tag{4.18}$$

Proof. If $f(\lambda) \in S^m_{\rho\#}(\mathbb{R})$, we can write $f = f_1 + f_2$ where $\hat{f}_1 \in \mathcal{E}'(\mathbb{R})$ is supported on $|t| < \pi/4$ and $f_2 \in \mathcal{S}(\mathbb{R})$, the Schwartz space of rapidly decreasing functions. Hence $f(H) = f_1(H) + f_2(H)$. Now

$$f_2(H) = H^{-k} f_{2,k}(H), \quad f_{2,k}(\lambda) = \lambda^k f_2(\lambda). \tag{4.19}$$

It follows that $f_2(H)$ is a smoothing operator, and

$$\|H^k f_2(H)\| \le \sup\, |\lambda^k f_2(\lambda)|. \tag{4.20}$$

As for $f_1(H)$, we have

$$f_1(H) = \int \hat{f}_1(t) e^{itH}\, dt, \tag{4.21}$$

and, using (4.15) with $|t| < \pi/4$, we have $f_1(H) = F_1(X, D)$, with

$$F_1(x,\xi) = 2^{-n} \int \hat{f}_1(\arctan s)(1+s^2)^{n/2} e^{is(|x|^2+|\xi|^2)}\, ds. \tag{4.22}$$

Here the distribution $\hat{f}_1(\arctan s)$ is written formally. Thus

$$F_1(x,\xi) = \varphi(|x|^2 + |\xi|^2), \tag{4.23}$$

with

$$\varphi(\lambda) = 2^{-n} \int \hat{f}_1(\arctan s)(1+s^2)^{n/2} e^{is\lambda}\, ds. \tag{4.24}$$

We see this is equal to the symbol (at $s = 0$) of a pseudodifferential operator obtained by applying a change of variable to $f_1(D_s) \in OPS^m_{\rho\#}(\mathbb{R})$. Thus, if $\rho \geq 1/2$,

$$\varphi(\lambda) - f(\lambda) \in S^{m-(2\rho-1)}_{\rho'\#}. \tag{4.25}$$

Our conclusions (4.37)–(4.38) follow immediately from this.

If $\rho > 1/2$, a complete asymptotic expansion can be given for $\varphi(\lambda)$. Note in particular that, if $f(\lambda)$ has an asymptotic expansion

$$f(\lambda) \sim \sum_{j\geq 0} f_{m-j}(\lambda), \tag{4.26}$$

with f_{m-j} homogeneous of degree $m - j$, then

$$F(x,\xi) \sim f(|x|^2 + |\xi|^2) + \sum_{j\geq 1} r_{2m-2j}(x,\xi) \in \mathcal{H}^{2m}_b. \tag{4.27}$$

It is well known that, if $F(x,\xi) \in S^0_{1,0}(\mathbb{R}^n)$, and in particular if $F(x,\xi) \in \mathcal{S}^0_1$, then $F(X,D)$ is continuous on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$. See, e.g., Nagase [N2] or Stein [S8]. As a consequence we deduce that, if $f(\lambda) \in S^0_{1\#}$, then $f(H) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, for $1 < p < \infty$. This result was proven by Mauceri [M2] by different means, involving setting up a Littlewood-Paley theory based on the semigroup e^{-tH}.

Proposition 4.2 is a special case of a result proven by Helffer and Robert [H4], by different means. The method of proof here, via

$$f(H) = \int \hat{f}(t) e^{itH}\, dt, \tag{4.28}$$

is a modification of the method used by the author to derive a functional calculus for first order elliptic self-adjoint pseudodifferential operators; see Chapter 12 of [T2], and also [T3]. Compare also some of the arguments in Appendix B of this paper.

One class of operators that Proposition 4.2 helps us analyze is the class of fractional powers of $-\mathcal{L}_0$. Note that, for $\alpha \in \mathbb{R}$,

$$\sigma_{(-\mathcal{L}_0)^\alpha}(\pm\lambda)(X,D) = \lambda^\alpha H^\alpha. \tag{4.29}$$

Now Proposition 4.2, together with (4.27), gives

$$H^\alpha \in OP\mathcal{H}_b^{2\alpha}, \tag{4.30}$$

if we let $f(\lambda)$ be an even element of $S^\alpha_{1\#}$ equal to $|\lambda|^\alpha$ for $\lambda \geq 1/2$ and smoothed out for $|\lambda| \leq 1/2$. Since $|\lambda| \leq 1/2$ does not intersect the spectrum of H, we have $H^\alpha = f(H)$. Now, we claim

$$H^\alpha = P_\alpha(X,D), \quad P_\alpha(x,\xi) \sim \sum_{j\geq 0} \varphi_{\alpha,j}(|x|^2 + |\xi|^2), \tag{4.31}$$

where $\varphi_{\alpha,0}(\lambda) = \lambda^\alpha$ and $\varphi_{\alpha,j}(\lambda)$ is homogeneous of degree $\alpha - 2j$ in λ; in other words, the odd terms in (4.26) are absent.

To prove (4.31) we may as well suppose $\alpha < 0$, since (4.31) could then be deduced for general α by multiplication by integral powers of H. In that case, we have $H^\alpha = F_1(X,D)$ mod $OP\mathcal{S}_1^{-\infty}$, with $F_1(X,D)$ given by (4.22) and

$$\hat{f}_1(\arctan s) = \psi(s)|\arctan s|^{1-\alpha} = \psi(s)|s|^{1-\alpha}\, g(s^2), \tag{4.32}$$

where $\psi \in C_0^\infty(-\pi/4, \pi/4)$, $\psi(s) = 1$ for $|s| \leq \pi/8$, say, and g is analytic. Since the factors $(1+s^2)^{n/2}$ and $g(s^2)$ are both analytic in s^2, for $|s| \leq \pi/4$, the result (4.31) follows. This gives

$$\sigma_{(-\mathcal{L}_0)^\alpha}(\pm 1)(X,D) = H^\alpha \in OP\mathcal{H}^{2\alpha}, \tag{4.33}$$

and hence Proposition 2.2 applies, to give the following result:

Proposition 4.3. *We have*

$$(-\mathcal{L}_0)^\alpha \in OP\Psi_0^{2\alpha}. \tag{4.34}$$

For more complicated functions of $\mathcal{L}_0$, Proposition 4.2 is not such an incisive tool, since it is not so effective in studying $f(\lambda H)$ for a large parameter λ, unless λ factors out, as it does in (4.39). However, using the identities given in Proposition 4.1, we can analyze the solution operator

$$e^{s\mathcal{L}_0} \tag{4.35}$$

to the "Heisenberg group heat equation"

$$\frac{\partial u}{\partial s} = \mathcal{L}_0 u. \tag{4.36}$$

In fact, we have

$$e^{s\mathcal{L}_0}\delta_0(t,q,p) = k_s(t,q,p) \tag{4.37}$$

with

$$\begin{aligned}\hat{k}_s(\pm\tau,y,\eta) &= \sigma_{e^{s\mathcal{L}_0}}(\pm\tau)(\pm\tau^{-1/2}y,\tau^{-1/2}\eta) \\ &= h_{s\tau}(\pm\tau^{-1/2}y,\tau^{-1/2}\eta),\end{aligned} \tag{4.38}$$

where $h_t(x,\xi)$ is given by (4.9). Hence, if

$$(\mathcal{F}_1 k_s)(\lambda,q,p) = \int_{-\infty}^{\infty} e^{-it\lambda}\, k_s(t,q,p)\,dt, \tag{4.39}$$

we have, by (4.8),

$$(\mathcal{F}_1 k_s)(\tau,q,p) = c_n \tau^n (\sinh s\tau)^{-n} \exp[-(\tau\coth\tau)(|q|^2+|p|^2)],$$

so

$$\begin{aligned}k_s(t,q,p) &= c_n s^{-n-1}\int_{-\infty}^{\infty} e^{i\tau(t/s)}\Big(\frac{\tau}{\sinh\tau}\Big)^n \exp[-(\tau\coth\tau)(|q|^2+|p|^2)/s]\,d\tau \\ &= s^{-n-1}\,k_1\Big(\frac{t}{s},\frac{q}{\sqrt{s}},\frac{p}{\sqrt{s}}\Big),\end{aligned} \tag{4.40}$$

where

$$k_1(t,q,p) = c_n\int_{-\infty}^{\infty} e^{it\tau}\Big(\frac{\tau}{\sinh\tau}\Big)^n \exp[-(\tau\coth\tau)(|q|^2+|p|^2)]\,d\tau. \tag{4.41}$$

Let us state this formally.

Proposition 4.4. *For $s>0$, we have*

$$e^{s\mathcal{L}_0}\delta_0(t,q,p) = k_s(t,q,p), \tag{4.42}$$

with

$$k_s(t,q,p) = s^{-n-1}k_1\Big(\frac{t}{s},\frac{q}{\sqrt{s}},\frac{p}{\sqrt{s}}\Big), \tag{4.43}$$

and $k_1\in\mathcal{S}(\mathbb{H}^n)$ given by (4.41).

This result was obtained by Gaveau [G1] by a different method, utilizing a diffusion process construction. It was also obtained, from a representation theory point of view, by

Hulanicki [H14] and in the Ph.D. thesis of Geller [G4]. From (4.40)–(4.43) it is a simple matter to calculate the kernel

$$(-\mathcal{L}_0)^{-1/2}e^{-s(-\mathcal{L}_0)^{1/2}}\delta_0(t,q,p)=P_s(t,q,p), \tag{4.44}$$

using the subordination identity

$$y^{-1}e^{-xy}=\pi^{1/2}\int_0^\infty e^{-x^2/4v}e^{-vy^2}v^{-1/2}\,dv, \tag{4.45}$$

valid for $x,y>0$. We obtain

$$P_s(t,q,p)=c_n'\int_{-\infty}^\infty\Big(\frac{\tau}{\sinh\tau}\Big)^n\big[\tfrac{1}{4}s^2+(\tau\coth\tau)(|q|^2+|p|^2)-it\tau\big]^{-n-1/2}\,d\tau. \tag{4.46}$$

We have developed certain aspects of the functional calculus for $\mathcal{L}_0$. It is useful to treat more general operators, of the form

$$P=\sum a_{jk}X_jX_k+i\alpha T=P_0+i\alpha T, \tag{4.47}$$

with (a_{jk}) real and positive definite, whose hypoellipticity was discussed in (2.68)–(2.72), as a consequence of Proposition 2.14. To construct the heat kernel, we can proceed as in (4.37)–(4.41), provided we know the Weyl symbol $h_t^Q(x,\xi)$ of

$$e^{-tQ(X,D)}=h_t^Q(X,D). \tag{4.48}$$

Here $Q(x,\xi)=\sum a_{jk}\chi_j\chi_k$, as in (2.72). Now, if $Q(x,\xi)$ takes the form

$$Q(x,\xi)=\sum\mu_j(x_j^2+\xi_j^2),\quad \mu_j>0, \tag{4.49}$$

which is equivalent to having

$$P_0=\sum\mu_j(L_j^2+M_j^2), \tag{4.50}$$

then the proof of Proposition 4.1 gives

$$h_t^Q(x,\xi)=\prod_{j=1}^n(\cosh t\mu_j)^{-1}\,\exp\Big(-\sum_{j=1}^n(x_j^2+\xi_j^2)(\tanh t\mu_j)\Big). \tag{4.51}$$

Consequently, for P_0 of the form (3.50), we have

$$e^{sP_0}\delta_0(t,q,p)=k_s^Q(t,q,p), \tag{4.52}$$

with

$$k_s^Q(t,q,p) = s^{-n-1} k_1^Q\Big(\frac{t}{s}, \frac{q}{\sqrt{s}}, \frac{p}{\sqrt{s}}\Big), \tag{4.53}$$

where $k_1^Q \in \mathcal{S}(\mathbb{R}^{2n+1})$ is given by

$$k_1^Q(t,q,p) = c_n \int_{-\infty}^{\infty} e^{it\tau} \prod_{j=1}^{n} \Big(\frac{\tau}{\sinh \mu_j \tau}\Big) \exp\Big(-\sum_{j=1}^{n} (\tau \coth \mu_j \tau)(q_j^2 + p_j^2)\Big) d\tau. \tag{4.54}$$

Now, given any P_0 of the form $\sum a_{jk} X_j X_k$, we can pick a symplectic basis of $\mathbb{R}^{2n}$ diagonalizing the quadratic form $Q(x,\xi) = \sum a_{jk} \chi_j \chi_k$. The resulting automorphism of $\mathbb{H}^n$ puts P_0 in the form (4.49), so the formula (4.54) for the heat kernel is valid generally, upon applying an automorphism to $\mathbb{H}^n$. If $a_{jk}(y)$ depends smoothly on a parameter y, automorphisms of $\mathbb{H}^n$ putting $P_0(y)$ into diagonal form may not be chosen to depend smoothly on y, so it is desirable to express the kernel (4.54) in an invariant form. We do this using the Hamilton map F_Q associated with the quadratic form Q, as defined in (2.48), i.e.,

$$Q(u,v) = \sigma(u, F_Q v), \tag{4.55}$$

where $Q(u,v)$ is the symmetric bilinear form such that $Q(u,u) = Q(u)$. The eigenvalues of F_Q are of the form $\pm i\mu_j$, where μ_j are as in (4.49), so

$$\det \sinh{(\tau/i)} F_Q = -\Big(\prod_{j=1}^{n} \sinh \mu_j \tau\Big)^2, \tag{4.56}$$

and hence

$$\prod_{j=1}^{n} \Big(\frac{\tau}{\sinh \mu_j \tau}\Big) = \big(-\tau^{-2n} \det \sinh{(\tau/i)} F_Q\big)^{-1/2}. \tag{4.57}$$

Now let

$$A_Q = (-F_Q^2)^{1/2}, \tag{4.58}$$

the unique square root of $-F_Q^2$ with positive spectrum. Then, with

$$(q,p) = z, \tag{4.59}$$

we can write

$$-\tau \sum_{j=1}^{n} (\coth \mu_j \tau)(q_j^2 + p_j^2) = -\tau Q(A_Q^{-1} \coth \tau A_Q z, z). \tag{4.60}$$

Consequently we can write the heat kernel (4.54) invariantly as

$$k_1^Q(t,z) = c_n \int_{-\infty}^{\infty} e^{it\tau}\, \Phi_Q(\tau,z)\, d\tau, \tag{4.61}$$

with

$$\Phi_Q(\tau,z) = \big(-\tau^{-2n} \det \sinh{(\tau/i)F_Q}\big)^{-1/2} \exp\big[-\tau Q(A_Q^{-1}\coth \tau A_Q z, z)\big]. \tag{4.62}$$

Note that, if $P_\alpha = P_0 + i\alpha T$, then

$$\begin{aligned} e^{sP_\alpha}\delta_0(t,q,p) &= e^{sP_0}\delta_0(t+is\alpha,q,p) \\ &= s^{-n-1} k_1^Q\Big(\frac{t}{s}+i\alpha, \frac{q}{\sqrt{s}}, \frac{p}{\sqrt{s}}\Big), \end{aligned} \tag{4.63}$$

where $k_1^Q(t+i\alpha,q,p)$ is defined from (4.61) by analytic continuation as long as

$$|\operatorname{Re}\alpha| < \sum_{j=1}^{n} \mu_j = \frac{1}{2}\operatorname{Tr} A_Q. \tag{4.64}$$

5. Remarks on the Heisenberg wave equation

Here we derive some results on solutions to the Heisenberg group wave equation

$$\frac{\partial^2 u}{\partial s^2} - \mathcal{L}_0 u = 0. \tag{5.1}$$

We can analyze this by analytically continuing (4.46). First, with $z = q + ip \in \mathbb{C}^n$, we can rewrite (4.46) as

$$P_s(t,z) = c_n \int_\gamma \Big(\frac{\tau}{\sin\tau}\Big)^n \big[s^2 + g_{A,B}(\tau)\big]^{-n-1/2}\, d\tau, \tag{5.2}$$

where the path γ is the imaginary axis, from $-i\infty$ to $+i\infty$, and

$$g_{A,B}(\zeta) = B\zeta\,\cot\zeta + A\zeta, \quad A = 4t, \quad B = 4|z|^2. \tag{5.3}$$

We can analytically continue $P_s(t,z)$ to Re $s > 0$, and pass to purely imaginary s, by deforming the contour γ. If γ is deformed to γ' in such a way that its image under $g_{A,B}$ hugs part of the positive real axis, then singularities of the deformed path γ' (where $g'_{A,B} = 0$) will correspond to singularities in the solution operator

$$(-\mathcal{L}_0)^{-1/2} e^{is(-\mathcal{L}_0)^{1/2}} \tag{5.4}$$

to (5.1) (with appropriate initial data). We refer to Chapter 1, §8 for details on this, but record the result here. Such a result was first obtained by Nachman [N1], by a different method.

Proposition 5.1. *The fundamental solution of the wave equation*

$$(-\mathcal{L}_0)^{-1/2} e^{is(-\mathcal{L}_0)^{1/2}} \delta_0(t,z) \tag{5.5}$$

has singularities only where, for some j

$$s^2 = g_{A,B}\big(x_j(A,B)\big), \tag{5.6}$$

where $g_{A,B}(\zeta)$ is given by (5.3) and the points $x_j(A,B)$ are the real zeros of $g'_{A,B}(x)$.

Such a result is also a special case of results on propagation of singularities proven by Melrose [M6] and Lascar [L1] for $\partial^2/\partial s^2 - P$ when P has symplectic characteristics of codimension 2 (including (5.5) on the three-dimensional Heisenberg group) and Lascar and Lascar [L2] in greater generality. These results imply the group $e^{is(-\mathcal{L}_0)^{1/2}}$ moves the wave front set of a distribution by the unique continuous (not smooth) extension of the Hamilton flow defined by the symbol of $(-\mathcal{L}_0)^{1/2}$ on the complement of the characteristic set. In particular, if P is any pseudodifferential operator on $\mathbb{H}^n$, the conjugate

$$e^{-is(-\mathcal{L}_0)^{1/2}} P e^{is(-\mathcal{L}_0)^{1/2}} = P(s) \tag{5.7}$$

must preserve wave front sets; $WF(P(s)u) \subset WF(u)$ for any $u \in \mathcal{E}'(\mathbb{H}^n)$. It is consequently of interest to analyze the conjugated operator as some sort of pseudodifferential operator, thus generalizing Egorov's theorem, which would describe such a conjugated operator if $\mathcal{L}_0$ were elliptic. We will look at some examples here where P is a convolution operator on $\mathbb{H}^n$. We will see that such conjugated operators are outside the classes of pseudodifferential operators we have defined so far, and further classes of operators will arise.

To start with what seems to be the simplest example, suppose $P \in OP\Psi_0^{m,\infty}$, so

$$\sigma_P(\pm\lambda)(X, D) = \lambda^m A_\pm, \quad A_\pm \in OP\mathcal{S}_1^{-\infty}. \tag{5.8}$$

It follows that $P(s)$, given by (5.7), satisfies

$$\sigma_{P(s)}(\pm\lambda)(X, D) = \lambda^m e^{-is\lambda^{1/2}H^{1/2}} A_\pm e^{is\lambda^{1/2}H^{1/2}} = \lambda^m B_\pm(s, \lambda). \tag{5.9}$$

Note that, for any integer k, the operator norm of $H^k B_\pm(s,\lambda) H^k$ on $L^2(\mathbb{R}^n)$ is bounded independently of λ, so, for $0 < \lambda < \infty$, $B_\pm(s,\lambda)$ is bounded in $OP\mathcal{S}_1^{-\infty}$ (as s runs over a bounded interval). If we take λ-derivatives, we see that

$$D_\lambda^j B_\pm(s,\lambda) = \lambda^{-j/2} e^{-is\lambda^{1/2}H^{1/2}} [(-\operatorname{ad} H^{1/2})^j (A_\pm)] e^{is\lambda^{1/2}H^{1/2}} + \cdots, \tag{5.10}$$

where the finite number of lower order terms in (5.10) all involve lower order powers of λ. Hence, for general $A_\pm$, $\lambda^{j/2} e^{is\lambda^{1/2}H^{1/2}}\, D_\lambda^j B_\pm(s,\lambda)\, e^{-is\lambda^{1/2}H^{1/2}}$ tends to a nonzero limit as $\lambda \to \infty$, so the best that can be said is:

$$\lambda^{j/2}\, D_\lambda^j B_\pm(s,\lambda) \quad \text{is bounded in } \ OP\mathcal{S}_1^{-\infty}, \text{ for } 1 \le \lambda < \infty. \tag{5.11}$$

Note that cutting off (5.9) on some bounded λ-interval alters $P(s)$ by a smoothing operator in this case, so we need only worry about $|\lambda| \ge 1$. Note that (5.11) is weaker than the hypothesis (3.41), in case $k = -\infty$, so $P(s)$ does not belong to $OP\Omega^{m,-\infty}$.

This example suggests the following class of operators. We say the right invariant operator A belongs to $OP\Omega_{1/2b}^{m,k}$ provided $b_\pm(\lambda, x, \xi) = \sigma_A(\pm\lambda)(x,\xi)$ satisfies the conditions

$$b_\pm(\lambda, x, \xi) \ \text{ is smooth and supported on } \ |x| + |\xi| \le C\lambda^{1/2}, \ \lambda \ge C', \tag{5.12}$$

and

$$\lambda^{j/2-m} D_\lambda^j b_\pm(\lambda, \cdot, \cdot) \ \text{ is bounded in } \ \mathcal{S}_1^k, \text{ for } \lambda \in \mathbb{R}^+. \tag{5.13}$$

Note the parallel with the conditions (3.40)–(3.41), defining $OP\Omega_b^{m,k}$. The analysis proving Proposition 3.10 shows that, for $Au = a * u$, the hypothesis (5.13) implies

$$|D_\tau^j D_y^\beta D_\eta^\alpha a(\tau, y, \eta)| \le C_{j\alpha\beta} \tau^{m-j/2-(|\alpha|+|\beta|)/2} \left(1 + |\tau^{-1/2} y| + |\tau^{-1/2}\eta|\right)^{k-|\alpha|-|\beta|}. \tag{5.14}$$

Thus we see that

$$OP\Omega_b^{m,k} \subset OP\Omega_{1/2b}^{m,k} \subset OPS_{1/2,1/2}^{m+\kappa/2}, \quad \kappa = \max(k, 0). \tag{5.15}$$

Also, in view of (5.14), it is reasonable to say a right invariant operator A belongs to $OP\Omega_{1/2}^{m,k}$ if it can be written as a sum of an element of $OP\Omega_{1/2b}^{m,k}$ and an element of $OPS_{1,0}^{m+k/2}$. It follows that

$$OP\Omega^{m,k} \subset OP\Omega_{1/2}^{m,k} \subset OPS_{1/2,1/2}^{m+\kappa/2}, \quad \kappa = \max(k, 0). \tag{5.16}$$

We should remark that $OP\Omega_{1/2}^{m,k}$ is not contained in Boutet de Monvel's class of operators $OPS^{m+k/2,k}(\mathbb{H}^n, \Lambda)$. If we let

$$OP\Omega_{1/2}^{m,-\infty} = \cap_{k>-\infty} OP\Omega_{1/2}^{m,k},$$

then what we have seen is that

$$P \in OP\Psi^{m,\infty} \Longrightarrow P(s) \in OP\Omega_{1/2}^{m,-\infty}, \tag{5.17}$$

where $P(s)$ is given by (3.71). More generally, one sees without much trouble that

$$P \in OP\Omega_{1/2}^{m,-\infty} \Longrightarrow P(s) \in OP\Omega_{1/2}^{m,-\infty}. \tag{5.18}$$

The assertions (5.17) and (5.18) follow directly from (5.10) and its natural generalization with $A_\pm$ replaced by $A_\pm(\lambda)$, in view of the fact that

$$E \in OP\mathcal{S}_1^{-\infty} \Longrightarrow e^{-isH^{1/2}} E e^{isH^{1/2}} \ \text{ is bounded in } \ OP\mathcal{S}_1^{-\infty}, \ s \in \mathbb{R}^+. \tag{5.19}$$

That this is true follows from the fact that a complete set of seminorms defining the topology of $OP\mathcal{S}_1^{-\infty}$ is given by the sequence of L^2-operator norms

$$P_k(E) = \|H^k E H^k\|, \quad k = 0, 1, 2, \dots, \tag{5.20}$$

each of which is invariant under conjugation by $e^{isH^{1/2}}$.

For finite k, it does not seem that $P \in OP\Omega_{1/2}^{m,k}$ implies $P(s) \in OP\Omega_{1/2}^{m,k}$. To see what sort of operator $P(s)$ is, we need to understand

$$E(t) = e^{itH^{1/2}} E_0 e^{-itH^{1/2}}, \quad t = s\lambda^{1/2}, \tag{5.21}$$

given $E_0 \in OP\mathcal{S}_1^k$. We need to have an analysis as $|t| \to \infty$ as well as for finite t. A good analysis for $|t|$ bounded is provided as follows. Set

$$E_0(t) = E_0, \tag{5.22}$$

and, for $j \geq 1$, let $E_j(t)$ be defined by

$$\frac{dE_j}{dt} = i[H^{1/2}, E_{j-1}(t)], \quad E_j(0) = 0. \tag{5.23}$$

Then, by induction, wee see that

$$E_j(t) = E_j t^j, \quad E_j = \frac{i^j}{j!}(\operatorname{ad} H^{1/2})^j E_0 \in OP\mathcal{S}_1^{k-j}. \tag{5.24}$$

If we write

$$E(t) = \sum_{j=0}^{N} E_j t^j + R_N(t) = F_N(t) + R_N(t), \tag{5.25}$$

we see that

$$\frac{dF_N}{dt} = i[H^{1/2}, F_N(t)] - it^N[H^{1/2}, E_N], \tag{5.26}$$

from which it follows, by Duhamel's principle, that

$$R_N(t) = \frac{i^{N+1}}{N!} \int_0^t e^{isH^{1/2}}[(\operatorname{ad} H^{1/2})^{N+1} \cdot E_0] e^{-isH^{1/2}} (t-s)^N \, ds. \tag{5.27}$$

Let us introduce the following notions. Set

$$\mathcal{D}_s = H^{-s/2}(L^2(\mathbb{R}^n)), \tag{5.28}$$

and say

$$T \in \mathcal{O}(m) \Longleftrightarrow T : \mathcal{D}_s \to \mathcal{D}_s, \quad \forall\ s \in \mathbb{R}. \tag{5.29}$$

Note that

$$OP\mathcal{S}_1^m \subset \mathcal{O}(m), \tag{5.30}$$

so we have $E_j \in \mathcal{O}(k-j)$ if $E_0 \in OP\mathcal{S}_1^k$, and, for each N,

$$(1+|t|)^{-N-1} R_N(t) \ \text{ is bounded in } \mathcal{O}(k-N), \quad \forall\ t \in \mathbb{R}. \tag{5.31}$$

This is not a very incisive result, due to the blow-up of $R_N(t)$ as $|t| \to \infty$.

We can do a little better by using a geometrical optics approach. Note that, if $e(t,x,\xi)$ is the full symbol of $E(t)$, it satisfies

$$\frac{\partial e}{\partial t} \sim H_g e + \sum_{j\geq 3,\text{odd}} \frac{2i}{j!}\{g,e\}_j, \tag{5.32}$$

where $\{g,e\}_j$ is given by (2.20) and g is the symbol of $H^{1/2}$:

$$g(X,D) = H^{1/2}. \tag{5.33}$$

We have $g \in \mathcal{H}_b^1$; in fact, by (4.31),

$$g(x,\xi) \sim g_1(x,\xi) + g_2(x,\xi) + \cdots, \tag{5.34}$$

with $g_{1+j}(x,\xi)$ homogeneous of degree $1-4j$ and

$$g_1(x,\xi) = (|x|^2 + |\xi|^2)^{1/2}. \tag{5.35}$$

We produce

$$e(t,x,\xi) \sim \sum_{j\geq 0} e_j(t,x,\xi), \tag{5.36}$$

with $e_0(t,x,\xi)$ satisfying

$$\frac{\partial e_0}{\partial t} = H_{g_1} e_0, \quad e(0,x,\xi) = E_0(x,\xi), \tag{5.37}$$

so

$$e_0(t,x,\xi) = E_0\big(\chi(t)(x,\xi)\big), \tag{5.38}$$

where

$$\chi(t) = \exp tH_{g_1} = \exp \frac{t}{g_1} \sum \frac{\partial}{\partial \theta_j}; \tag{5.39}$$

here

$$\sum \frac{\partial}{\partial \theta_j} = \frac{1}{2} H_{|x|^2+|\xi|^2} \tag{5.40}$$

generates a group

$$\rho(t) = \exp\, t \sum \frac{\partial}{\partial \theta_j} \tag{5.41}$$

of rotations on $\mathbb{R}^{2n}$, and

$$e_0(t,x,\xi)=E_0\bigl(\rho(t/g_1)(x,\xi)\bigr). \tag{5.42}$$

Note that $e_0(t,x,\xi)$ satisfies estimates of the form

$$|D_\xi^\alpha D_x^\beta e_0(t,x,\xi)|\le C_{\alpha\beta}(1+|x|+|\xi|)^{k-|\alpha|-|\beta|}\Bigl(1+\frac{|t|}{g_1}\Bigr)^{|\alpha|+|\beta|}. \tag{5.43}$$

Then

$$\frac{\partial e_0}{\partial t}(t,X,D)-i[H^{1/2},e_0(t,X,D)]=A_1(t,X,D), \tag{5.44}$$

where $A_1(t,x,\xi)$ satisfies estimates of the form

$$|D_\xi^\alpha D_x^\beta A_1(t,x,\xi)|\le C_{\alpha\beta}(1+|x|+|\xi|)^{k-5-|\alpha|-|\beta|}\Bigl(1+\frac{|t|}{g_1}\Bigr)^{3+|\alpha|+|\beta|}. \tag{5.45}$$

One can hence solve for $e_1(t,x,\xi)$ the equation

$$\frac{\partial e_1}{\partial t}-H_{g_1}e_1=-A_1(t,x,\xi),\quad e_1(0,x,\xi)=0, \tag{5.46}$$

and continue in this fashion, to get a more accurate approximation to $e(t,X,D)=E(t)$. Such an approach still does not give an incisive analysis of (5.7) for $P\in OP\Psi_0^{m,k}$, with k finite, but it does allow one to analyze (5.7) in case $P\in OP\Sigma^m$ has symbol vanishing on a conic neighborhood of Λ. In that case we are reduced to understanding (5.21) when the symbol of E_0 is supported on

$$M^{-1}\lambda^{1/2}\le(|x|^2+|\xi|^2)^{1/2}\le M\lambda^{1/2}, \tag{5.47}$$

and, as the right sides of (5.43) and (5.45) indicate, the growth in t does not present a problem in this case. Since the symbol of P is supported where $\mathcal{L}_0$ is elliptic, in this case, this analysis merely reproduces standard results of geometrical optics, so we will not dwell on it.

The correct class of (not necessarily right invariant) pseudodifferential operators, invariant under conjugation by such unitary operators as $e^{is(-\mathcal{L}_0)^{1/2}}$, and the corresponding extension of Egorov's theorem, remain to be achieved.

6. A hypoellipticity result of Rothschild

As we saw in §2, if $K_0 \in OP\Psi_0^m$ has the property that the operators $\sigma_{K_0}(X, D)$ are elliptic but not both injective, K_0 will not be hypoelliptic, but it is possible for $K_0 + K_1$ to be hypoeliptic, for some $K_1 \in OP\Psi^{-1}$. In [R4], Rothschild proved the following surprising result, giving a complete analysis for such operators that are differential operators.

Proposition 6.1. *Let K be a right invariant differential operator on $\mathbb{H}^n$. Suppose $K = K_0 + K_1$ with $K_0 \in OP\Psi_0^m$, $K_1 \in OP\Psi^{m-1}$, and $\sigma_{K_0}(\pm 1)(X, D)$ elliptic. Suppose furthermore that, for some M,*

$$\sigma_K(\pm\lambda)(X, D) \quad \textit{is injective, whenever } \lambda \geq M. \tag{6.1}$$

Then K is hypoelliptic.

This section will be devoted to a proof of the following natural generalization of Proposition 4.1.

Proposition 6.2. *Suppose $K \in OP\Psi^m$ is a pseudodifferential operator given by a finite sum*

$$K = K_0 + K_1 + \cdots + K_\mu, \quad K_j \in OP\Psi_0^{m-j}. \tag{6.2}$$

Suppose $\sigma_{K_0}(\pm 1)(X, D)$ elliptic, and suppose (6.1) holds. Then K is hypoelliptic.

The proof of Proposition 6.2 will be along the same lines as Rothschild's proof, with some simplifications, due to the use of the machinery developed in §2.

Of course, the hypoellipticity of K fails to follow from Theorem 2.17 only in the case when at least one of the operators $\sigma_{K_0}(\pm 1)(X, D)$ has a nontrivial kernel. Let $\pi_\pm$ denote the orthogonal projections of $L^2(\mathbb{R}^n)$ onto these kernels. (One of these operators might be zero, but not both.)

We make some preliminary simplifications. First, we can assume K is self-adjoint, since the hypoellipticity of K^*K implies the hypoellipticity of K, and hypothesis (6.1) for K implies the same sort of hypothesis for K^*K. Thus each K_j in (6.2) is self-adjoint, and $\sigma_{K_j}(\pm\lambda)(X, D)$ is formally self-adjoint. Now, if we define $\widetilde{K}_0 \in OP\Psi_0^m$ by

$$\sigma_{\widetilde{K}_0}(\pm\lambda)(X, D) = \sigma_{K_0}(\pm\lambda)(X, D) + \lambda^{m/2}\pi_\pm, \tag{6.3}$$

then $\widetilde{K}_0$ has an inverse $\widetilde{G}_0 \in OP\Psi_0^{-m}$, and we obtain a new operator

$$\begin{aligned} E &= \widetilde{G}_0 K = \widetilde{G}_0 K_0 + \cdots + \widetilde{G}_0 K_\mu \\ &= E_0 + E_1 + \cdots + E_\mu \in OP\Psi^0, \end{aligned} \tag{6.4}$$

with $E_j \in OP\Psi_0^{-j}$. To prove hypoellipticity of K, it will suffice to construct a parametrix for E. Note that

$$\sigma_{E_0}(\pm 1)(X, D) = I - \pi_\pm. \tag{6.5}$$

By the observation above, to construct a parametrix for E, it suffices to do so for E^*E, so we can assume without loss of generality that

$$E \in OP\Psi^0 \ \text{ is self-adjoint, with expansion (6.4),} \tag{6.6}$$

and the operator we have still satisfies (6.5), and, in addition:

$$\sigma_E(\pm\lambda)(X, D) \ \text{ is invertible, for } \lambda \geq M. \tag{6.7}$$

Consequently, we are reduced to constructing a parametrix for E, under assumptions (6.5)–(6.7). Note that $\pi_\pm$ are projections onto finite dimensional subspaces of $\mathcal{S}(\mathbb{R}^n)$ and hence are operators in $OP\mathcal{S}_1^{-\infty}$; this follows from general results on the kernel of an elliptic operator in $OP\mathcal{H}_b^0 \subset OP\mathcal{S}_1^0$; see Grusin [G12] or Beals [B4].

Note that, if

$$\sigma_{E_j}(\pm 1)(X, D) = A_j^\pm \in OP\mathcal{H}^{-j} \subset OP\mathcal{S}_1^{-j}, \tag{6.8}$$

then

$$\sigma_E(\pm\lambda)(X, D) = I - \pi_\pm + \sum_{j=1}^{\mu} \lambda^{-j/2} A_j^\pm. \tag{6.9}$$

The operators $A_j^\pm$ in (6.9) are all compact and self-adjoint. Now, if we set

$$\varepsilon = \lambda^{-1/2}, \quad |\varepsilon| \leq M^{-1/2}, \tag{6.10}$$

then, according to (6.9), $\sigma_E(\pm\lambda)(X, D)$ is equal to

$$P_\pm(\varepsilon) = I - \pi_\pm + \sum_{j=1}^{\mu} A_j^\pm \varepsilon^j = I - \pi_\pm + A^\pm(\varepsilon), \tag{6.11}$$

an analytic function (in fact, a polynomial) in ε, taking values in $OP\mathcal{H}^0$. Our hypothesis (6.7) states that the operators $P_\pm(\varepsilon)$ are invertible on $L^2(\mathbb{R}^n)$, for sufficiently small ε. We wish to study these inverses.

Clearly $A^\pm(\varepsilon)$ has small operator norm if $|\varepsilon|$ is small, so the spectrum of $P_\pm(\varepsilon)$ is concentrated near the points 0 and 1, say within a distance 1/4. Let γ denote the circle of radius 1/2 centered about 0, and set

$$\widetilde{\pi}_\pm(\varepsilon) = \frac{1}{2\pi i} \int_\gamma \big(\zeta - P_\pm(\varepsilon)\big)^{-1} \, d\zeta, \tag{6.12}$$

and

$$Q_\pm(\varepsilon) = I - \widetilde{\pi}_\pm(\varepsilon) = \frac{1}{2\pi i}\int_{\gamma'} \bigl(\zeta - P_\pm(\varepsilon)\bigr)^{-1}\, d\zeta, \tag{6.13}$$

where γ' denotes the circle of radius $1/2$ centered about 1. The operators $\widetilde{\pi}_\pm(\varepsilon)$ are analytic in ε, taking values in $\mathcal{L}(L^2(\mathbb{R}^n))$, and are all projections. Note that $\widetilde{\pi}_\pm(\varepsilon) \to \pi_\pm$ in operator norm as $\varepsilon \to 0$, so all are projections onto finite dimensional spaces, of dimension equal to the range of $\pi_\pm$, for ε small. Let us write

$$\widetilde{\pi}_\pm(\varepsilon) = \sum_{j=0}^{\infty} \kappa_j^\pm \varepsilon^j, \quad \kappa_0^\pm = \pi_\pm, \tag{6.14}$$

so $\kappa_j^\pm$ are bounded operators on $L^2(\mathbb{R}^n)$.

Lemma 6.3. *We have*

$$\kappa_j^\pm \in OP\mathcal{S}_1^{-\infty}.$$

Proof. The operators $Q_\pm(\varepsilon)$ are analytic in ε with values in $OP\mathcal{H}^0$, all with principal symbol equal to 1, so in fact $\widetilde{\pi}_\pm(\varepsilon)$ are analytic in ε with values in $OP\mathcal{S}_1^{-1}$, and hence

$$\widetilde{\pi}_\pm(\varepsilon) = \widetilde{\pi}_\pm(\varepsilon)^N$$

is analytic in ε with values in $OP\mathcal{S}_1^{-N}$, for each $N \in \mathbb{Z}^+$, which yields the desired result.

Suppose that $W_\pm(\varepsilon)$ inverts $P_\pm(\varepsilon)$ on the range of $\widetilde{\pi}_\pm(\varepsilon)$, and

$$W_\pm(\varepsilon) = \widetilde{\pi}_\pm(\varepsilon) W_\pm(\varepsilon) = W_\pm(\varepsilon)\widetilde{\pi}_\pm(\varepsilon). \tag{6.15}$$

This uniquely characterizes $W_\pm(\varepsilon)$. Then

$$(I + W_\pm(\varepsilon))P_\pm(\varepsilon) = I + (\widetilde{\pi}_\pm(\varepsilon) - \pi_\pm) + A^\pm(\varepsilon) = I + \widetilde{A}^\pm(\varepsilon), \tag{6.16}$$

where $\widetilde{A}^\pm(\varepsilon)$ is an analytic function of ε, with values in $OP\mathcal{S}_1^{-1}$. In fact,

$$\widetilde{A}^\pm(\varepsilon) = \sum_{j=1}^{\infty} \widetilde{A}_j^\pm \varepsilon^j, \quad \widetilde{A}_j^\pm \in OP\mathcal{H}^{-j}, \tag{6.17}$$

by (6.8) and Lemma 6.3. Consequently, if we define an operator $\mathcal{W}$ by

$$\sigma_{\mathcal{W}}(\pm\lambda)(X, D) = \psi(\varepsilon) W_\pm(\varepsilon), \quad \varepsilon = \lambda^{-1/2}, \tag{6.18}$$

where $\psi(\varepsilon)$ is smooth, with small support, and equal to 1 for ε very small, we have

$$(I + \mathcal{W})E = I + \mathcal{A}, \tag{6.19}$$

with

$$\sigma_{\mathcal{A}}(\pm\lambda)(X, D) = \widetilde{A}^\pm(\varepsilon), \quad \varepsilon = \lambda^{-1/2}, \ \text{ small}, \tag{6.20}$$

and hence, in light of (6.17),

$$\mathcal{A} \in OP\Psi^{-1}. \tag{6.21}$$

The following information on $\mathcal{W}$ will be central for our parametrix construction.

Lemma 6.4. *We have*

$$\mathcal{W} \in OP\Psi^{\kappa,\infty}, \tag{6.22}$$

for some $\kappa \in \mathbb{Z}^+$.

Proof. To prove (6.22), it suffices to show that for some κ

$$\varepsilon^\kappa W_\pm(\varepsilon) \ \text{ is analytic in } \varepsilon, \text{ with values in } OP\mathcal{S}_1^{-\infty}. \tag{6.23}$$

We transform the analysis of $W_\pm(\varepsilon)$ to a finite dimensional problem. Consider the operators

$$U_\pm(\varepsilon) = \pi_\pm \widetilde{\pi}_\pm(\varepsilon) + (I - \pi_\pm)(I - \widetilde{\pi}_\pm(\varepsilon)). \tag{6.24}$$

It is easy to see that

$$U_\pm(\varepsilon) : \mathcal{R}\big(\widetilde{\pi}_\pm(\varepsilon)\big) \longrightarrow \mathcal{R}(\pi_\pm), \tag{6.25}$$

where $\mathcal{R}(T)$ denotes the range of the operator T. Also,

$$\ker U_\pm(\varepsilon) = \ker \widetilde{\pi}_\pm(\varepsilon) \cap \mathcal{R}(\pi_\pm) \oplus \mathcal{R}\big(\widetilde{\pi}_\pm(\varepsilon)\big) \cap \ker \pi_\pm = \ker U_\pm(\varepsilon)^*. \tag{6.26}$$

Furthermore,

$$U_\pm(\varepsilon) = I + (I + 2\pi_\pm)\rho_\pm(\varepsilon) = I + K_\pm(\varepsilon), \tag{6.27}$$

where

$$\rho_\pm(\varepsilon) = \widetilde{\pi}_\pm(\varepsilon) - \pi_\pm, \tag{6.28}$$

and it is easy to verify from (6.26) that $\ker U_\pm(\varepsilon) = \ker U_\pm(\varepsilon)^*$ is spanned by the $+1$ and -1 eigenspaces of $K_\pm(\varepsilon)$. However, $\|K_\pm(\varepsilon)\|$ is small if ε is, so consequently $U_\pm(\varepsilon)$ is invertible for small ε, and hence the map (6.25) is bijective. It follows that

$$W_\pm(\varepsilon) = U_\pm(\varepsilon)^{-1} X_\pm(\varepsilon) U_\pm(\varepsilon), \tag{6.29}$$

where $X_\pm(\varepsilon)$ is the inverse, on the range of $\pi_\pm$, of

$$Y_\pm(\varepsilon) = U_\pm(\varepsilon) P_\pm(\varepsilon) U_\pm(\varepsilon)^{-1}. \tag{6.30}$$

Note that $U_\pm(\varepsilon)$ and $U_\pm(\varepsilon)^{-1}$ are analytic functions of ε with values in $OP\mathcal{S}_1^0$.

Now, on the range of $\pi_\pm$, $Y_\pm(\varepsilon)$ is an analytic family of $L_\pm \times L_\pm$ matrices, where $L_\pm$ is the dimension of the range of $\pi_\pm$. Since $P_\pm(\varepsilon)$ is a polynomial in ε, and since $U_\pm(\varepsilon)$ is invertible, uniformly as $\varepsilon \to 0$, it follows that these matrices have determinants that are analytic functions of ε, and, being not identically zero, they vanish to finite order, say to order κ, at $\varepsilon = 0$. Thus Cramer's rule applied to the construction of their inverses yields meromorphic matrix valued functions with poles of order at most κ at $\varepsilon = 0$. This proves the assertion (6.23) and establishes the lemma.

Now $I + \mathcal{A}$, arising in (6.19), has a parametrix $I + \mathcal{B}$, $\mathcal{B} \in OP\Psi^{-1}$, and hence

$$(I + \mathcal{B})(I + \mathcal{W})E = I, \tag{6.31}$$

modulo a smoothing operator. Since $(I+\mathcal{B})(I+\mathcal{W}) \in OP\Psi^\kappa$, this proves the hypoellipticity of E. In fact, we obtain the following more precise version of Proposition 6.2.

Proposition 6.5. *If $K \in OP\Psi^m$ satisfies the conditions of Proposition 6.2, then K is hypoelliptic, with a parametrix of the form*

$$L_1 + L_2, \tag{6.32}$$

where

$$L_1 \in OP\Psi^{-m}, \quad L^2 \in OP\Psi^{-m+\kappa,\infty}. \tag{6.33}$$

Finally we show that the condition (6.1) is necessary for hypoellipticity, granted the other hypotheses of Proposition 6.2. Indeed, passing from K to $E \in OP\Psi^0$ as before, we can still obtain the $L_\pm \times L_\pm$ matrix valued analytic function $X_\pm(\varepsilon)$, given by (6.29), making a slight change in the argument in Lemma 6.4 if E is not assumed to be self-adjoint. (The self-adjointness was used in a minimal way, in (6.26), and can be avoided.) Now, if the hypothesis (6.1) does not hold, then either $X_+(\varepsilon)$ or $X_-(\varepsilon)$ must have determinant identically zero (for small ε). Now we have the following result.

Lemma 6.6. *Assume $X(\varepsilon)$ is an analytic $L \times L$ matrix valued function, near $\varepsilon = 0$, with determinant identically zero. Then there is an analytic function $u(\varepsilon)$ with values in $\mathbb{C}^L$ such that $u(0) \neq 0$ and $X(\varepsilon)u(\varepsilon) \equiv 0$.*

Proof. Replacing $X(\varepsilon)$ by $X(\varepsilon)^* X(\varepsilon)$, we can suppose that $X(\varepsilon)$ itself is self adjoint. Then the result follows from Theorem 1.10, p. 71, of Kato [K1].

So if $X_+(\varepsilon)$ (say) has determinant identically zero, for $|\varepsilon|$ small, define $\widetilde{V}_+(\varepsilon)$ to be the orthogonal projection onto such $u(\varepsilon)$, thought of as an analytic function of ε with values in $\mathcal{R}(\pi_+) \subset \mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$, and let $V_+(\varepsilon) = U_+(\varepsilon)^{-1}\widetilde{V}_+(\varepsilon)U_+(\varepsilon)$, an analytic function of ε with values in $\mathcal{S}_1^{-\infty}$. Then define $S_0 \in OP\Psi^{0,\infty}$:

$$\sigma_{S_0}(\lambda)(X, D) = \psi(\lambda)V_+(|\lambda|^{-1/2}), \tag{6.34}$$

where $\psi(\lambda) = 1$ for $\lambda \geq 2M$, $\psi(\lambda) = 0$ for $\lambda \leq M$, M picked sufficiently large. Then we see that S_0 is not a smoothing operator, but clearly

$$KS_0 = 0, \quad \text{mod } OPS^{-\infty}. \tag{6.35}$$

This establishes the following converse to Proposition 6.2.

Proposition 6.7. *If $K \in OP\Psi^m$ is of the form (6.2) with $\sigma_{K_0}(\pm 1)(X, D)$ elliptic, then the hypothesis (6.1) is necessary as well as sufficient for hypoellipticity of K.*

Chapter III. Pseudodifferential operators on contact manifolds

A contact structure on a manifold M, of dimension $2n+1$, is given by a line bundle Λ in T^*M that is symplectic in $T^*M \setminus 0$, i.e., the symplectic form on $T^*M \setminus 0$ is nondegenerate acting on tangent vectors to Λ. An alternative characterization is the following: for a local nonvanishing section α of Λ, we demand that $\alpha \wedge d\alpha \wedge \cdots \wedge d\alpha \neq 0$, where there are n factors of $d\alpha$. Darboux' theorem implies that any two contact manifolds of the same dimension are locally diffeomorphic via a map preserving these contact structures, i.e., preserving α up to a scalar factor.

The Heisenberg group $\mathbb{H}^n$ forms a convenient local model for a contact manifold of dimension $2n+1$, as was emphasized by Dynin [D2], following Folland and Stein [F4], who used $\mathbb{H}^n$ as an "infinitesimal" model for constructing kernels on certain CR-manifolds, such as the boundary of a strictly pseudoconvex domain in $\mathbb{C}^{n+1}$. The contact structure we put on $\mathbb{H}^n$ is the line bundle, invariant by right translations, whose fiber over the identity in $\mathbb{H}^n$ is spanned by dt, in the coordinates on $\mathbb{H}^n$ used in Chapter II. This is also the characteristic set of the "Heisenberg Laplacian" $\mathcal{L}_0$, discussed in that chapter.

In the definition of a contact manifold M, associated with the line bundle Λ there is the orthogonal bundle $B \subset TM$, of fiber dimension $2n$. A CR-manifold (of maximal complex dimension) with nondegenerate Levi form, is a contact manifold with a family of complex structures on the fibers of B, which satisfy a certain integrability condition. For example, if M is a hypersurface in $\mathbb{C}^{n+1}$, B_x consists of the vectors $v \in \mathbb{C}^{n+1}$, tangent to M at x, such that iv is also tangent to M. In this case T_xM is naturally identified as an $\mathbb{R}$-linear subspace of $\mathbb{C}^n$. The nondegeneracy condition $\alpha \wedge d\alpha \cdots \wedge d\alpha \neq 0$ is equivalent to the nondegeneracy of the Levi form. On such a CR-manifold is a sequence of Kohn Laplacians $\square_b$, some of which are hypoelliptic with loss of one derivative, in the strictly pseudoconvex case. The analysis of such operators via analysis on $\mathbb{H}^n$ provides a nice tool for constructing parametrices of $\square_b$ in such cases. Such parametrices lie in classes of operators determined by the contact structure on M. As concerns the ease of constructing such parametrices, we are fortunate that they lie in classes of operators rather insensitive to the finer CR-structure, since two CR-manifolds need not be locally isomorphic, and classifying these structures may be hopeless. (On the infinitesimal level, one does have the Moser normal form.)

In §1 we develop several classes of pseudodifferential operators on contact manifolds. We use the Heisenberg group as a model. Operator classes studied include $OP\widetilde{\Psi}^m$, $OP\widetilde{\Psi}^{m,k}$, of pure Heisenberg type, the classical operators $OP\widetilde{\Sigma}^m = OPS^m$, and amalgamations, $OP\widetilde{\Omega}^{m,k}$ and $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$. We establish invariance of these operator classes under contact diffeomorphisms, to conclude that they are naturally defined on contact manifolds. Section 2 gives some technical results on the expansion of the symbol of a product.

Section 3 constructs parametrices for a natural class of subelliptic operators on contact manifolds, with double characteristics on the contact line bundle Λ. Second order subelliptic differential operators, such as the Kohn Laplacian $\Box_b$, have parametrices in $OP\widetilde{\Psi}^{-2}$. More general operators $P \in OPS^m$, doubly characteristic on Λ, are seen to be hypoelliptic, under a certain condition on the subprincpal symbol. We obtain parametrices in $OP\widetilde{\Omega}^{-m+1,-2}$. In §4 we look at $\Box^+$, the Neumann operator for the $\overline{\partial}$-Neumann problem; here $\Box^+ \in OPS^1(M)$ with $M = \partial\mathcal{O}$, where $\mathcal{O} \subset \mathbb{C}^n$ is a strongly pseudoconvex domain, and $\Box^+$ is doubly characteristic on one componentΛ^+ of $\Lambda \setminus 0$. We show that $\Box^+$ has a parametrix $E \in OP\widetilde{\mathfrak{H}}^{-2,1}_{\alpha,\delta}$, which is a more precise result than $E \in OP\widetilde{\Omega}^{0,-2}$. In fact, we obtain an even more precise analysis of E, as a sum of three terms, in $OP\widetilde{\Psi}^{0,1}$, $OP\widetilde{\Psi}^{-1}$, and $OP\widetilde{\mathfrak{H}}^{0,-1}_{\alpha,\delta}$.

In §5 we construct a parametrix for the heat semigroup e^{tP}, when P is a negative, self-adjoint second order subelliptic operator on a contact manifold M, doubly characteristic on Λ and with an appropriate restriction on its subprincipal symbol; the case of the Kohn Laplacian on the boundary of a strongly pseudoconvex domain is included. We obtain an asymptotic expansion for the trace of e^{tP}, which yields eigenvalue asymptotics.

In §6 we study the Szegö projector S, the orthogonal projection of $L^2(\partial\mathcal{O})$ onto the space of boundary values of holomorphic functions on $\mathcal{O}$, a strongly pseudoconvex domain in $\mathbb{C}^n$. We show that $S \in OP\widetilde{\Psi}^{0,\infty}$ and examine its symbol. We draw conections with other studies of S, particularly of Boutet de Monvel and Sjöstrand [B12].

1. Operator classes on contact manifolds

Since we are interested in a local analysis of pseudodifferential operators on a contact manifold M, we can implement Darboux' theorem and suppose $M=\mathbb{H}^n$, with the contact structure given by the right invariant line bundle $\Lambda=\operatorname{char}\mathcal{L}_0$, discussed in Chapter II. Thus our operator classes will be obtained from the classes of convolution operators developed in Chapter II, via the machinery fashioned in Chapter I. Recall that we developed in Chapter II the classes $OP\Psi^m$, $OP\Sigma^m$, and $OP\Omega^{m,k}$. Thus the theory of Chapter I gives us the following operator classes:

$$OP\widetilde{\Psi}^m, \quad OP\widetilde{\Psi}^{m,k}, \quad OP\widetilde{\Sigma}^m, \quad OP\widetilde{\Omega}^{m,k}. \tag{1.1}$$

We also make a further study of the classes $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$, introduced in Chapter I, §2, now specialized to $G=\mathbb{H}^n$. Of these, the class $OP\widetilde{\Sigma}^m$ has a transparent behavior. Indeed, the proof of Proposition 1.4 of Chapter I gives immediately:

Proposition 1.1. *Locally, and modulo smoothing operators, we have*

$$OP\widetilde{\Sigma}^m=OPS^m. \tag{1.2}$$

The other classes require further study. Of course, in light of (2.4) of Chapter II and the discussion following it, we have

$$OP\widetilde{\Psi}^m\subset OPS^m_{1/2,1/2}, \quad \text{if } m\ge 0, \quad OPS^{m/2} \quad \text{if } m\le 0, \tag{1.3}$$

as a corollary to Proposition 1.1 of Chapter I. Similarly, (2.18) of Chapter II implies

$$OP\widetilde{\Omega}^{m,k}\subset OPS^{m+\kappa/2}_{1/2,1/2}, \quad \kappa=\max(k,0). \tag{1.4}$$

Now, the whole point of our analysis is not to make such use of the rather weak results (1.3) and (1.4), but to develop symbolic operator calculi for $OP\widetilde{\Psi}^m$ and $OP\widetilde{\Omega}^{m,k}$. In order to do this, we need to verify the hpyotheses that figured in Propositions 1.2, 1.3, and 1.5 of Chapter I; these hypotheses are given in (1.25), (1.26), (1.27), (1.42), (1.43), and (1.46) of Chapter I. We recall them here:

$$\mathfrak{X}^m\subset S^m_{\rho\#} \quad \text{for some} \quad \rho\in(0,1],\ m\ge 0, \tag{(1.25)}$$

$$\mathfrak{X}^m\subset S^{m\sigma}_{\rho\#} \quad \text{if} \quad m<0,\ \text{for some } \sigma\in(0,1], \tag{(1.26)}$$

$$A\in OP\mathfrak{X}^m,\ B\in OP\mathfrak{X}^\mu\Longrightarrow AB\in OP\mathfrak{X}^{m+\mu}, \tag{(1.27)}$$

$$p(\xi)\in\mathfrak{X}^m\Longrightarrow D^\alpha_\xi p(\xi)\in\mathfrak{X}^{m-\tau|\alpha|}, \quad \text{for some} \quad \tau\in(0,1], \tag{(1.42)}$$

$$K_j\in\mathfrak{X}^{m-\tau j}\Longrightarrow\exists\ K\in\mathfrak{X}^m,\ K\sim K_0+K_1+\cdots, \tag{(1.43)}$$

$$p(\xi)\in\mathfrak{X}^m\Longrightarrow\overline{p(\xi)}\in\mathfrak{X}^m. \tag{(1.46)}$$

In the cases of Ψ^m, $\Psi^{m,k}$, and $\Omega^{m,k}$, all these results are either immediate from the definitions or taken care of explicitly in Chapter II, §2, except for ((1.42)), which we take care of now.

Proposition 1.2. *If* $\hat{k}(\tau, y, \eta) \in \Psi_0^m$, *then*

$$D_\tau^j D_y^{\gamma_1} D_\eta^{\gamma_2} \in \Psi_0^{m-2j-|\gamma_1|-|\gamma_2|}. \tag{1.5}$$

Proof. This is an immediate consequence of differentiating the identity

$$\hat{k}(\tau\lambda, \tau^{1/2}y, \tau^{1/2}\eta) = \tau^{m/2}\hat{k}(\lambda, y, \eta),$$

which defines membership in Ψ_0^m.

Proposition 1.3. *If* $\hat{k} \in \Omega^{m,k}$, *then*

$$D_\tau^j D_y^{\gamma_1} D_\eta^{\gamma_2} \hat{k}(\tau, y, \eta) \in \Omega^{m-j-(|\gamma_1|+|\gamma_2|)/2, k-|\gamma_1|-|\gamma_2|}. \tag{1.6}$$

Proof. This follows immediately from the characterization (3.45) of $\Omega^{m,k}$ given in Chapter II.

If $K(w)$ is a smooth function of $w = (t, q, p) \in \mathbb{H}^n$ with values in one of these classes, we denote the symbol of

$$\mathfrak{K}u(w) = K(w)u(w) \tag{1.7}$$

by

$$\sigma_{\mathfrak{K}}(w, \pm\lambda)(X, D) = \pi_{\pm\lambda}(K(w)). \tag{1.8}$$

Using Proposition 1.2 and the machinery of Chapters I and II, we deduce the following:

Proposition 1.4. *If* $A \in OP\widetilde{\Psi}^m$, $B \in OP\widetilde{\Psi}^\mu$, *then* $AB \in OP\widetilde{\Psi}^{m+\mu}$. *If* $C \in OP\widetilde{\Psi}^{m+\mu}$ *is defined by*

$$\sigma_C(w, \pm\lambda)(X, D) = \sigma_A(w, \pm\lambda)(X, D)\sigma_B(w, \pm\lambda)(X, D), \tag{1.9}$$

then

$$AB - C \in OP\widetilde{\Psi}^{m+\mu-1}. \tag{1.10}$$

To apply Proposition 1.3, note that

$$\Omega^{\mu-\nu/2, k-\nu} \subset \Omega^{\mu,k} \cap S_{1/2,1/2}^{\mu-\nu/2+\kappa(\nu)}, \quad \kappa(\nu) = \max(k-\nu, 0), \tag{1.11}$$

and that a formal series

$$\sum_{\nu \geq 0} P_\nu, \quad P_\nu \in OP\Omega^{\mu-\nu/2, k-\nu} \tag{1.12}$$

asymptotically sums to an element of $OP\Omega^{\mu,k}$. Also

$$\Omega^{\mu-1/2, k} \subset \Omega^{\mu, k-1}. \tag{1.13}$$

Hence we have:

Proposition 1.5. *If $A \in OP\widetilde{\Omega}^{m,k},\ B \in OP\widetilde{\Omega}^{\mu,\ell}$, then*

$$AB \in \widetilde{\Omega}^{m+\mu,k+\ell}. \tag{1.14}$$

Furthermore, if C is defined by (1.9), then

$$AB - C \in OP\widetilde{\Omega}^{m+\mu-1/2,k+\ell-1}. \tag{1.15}$$

We turn to a consideration of the invariance of $OP\widetilde{\Psi}^m$ and $OP\widetilde{\Omega}^{m,k}$ under coordinate changes. Let $\varphi : \mathbb{H}^n \to \mathbb{H}^n$ be a C^∞ diffeomorphism that preserves the contact structure, with inverse denoted ψ. If A is an operator on functions on $\mathbb{H}^n$ given by

$$Au(w) = \int a(w, wz^{-1})u(z)\,dz, \quad w, z \in \mathbb{H}^n, \tag{1.16}$$

then $B = \psi^* A \varphi^*$ is given by formula (1.60) of Chapter I:

$$Bu(w) = \int b\big(w, \Psi(w, z, wz^{-1})\big)u(z)\widetilde{H}(z)\,dz, \tag{1.17}$$

where $b(w, z) = a(\psi(w), z)$ and $\Psi(w, z, y)$ is linear in y, in exponential coordinates (which for $\mathbb{H}^n$ are the standard coordinates on $\mathbb{R}^{2n+1}$). In the formal expansion

$$b\big(w, \Psi(w, z, wz^{-1})\big) \sim \sum_{\gamma \geq 0, |\sigma| \geq |\gamma|} C_{\sigma\gamma}\,(wz^{-1})^{\gamma+\sigma}\, b_{(\gamma)}\big(w, \Psi(w)(wz^{-1})\big), \tag{1.18}$$

if we set $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ and

$$D^\gamma \hat{b}(\tau, y, \eta) = D_\tau^{\gamma_0} D_y^{\gamma_1} D_\eta^{\gamma_2} \hat{b}(\tau, y, \eta),$$

then the Fourier transform of the general term in (1.18) is

$$C_{\sigma\gamma}\, D_\tau^{\gamma_0+\sigma_0} D_y^{\gamma_1+\sigma_1} D_\eta^{\gamma_2+\sigma_2} \Big[\tau^{\gamma_0} y^{\gamma_1} \eta^{\gamma_2} \hat{b}\big(w, \Psi(w)^t(\tau, y, \eta)\big)\Big]. \tag{1.19}$$

Now the hypothesis that φ preserves the contact structure on $M = \mathbb{H}^n$ is equivalent to the hypothesis that

$$\Psi(w)^t = D\psi(w)^t \;\text{ preserves the space } \{y = \eta = 0\}. \tag{1.20}$$

We claim that, in this case, if $A \in OP\widetilde{\Psi}_0^m$, then (1.20) is a smooth function of $w \in \mathbb{H}^n$ with values in $\Psi^{m-2\sigma_0-|\sigma_1|-|\sigma_2|} \subset \Psi^{m-|\gamma|}$. This follows from Proposition 1.2 if we know that

$$\hat{b}\big(w, \Psi(w)^t(\tau, y, \eta)\big) \in \Psi^m,$$

for each w, which follows easily from (1.20). Similarly, using Proposition 1.3, one sees that if $A \in OP\widetilde{\Omega}^{m,k}$, then (1.19) is a smooth function of w with values in

$$\Omega^{m-\sigma_0-(|\sigma_1|+|\sigma_2|)/2,k-|\sigma_1|-|\sigma_2|} \subset \Omega^{m-|\gamma|/2,k}.$$

If the sum in (1.18) is restricted to $0 \leq |\gamma| \leq N,\ |\gamma| \leq |\sigma| \leq 2N$, it is not hard to see that the remainder term represents an arbitrarily smooth kernel if N is sufficiently large. We have the following result.

Proposition 1.6. *Let $\varphi : \mathbb{H}^n \to \mathbb{H}^n$ be a diffeomorphism that preserves the contact structure, with inverse ψ. Then*

$$A \in OP\widetilde{\Psi}^m \Longrightarrow \psi^* A \varphi^* \in OP\widetilde{\Psi}^m$$

and

$$A \in OP\widetilde{\Omega}^{m,k} \Longrightarrow \psi^* A \varphi^* \in OP\widetilde{\Omega}^{m,k}.$$

We now turn our attention to $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$, which was studied gor general 2 step nilpotent Lie groups in Chapter I, §2. In the case $G = \mathbb{H}^n$, $\alpha(s)(t,q,p) = (e^{2s}t, e^s q, e^s p)$, we can say a bit more. Recall that to say $\mathfrak{P}u(w) = P(w)u(w)$ defines $\mathfrak{P} \in OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$ is equivalent to saying that $P(w)u(w) = p_y * u(w)|_{y=w}$ and that $\hat{p}(w,\zeta) = \hat{p}_w(\zeta)$ has an asymptotic expansion

$$\hat{p}(w,\zeta) \sim \sum_{j\geq 0} a_j(w,\zeta) b_j(w,\zeta), \tag{1.21}$$

where

$$a_j(w,\zeta) \in \Psi^{m_j}, \quad b_j(w,\zeta) \in \Sigma^{\mu_j} \quad (\text{smooth in } j), \tag{1.22}$$

with

$$m_j \leq m, \quad \mu_j \leq \mu, \quad m_j + \mu_j \to -\infty. \tag{1.23}$$

As shown in Proposition 2.12 of Chapter I, $\mathfrak{P} \in OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$ if and only if

$$\mathfrak{P} \sim \sum_{j\geq 0} \mathfrak{K}_j \mathfrak{L}_j, \tag{1.24}$$

with

$$\mathfrak{K}_j \in OP\widetilde{\Psi}^{m_j}, \quad \mathfrak{L}_j \in OPS^{\mu_j}, \tag{1.25}$$

and m_j, μ_j satisfying (1.23). In the case $G = \mathbb{H}^n$, recall that, outside any conic neighborhood of $\Lambda \subset T^*\mathbb{H}^n \setminus 0$, elements of $OP\widetilde{\Psi}^m$ belong to OPS^m. Thus, microlocally away from Λ, $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$ coincides with $OPS^{m+\mu}$. In order to understand such operators better near Λ, we can use Proposition 3.4 from Chapter II. This yields:

Proposition 1.7. *If $K \in OP\widetilde{\Psi}^m$ and $L \in OPS^\mu$, the symbol of L being supported near Λ, then*

$$\sigma_{KL}(w, \pm\lambda) \sim \lambda^{m/2+\mu} \sum_{j\geq 0} \lambda^{-j/2} e_j^{\pm}(w,\lambda,X,D), \tag{1.26}$$

where

$$e_0^{\pm}(w,\lambda,x,\xi) = \sigma_K(w,\pm 1)(x,\xi)\ \sigma_L(w,\pm 1)(\lambda^{-1/2}x,\lambda^{-1/2}\xi), \tag{1.27}$$

and for $j \geq 1$

$$e_j^{\pm}(w,\lambda,x,\xi) = \sum_{k=1}^{K(j)} p_{kj}^{\pm}(w,\lambda^{-1/2}x,\lambda^{-1/2}\xi)a_{kj}^{\pm}(w,x,\xi), \tag{1.28}$$

with

$$\begin{aligned} &p_{kj}^{\pm}(w,x,\xi) \in C^\infty, \quad \textit{compactly supported in } (x,\xi), \\ &a_{kj}^{\pm}(w,x,\xi) \in \mathcal{H}_b^{m-j}. \end{aligned} \tag{1.29}$$

In a similar fashion we have

$$\sigma_{LK}(w,\pm\lambda)(X,D) \sim \lambda^{m/2+\mu} \sum_{j\geq 0} \lambda^{-j/2} f_j^{\pm}(w,\lambda,X,D), \tag{1.30}$$

where

$$f_0^{\pm} = e_0^{\pm}, \tag{1.31}$$

and $f_j^{\pm}(w,\lambda,x,\xi)$ *has an expression similar in form to (1.28)–(1.29).*

In light of the characterization (1.24)–(1.25) of $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$, we have:

Corollary 1.8. *Let the symbol of* $\mathfrak{P}$ *be supported near* Λ. *Then* $\mathfrak{P} \in OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$ *if and only if* $\sigma_{\mathfrak{P}}(w,\pm\lambda)(X,D)$ *has the form*

$$\sigma_{\mathfrak{P}}(w,\pm\lambda)(x,\xi) \sim \lambda^{m/2+\mu} \sum_{j\geq 0} \lambda^{-j/2} e_j^{\pm}(w,\lambda,x,\xi), \tag{1.32}$$

where $e_j^{\pm}(w,\lambda,x,\xi)$ *has the form (1.28)–(1.29).*

This result makes precise the inclusion

$$OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta} \subset OP\widetilde{\Omega}^{m/2+\mu,m}. \tag{1.33}$$

We also remark that, in light of the characterization (1.24)–(1.25), Proposition 1.6 extends to $A \in OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$.

2. More on symbol expansions of products

In this section we will derive further results on the expansion of a product

$$ABu(x) = \sum_{\gamma \geq 0} A^{[\gamma]}(x)\, B_{[\gamma]}(x) u(x), \tag{2.1}$$

derived in (1.41) of Chapter I. This material will be useful in some analyses of Szegö operators and Toeplitz operators, in §6. We desire to obtain explicit information on the major terms after the principal term. In particular, we will look at the cases

$$\gamma = (1,0,0), \quad \text{and} \;\; \gamma = (0,\gamma_1,\gamma_2), \quad |\gamma_1| + |\gamma_2| = 1 \;\; \text{or} \;\; 2. \tag{2.2}$$

We want to understand $A^{[\gamma]}$ on the symbol level. Recall from (1.37) of Chapter I that, if $Au(x) = a * u(x)$, then $A^{[\gamma]}u = a^{[\gamma]} * u$ with

$$\widehat{a^{[\gamma]}}(\xi) = D^{\gamma} \hat{a}(\xi). \tag{2.3}$$

In case $G = \mathbb{H}^n$, with coordinates (t,q,p) and dual coordinates (τ, y, η), this reads

$$\widehat{a^{[\gamma]}}(\tau,y,\eta) = D_{\tau}^{\gamma_0} D_y^{\gamma_1} D_{\eta}^{\gamma_2} \hat{a}(\tau,y,\eta), \quad \gamma = (\gamma_0,\gamma_1,\gamma_2). \tag{2.4}$$

Now recall that

$$\hat{a}(\pm\tau, y, \eta) = \sigma_A(\pm\tau)(\pm\tau^{-1/2} y, \tau^{-1/2}\eta), \tag{2.5}$$

and

$$\sigma_A(\pm\lambda)(x,\xi) = \hat{a}(\pm\lambda, \pm\lambda^{1/2} x, \lambda^{1/2}\xi). \tag{2.6}$$

It follows easily that

$$\sigma_{A^{[(0,\gamma_1,\gamma_2)]}}(\pm\lambda)(x,\xi) = \lambda^{-|\gamma|/2} (\pm D_x)^{\gamma_1} D_{\xi}^{\gamma_2} \sigma_A(\pm\lambda)(x,\xi). \tag{2.7}$$

In particular,

$$\begin{aligned} D_x^{\gamma_1} = D_{x_j} \Rightarrow \sigma_{A^{[(0,\gamma_1,0)]}}(X,D) &= \pm\lambda^{-1/2} (D_{x_j}\sigma_A)(\pm\lambda)(X,D) \\ &= \pm\lambda^{-1/2}\, [D_j, \sigma_A(\pm\lambda)(X,D)], \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} D_{\xi}^{\gamma_2} = D_{\xi_j} \Rightarrow \sigma_{A^{[(0,0,\gamma_2)]}}(X,D) &= \lambda^{-1/2}\, (D_{\xi_j}\sigma_A)(\pm\lambda)(X,D) \\ &= \lambda^{-1/2}\, [X_j, \sigma_A(\pm\lambda)(X,D)]. \end{aligned} \tag{2.9}$$

Equivalently, for right invariant A on $\mathbb{H}^n$, we have, in case (2.8),

$$A^{[(0,\gamma_1,0)]} = T^{-1}[M_j, A], \tag{2.10}$$

and in case (2.9),

$$A^{[(0,0,\gamma_2)]} = T^{-1}[L_j, A]. \tag{2.11}$$

Recall that L_j and M_j are the right invariant vector fields generating the Lie algebra of $\mathbb{H}^n$, and T spans its center.

Next we look at $A^{[(1,0,0)]}$. From (2.5) we have

$$\sigma_{A^{[(1,0,0)]}}(\pm\lambda)(x,\xi) = \bigl[-\tfrac{1}{2}\lambda^{-1}(x\cdot D_x + \xi\cdot D_\xi) + D_\lambda\bigr]\sigma_A(\pm\lambda)(x,\xi). \tag{2.12}$$

In particular, if $A \in OP\Psi_0^m$,

$$\begin{aligned}\sigma_{A^{[(1,0,0)]}}(\pm\lambda)(x,\xi) &= -\tfrac{1}{2}\lambda^{-1}(x\cdot D_x + \xi\cdot D_\xi - m)\sigma_A(\pm\lambda)(x,\xi)\\ &= -\frac{1}{2}\lambda^{-1}\Bigl(r\frac{\partial}{\partial r} - m\Bigr)\sigma_A(\pm\lambda)(x,\xi),\end{aligned} \tag{2.13}$$

where r denotes the radial variable; $r^2 = |x|^2 + |\xi|^2$. Note that the principal term in $\sigma_A(\pm\lambda)(x,\xi)$, homogeneous of degree m in (x,ξ), is annihilated by $r\,\partial/\partial r - m$.

We turn now to $B_{[\gamma]}(y)$, given by

$$B_{[\gamma]}(y) = \frac{1}{\gamma!}\, D_w^\gamma B((\exp w)y)\big|_{w=0}, \quad y \in \mathbb{H}^n, \tag{2.14}$$

in light of the formula (1.34) of Chapter I. If $|\gamma| = 1$, the right side of (2.14) clearly involves a right invariant vector field; we have

$$D_x^{\gamma_1} = D_{x_j} \Rightarrow B_{[(0,\gamma_1,0)]}(y) = (L_j B)(y), \tag{2.15}$$

and

$$D_\xi^{\gamma_2} = D_{\xi_j} \Rightarrow B_{[(0,0,\gamma_2)]} = (M_j B)(y). \tag{2.16}$$

To simplify notation, let $X_\gamma = L_j$ in case (2.15) and $X_\gamma = M_j$ in case (2.16), so we get

$$\gamma_0 = 0,\ |\gamma| = 1 \Longrightarrow B_{[\gamma]}(y) = (X_\gamma B)(y). \tag{2.17}$$

To complete the analysis of $B_{[\gamma]}(y)$ for $|\gamma| = 1$, note that

$$\gamma = (1,0,0) \Longrightarrow B_{[\gamma]}(y) = (TB)(y). \tag{2.18}$$

Let us now look at $B_{[(0,\gamma_1,\gamma_2)]}(y)$ in case $|\gamma_1|+|\gamma_2|=2$. We can write $(0,\gamma_1,\gamma_2)=\gamma+\gamma'$ with $|\gamma|=|\gamma'|=1$, so both γ and γ' satisfy the condition (2.17). Note that

$$X_{\gamma'}X_\gamma B(y)=D^{\gamma'}_{w'}D^\gamma_w B\big((\exp w)(\exp w')y\big)\big|_{w,w'=0}. \tag{2.19}$$

If we use the Campbell-Hausdorff formula

$$(\exp w)(\exp w')=\exp\big(w+w'+\tfrac{1}{2}[w,w']+\cdots\big) \tag{2.20}$$

and plug the right side of (2.19) into

$$B((\exp w)y)\sim\sum_{\gamma\ge0}w^\gamma\,B_{[\gamma]}(y), \tag{2.21}$$

we get

$$\begin{aligned}2B_{[\gamma+\gamma']}(y)&=X_{\gamma'}X_\gamma B(y)-\tfrac{1}{2}[X_{\gamma'},X_\gamma]B(y)\\&=\tfrac{1}{2}(X_{\gamma'}X_\gamma+X_\gamma X_{\gamma'})B(y).\end{aligned} \tag{2.22}$$

Note that, precisely when γ' and γ are complementary, e.g., $D^\gamma_{x,\xi}=D_{x_j},\ D^{\gamma'}_{x,\xi}=D_{\xi_j}$, we have $[X_{\gamma'},X_\gamma]$ nonzero, and then this commutator is $\pm T$.

3. Subelliptic operators on contact manifolds

Our main goal in this section is to show that the classes of operators considered in §1 contain the parametrices of classical pseudodifferential operators on a contact manifold M that are doubly characteristic on the contact line bundle and satisfy a certain condition, which will be given below, that will guarantee hypoellipticity with loss of one derivative. As we have seen, we can use local diffeomorphisms preserving the contact structure, and assume M is $\mathbb{H}^n$, with the right invariant contact structure described in the introduction to this chapter.

First consider a second order differential operator on $\mathbb{H}^n$, with nonnegative principal symbol vanishing to precisely second order on $\Lambda \subset T^*\mathbb{H}^n \setminus 0$. Then we have (generalizing (2.70) of Chapter II):

$$P = \sum_{j,k} a_{jk}(w) X_j X_k + i\alpha(w)T + B(w, D_w), \quad w \in \mathbb{H}^n. \tag{3.1}$$

Here, $a_{jk}(w) = a_{kj}(w)$ is a smooth function of w, forming a positive definite, real matrix, $\alpha(w) \in C^\infty(\mathbb{H}^n)$, and $B = B(w, D_w)$ is a first order differential operator on $\mathbb{H}^n$ whose principal symbol vanishes on Λ. Hence $B \in OP\widetilde{\Psi}^1$. The vector fields X_j are as in (2.69) of Chapter II. Our first result follows immediately from Theorem 2.17 of Chapter II and the machinery of §1 of this chapter.

Theorem 3.1. *Suppose that, for all $y \in \Omega \subset \mathbb{H}^n$, the symbols*

$$\sigma_{P_2}(\pm 1)(X, D) \tag{3.2}$$

are elliptic in $OP\mathcal{H}^2$ and invertible on $L^2(\mathbb{R}^n)$, where $P_2(y)$ is the right invariant differential operator

$$P_2(y) = \sum_{j,k} a_{jk}(y) X_j X_k + i\alpha(y)T, \tag{3.3}$$

which is to say, $\pm\alpha(y)$ avoids the discrete set determined by the spectrum of the second order operator $Q(y, X, D)$ associated with the quadratic form derived from the double sum in (3.3). (See Proposition 2.14 of Chapter II.) Then P, defined by (3.2), is hypoelliptic on Ω, with parametrix in $OP\widetilde{\Psi}^{-2}$.

Proof. Let $A \in OP\widetilde{\Psi}^{-2}$ be defined by

$$\sigma_A(w, \pm\lambda)(X, D) = \lambda^{-1}\, \sigma_{P_2(w)}(\pm 1)(X, D)^{-1}. \tag{3.4}$$

The Proposition 1.5 yields

$$PA = I + R_1, \quad AP = I + R_2, \quad R_j \in OP\widetilde{\Psi}^{-1}. \tag{3.5}$$

From here the standard construction of a parametrix for P goes through.

We remark that the analysis above works if in (3.1) $\alpha(w)$ and B are matrix valued, as long as $a_{jk}(w)$ are scalar. The condition for hypoellipticity is then that all eigenvalues of $\pm\alpha(y)$ avoid the spectrum of $Q(y,X,D)$, for all $y \in \Omega \subset \mathbb{H}^n$. Such a result applies to $\Box_b$, the Kohn Laplacian, on the boundary of a strictly pseudoconvex domain, in cases where one has hypoellipticity with loss of one derivative.

We take the space to explain here the phrase "hypoellipticity with loss of one derivative." If $P \in OPS^m$ is elliptic, we have the regularity result

$$Pu \in H^s_{\text{loc}} \Longrightarrow u \in H^{s+m}_{\text{loc}}.$$

Now, in the case covered by Proposition 3.1, since $OP\widetilde{\Psi}^{-2} \subset OPS^{-1}_{1/2,1/2}$, we have

$$Pu \in H^s_{\text{loc}} \Longrightarrow u \in H^{s+1}_{\text{loc}}, \tag{3.6}$$

rather than $u \in H^{s+2}_{\text{loc}}$. This result cannot be improved, and this explains the terminology.

We now consider the more general situation, where $P \in OPS^m$ is a classical pseudodifferential operator on $M = \mathbb{H}^n$ whose principal symbol is nonnegative and vanishes to exactly second order on Λ. We assume the Hessian of p_m transverse to Λ is nondegenerate. For the purpose of constructing a parametrix, we can compose P with an elliptic operator in OPS^{2-m}, and suppose without loss of generality that $P \in OPS^2$. We want to construct a microlocal parametrix belonging to $OP\widetilde{\Omega}^{-1,-2}$, under appropriate hypotheses. In the following, Λ^+ denotes a particular connected component of Λ. We will be working microlocally near Λ^+.

First consider the case when P is right invariant on $\mathbb{H}^n$. We can find operators $(A_{jk}) \in OP\Sigma^0$, whose symbol at a point of Λ^+ is a positive definite, real matrix, and $B \in OP\Sigma^1$, such that

$$P = \sum_{j,k} A_{jk} X_j X_k + B. \tag{3.7}$$

Here X_j are as in (3.1). Now write

$$A_{jk} = a_{jk} + B_{jk}, \tag{3.8}$$

where a_{jk} are real constants and $B_{jk} \in OP\Sigma^0$ have symbols vanishing on Λ^+. Similarly, write

$$B = i\alpha T + \sum B_j X_j + B_0, \quad B_j \in OP\Sigma^0,\ 0 \le j \le 2n. \tag{3.9}$$

Thus we have

$$\begin{aligned} P &= \sum_{j,k} a_{jk} X_j X_k + i\alpha T + \sum B_{jk} X_j X_k + \sum B_j X_j + B_0 \\ &= P_\alpha + \sum B_{jk} X_j X_k + \sum B_j X_j + B_0. \end{aligned} \tag{3.10}$$

Recall that $B_j,\ B_{jk} \in OP\Sigma^0$, and the symbols of B_{jk} vanish on Λ^+. Since by hypothesis P is elliptic off Λ^+, to construct a parametrix in $OP\Omega^{-1,-2}$, it will suffice to construct a parametrix for (3.10) where B_{jk}, B_j, and B_0 are cut off to have symbols supported in a small conic neighborhood of Λ^+.

The operator

$$P_\alpha = \sum_{j,k} a_{jk} X_j X_k + i\alpha T \tag{3.11}$$

belongs to $OP\Psi_0^2$. Suppose α avoids the appropriate discrete set, so there is

$$P_\alpha^{-1} \in OP\Psi_0^{-2}, \quad P_\alpha P_\alpha^{-1} = P_\alpha^{-1} P_\alpha = I \ \text{ microlocally near } \ \Lambda^+. \tag{3.12}$$

Now look at

$$Q = PP_\alpha^{-1} = I + \sum B_{jk} X_j X_k P_\alpha^{-1} + \sum B_j X_j P_\alpha^{-1} + B_0 P_\alpha^{-1}. \tag{3.13}$$

Its symbol is

$$\begin{aligned} \sigma_Q(\pm\lambda)(X,D) = I &+ \sum b_{jk}^{\pm}(\lambda^{-1/2}X, \lambda^{-1/2}D)\, a_{jk}^{\pm}(X,D) \\ &+ \lambda^{-1/2} \sum b_j^{\pm}(\lambda^{-1/2}X, \lambda^{-1/2}D)\, a_j^{\pm}(X,D) \\ &+ \lambda^{-1} b_0^{\pm}(\lambda^{-1/2}X, \lambda^{-1/2}D)\, a_0^{\pm}(X,D). \end{aligned} \tag{3.14}$$

Here

$$b_{jk}^{\pm}(x,\xi),\ b_j^{\pm}(x,\xi) \in C_0^\infty(\mathbb{R}^{2n}) \tag{3.15}$$

and

$$a_{jk}^{\pm}(x,\xi) \in \mathcal{H}^0, \quad a_0^{\pm}(x,\xi) \in \mathcal{H}^{-2}, \quad a_j^{\pm}(x,\xi) \in \mathcal{H}^{-1},\ j \ge 1. \tag{3.16}$$

We deduce that

$$\sigma_Q(\pm\lambda)(X,D) = I + r_{\pm}(\lambda, X, D) \tag{3.17}$$

where

$$r_{\pm}(\lambda, x, \xi) \ \text{ is supported in } \ |x| + |\xi| \le C_1 \lambda^{1/2},\ \lambda \ge C, \tag{3.18}$$

and

$$\lambda^k D_\lambda^k r_{\pm}(\lambda, \cdot, \cdot) \ \text{ is bounded in } \ \mathcal{S}_1^0, \quad \text{for } \lambda \ge 1. \tag{3.19}$$

Since we are working microlocally near Λ^+, we could drop the $\pm$ subscripts, but we will instead retain the $\pm$ notation.

Now, in (3.19), we can suppose $r_\pm(\lambda, x, \xi)$ is small. In fact, if B_{jk} are cut off sufficiently near Λ, we can suppose, the ε_0 picked small

$$\sup_{x,\xi} |r_\pm(\lambda, x, \xi)| \le C_0(\varepsilon_0 + \lambda^{-1/2}), \tag{3.20}$$

and, for $\alpha > 0$,

$$\sup_{x,\xi} |D^\alpha_{x,\xi}\, r_\pm(\lambda, x, \xi)| \le C_\alpha \lambda^{-1/2}. \tag{3.21}$$

An operator norm estimate proven in Appendix A implies

$$\|r_\pm(\lambda, X, D)\| \le C(\varepsilon_0 + \lambda^{-1/2}). \tag{3.22}$$

Thus, for λ large, $I + r_\pm(\lambda, X, D)$ is invertible on $L^2(\mathbb{R}^n)$, so we can define

$$I + s_\pm(\lambda, X, D) = \big(I + r_\pm(\lambda, X, D)\big)^{-1}. \tag{3.23}$$

We now undertake to show that $s_\pm(\lambda, x, \xi)$ satisfies the conditions (3.19)–(3.21) and a slight midification of (3.18).

As a preparation for the analysis of $s_\pm(\lambda, x, \xi)$, let us define $\sigma_\pm(\lambda, x, \xi)$ by the identity

$$1 + \sigma_\pm(\lambda, x, \xi) = \big(1 + r_\pm(\lambda, x, \xi)\big)^{-1}, \quad \lambda \ \text{ large}. \tag{3.24}$$

It is easy to see that $\sigma_\pm(\lambda, x, \xi)$ satisfies (3.18)–(3.21). In particular, the L^2 operator norm $\|\sigma_\pm(\lambda, X, D)\|$ is small, for λ large. Thus we have

$$(I + r_\pm(\lambda, X, D))(I + \sigma_\pm(\lambda, X, D)) = I - \rho_\pm(\lambda, X, D), \tag{3.25}$$

with

$$\|\rho_\pm(\lambda, X, D)\| \ \text{ small, for } \lambda \text{ large}. \tag{3.26}$$

Also $\rho_\pm(\lambda, x, \xi) \in \mathcal{S}^{-1}_1$, and more generally

$$\lambda^k\, D^k_\lambda \rho_\pm(\lambda, \cdot, \cdot) \ \text{ is bounded in } \mathcal{S}^{-1}_1, \quad \text{for } \lambda \text{ large}. \tag{3.27}$$

Furthermore, $\rho_\pm(\lambda, x, \xi)$ satisfies the estimate (3.20). Using $\rho_\pm(\lambda)$ as shorthand for $\rho_\pm(\lambda, X, D)$, we have

$$(I - \rho_\pm(\lambda))^{-1} = I + \rho_\pm(\lambda) + \cdots + \rho_\pm(\lambda)^k + \rho_\pm(\lambda)^\ell (I - \rho_\pm(\lambda))^{-k} \rho_\pm(\lambda)^{k-\ell}, \tag{3.28}$$

for $0 \le \ell \le k$. It follows that

$$(I - \rho_\pm(\lambda))^{-1} = I + \tau_\pm(\lambda, X, D) \tag{3.29}$$

with $\lambda^k D_\lambda^k \tau_\pm(\lambda, \cdot, \cdot)$ bounded in $\mathcal{S}_1^{-k}$ for λ large, and $\tau_\pm$ satisfies (3.20)–(3.21). Since

$$I + s_\pm(\lambda, X, D) = (I + \sigma_\pm(\lambda, X, D))(I + \tau_\pm(\lambda, X, D)), \tag{3.30}$$

we see that $s_\pm(\lambda, X, D)$ satisfies conditions (3.19)–(3.20).

We proceed to modify $s_\pm(\lambda, x, \xi)$ to obtain (3.18). Note that (3.23) is equivalent to

$$s_\pm(\lambda, X, D) = -r_\pm(\lambda, X, D) - r_\pm(\lambda, X, D) s_\pm(\lambda, X, D). \tag{3.31}$$

Since $r_\pm(\lambda, X, D)$ satisfies (3.18), we can find $\chi \in C_0^\infty(\mathbb{R}^{2n})$ with $\chi(\lambda^{-1/2}x, \lambda^{-1/2}\xi) = 1$ on supp $r_\pm(\lambda, x, \xi)$, and then

$$r_\pm(\lambda, X, D)\chi(\lambda^{-1/2}X, \lambda^{-1/2}D) - r_\pm(\lambda, X, D) = v_\pm(\lambda, X, D) \tag{3.32}$$

with $D_\lambda^k v_\pm(\lambda, x, \xi) = O(\lambda^{-\infty})$ in $\mathcal{S}_1^{-\infty}$, for large λ, with a similar result for the products in the reverse order. Thus if we set

$$\tilde{s}_\pm(\lambda, X, D) = s_\pm(\lambda, X, D)\chi(\lambda^{-1/2}X, \lambda^{-1/2}D), \tag{3.33}$$

formula (3.31) shows that

$$\tilde{s}_\pm(\lambda, X, D) = -r_\pm(\lambda, X, D) - r_\pm(\lambda, X, D)\tilde{s}_\pm(\lambda, X, D) + \tilde{v}_\pm(\lambda, X, D), \tag{3.34}$$

with $D_\lambda^k \tilde{v}_\pm = O(\lambda^{-\infty})$ in $\mathcal{S}_1^{-\infty}$. This is equivalent to

$$(I + r_\pm(\lambda, X, D))(I + \tilde{s}_\pm(\lambda, X, D)) = I + \delta_\pm(\lambda, X, D), \tag{3.35}$$

with $\delta_\pm(\lambda, x, \xi) = O(\lambda^{-\infty})$ in $\mathcal{S}_1^{-\infty}$. Thus

$$\sigma_D(\pm\lambda)(X, D) = \delta_\pm(\lambda, X, D)\psi(\lambda) \tag{3.36}$$

defines a smoothing operator on $\mathbb{H}^n$, where $\psi(\lambda)$ is a smooth cut-off, equal to 1 for large λ, 0 for small λ. Now $\tilde{s}_\pm(\lambda, X, D)$ defines an operator $\widetilde{S} \in OP\Omega_b^{0,0}$ by

$$\sigma_{\widetilde{S}}(\pm\lambda)(X, D) = \tilde{s}_\pm(\lambda, X, D), \tag{3.37}$$

and (3.17) and (3.35) yield

$$PP_\alpha^{-1}(I + \widetilde{S}) = I + D, \tag{3.38}$$

with D a smoothing operator on $\mathbb{H}^n$. A similar argument yields a left parametrix for P, and the two are seen to be equal, modulo a smoothing operator, so a two-sided parametrix for P is given by

$$E = P_\alpha^{-1}(I + \widetilde{S}) \in OP\Omega^{-1,-2}. \tag{3.39}$$

Given this work on the right invariant case, we can now easily obtain the following general result.

Theorem 3.2. *Let $P \in OPS^m$ have nonnegative principal symbol, doubly characteristic on the variety $\Lambda \subset T^*\mathbb{H}^n \setminus 0$, satisfying the condition*

$$p_m(z,\zeta) \geq C|\zeta|^{m-2} dist((z,\zeta),\Lambda)^2.$$

Write P in the form

$$Pu(w) = P(w)u(w) \tag{3.40}$$

where $P(y)$ is a right invariant (convolution) operator in OPS^m, for each $y \in \mathbb{H}^n$. Construct, for each $y \in \mathbb{H}^n$, the second order differential operator $P_\alpha(y)$, according to the prescription (3.7)–(3.11). Suppose that, for each $y \in \Omega \subset \mathbb{H}^n$, the operator $P_\alpha(y)$ satisfies the condition for hypoellipticity (microlocally near Λ^+) given in Theorem 3.1. Then P has a parametrix on Ω, microlocally near Λ^+:

$$\mathfrak{E} \in OP\widetilde{\Omega}^{-m+1,-2}, \tag{3.41}$$

and consequently P is hypoelliptic on Ω, microlocally near Λ^+.

Proof. The argument just given leads to $E(y)$, a smooth function of y with values in $OP\Omega^{-m+1,-2}$, such that $E(y)$ is a parametrix for $P(y)$, for each y. Define $\mathfrak{E}_1 \in OP\widetilde{\Omega}^{-m+1,-2}$ by

$$\mathfrak{E}_1 u(w) = E(w)u(w). \tag{3.42}$$

Then, by Proposition 1.5, especially (1.15), we have

$$\mathfrak{E}_1 P = I + R, \quad R \in OP\widetilde{\Omega}^{-1/2,-1}. \tag{3.43}$$

Since $OP\widetilde{\Omega}^{-1/2,-1} \subset OPS^{-1/2}_{1/2,1/2}$, R has negative order, so the Neumann series yields a parametrix in $OP\widetilde{\Omega}^{0,0}$ for I_R:

$$(I+R)^{-1} = I + S, \quad S \in OP\widetilde{\Omega}^{-1/2,-1}. \tag{3.44}$$

Then a parametrix for P is given by

$$\mathfrak{E} = (I+S)\mathfrak{E}_1. \tag{3.45}$$

This finishes the proof.

Since $OP\widetilde{\Omega}^{-m+1,-2} \subset OPS^{-m+1}_{1/2,1/2}$, we get the regularity results

$$Pu \in H^s_{\text{loc}} \Longrightarrow u \in H^{s+m-1}_{\text{loc}}, \tag{3.46}$$

as in (3.6), i.e., hypoellipticity with loss of one derivative. It is useful to complement this with a couple of other results. If $\mathfrak{E} \in OP\widetilde{\Omega}^{-m+1,-2}$, then, since $X_j \in OP\Psi^1_0$, we have

$$X_j \mathfrak{E} \in OP\widetilde{\Omega}^{-m+3/2,-1} \subset OPS^{-m+3/2}_{1/2,1/2}, \tag{3.47}$$

and

$$X_j X_k \mathfrak{E} \in OP\widetilde{\Omega}^{-m+2,0} \subset OPS^{-m+2}_{1/2,1/2}. \tag{3.48}$$

Thus, in addition to (3.46), we have

$$Pu \in H^s_{\text{loc}} \Longrightarrow X_j u \in H^{s+m-3/2}_{\text{loc}}, \; X_j X_k u \in H^{s+m-2}_{\text{loc}}, \tag{3.49}$$

providing the condition for hypoelliptiticy in Theorem 3.2 is satisfied.

We next want to restate the condition for hypoellipticity with loss of one derivative given in Theorem 3.2, for an operator $P \in OPS^m$, doubly characteristic on $\Lambda \subset T^*\mathbb{H}^n \setminus 0$, more explicitly in terms of its symbol. We will produce some invariants associated with the symbol of such an operator. These quantities played a role in the papers of Sjöstrand [S6], Boutet de Monvel [B7], Ivrii and Petkov [I1], Hörmander [H9], and others, and they are also discussed in Chapters 13 and 15 of [T2]. For the purposes of this discussion, we shall assume $m = 2$, which involves no loss of generality.

If $P \in OPS^2$ has principal p_2, nonnegative and vanishing to second order on Λ, with nondegenerate transverse Hessian, then, microlocally near any given point of Λ, we can write

$$p_2(x,\xi) = \sum_{j=1}^{\mu} a_j(x,\xi)^2, \tag{3.50}$$

where $a_j(x,\xi)$ are real, homogeneous of degree 1, vanishing on Λ, with da_j linearly independent at each point of Λ. Hence

$$P = \sum_{j=1}^{\mu} a_j(x,D)^2 + B(x,D), \quad B \in OPS^1. \tag{3.51}$$

On the set Λ, we have

$$\begin{aligned} B_1(x,\xi) &= p_1(x,\xi) - \frac{1}{i} \sum_{|\alpha|=1} a_j^{(\alpha)}(x,\xi)\, a_{j(\alpha)}(x,\xi) \\ &= p_1(x,\xi) + \frac{i}{2} \sum_{\nu} \frac{\partial^2 p_2}{\partial x_\nu \partial \xi_\nu}, \end{aligned} \tag{3.52}$$

The right side of (3.52) is called the subprincipal symbol of P, and is denoted

$$\text{sub}\sigma(P)(x,\xi). \tag{3.53}$$

It is easy to see that, for any $A \subset OPS^0$, with principal symbol $a(x, \xi)$,

$$\text{sub}\sigma(AP) = \text{sub}\sigma(PA) = a(x, \xi)\,\text{sub}\sigma(P)(x, \xi), \quad \text{on } \Lambda. \tag{3.54}$$

Furthermore, if J is an elliptic Fourier integral operator with associated canonical transformation $\mathcal{J}$, then

$$\text{sub}\sigma(JPJ^{-1})(\mathcal{J}(x, \xi)) = \text{sub}\sigma(P)(x, \xi), \quad \text{on } \Lambda. \tag{3.55}$$

For two proofs of this, see Chapter 15 of [T2].

Also, at each point $(x_0, \xi_0) \in \Lambda$, we define a Hamilton map F, which is associated to the Hessian of $p_2(x, \xi)$ at (x_0, ξ_0); see the process by which a Hamilton matrix is associated to a quadratic form on a symplectic vector space, as described in Proposition 2.14 of Chapter II. It is straightforward that, for the special case

$$P = \sum_{j=1}^{n} \mu_j (L_j^2 + M_j^2) + i\alpha T + B = \sum_{j=1}^{n} \mu_j (X_j^2 + X_{j+n}^2) + i\alpha T + B, \tag{3.56}$$

where B is a first order differential operator whose principal symbol vanishes on Λ, we have (with (τ, y, η) denoting dual coordinates to (t, q, p)), on Λ^+,

$$\text{sub}\sigma(P)((t, q, p), (\tau, y, \eta)) = -\alpha|\tau|, \tag{3.57}$$

and

$$\text{spec}\, i^{-1} F = \{\pm \mu_j |\tau|\} = \{\pm \mu_j^{\#}\}, \tag{3.58}$$

the last equality serving to define $\mu_j^{\#}$. In particular, if we denote by $\text{Tr}^+ F$ the sum of the positive eigenvalues of F/i (counting multiplicities), we have

$$\text{Tr}^+ F = \sum \mu_j |\tau|. \tag{3.59}$$

In this case Theorem 3.2 reduces to a special case of the analysis of the right invariant operators in (2.70) of Chapter II. The hypoellipticity condition derived there, in light of (3.57)–(3.58) of this section, is equivalent to the following condition: for all nonnegative integers α_ν, the quantity

$$\text{sub}\sigma(P) + \sum_\nu (2\alpha_\nu + 1)\mu_\nu^{\#} \tag{3.60}$$

is nonvanishing on Λ; here $\{\mu_\nu^{\#}\}$ is the set of positive eigenvalues of F/i. More generally, if P is the second order right invariant differential operator

$$P = \sum a_{jk} X_j X_k + i\alpha T + B, \tag{3.61}$$

where B is a first order differential operator whose principal symbol vanishes on Λ, then there is a linear map on $\mathbb{H}^n$, given in fact by a symplectic transformation on $\mathbb{R}^{2n} = \{(q,p)\}$, which takes P to the form (3.56). Such a transformation is a group automorphism of $\mathbb{H}^n$. By the preceeding discussion, the quantity (3.60) is invariant under this transformation, so Theorem 2.17 and Proposition 2.14 of Chapter II imply that the right invariant differential operator (3.61) is hypoelliptic on $\mathbb{H}^n$, with parametrix in $OP\Psi^{-2}$, if and only if the quantity (3.60) is nonvanishing on Λ.

Now let us pass to general $P \in OPS^2$, $p_2 \geq 0$, vanishing on Λ^+, with nondegenerate transverse Hessian. Then P has the form (generalizing (3.7)):

$$Pu(z) = \sum A_{jk}(z) X_j X_k u + i\alpha(z) T u + Bu, \quad z = (t,q,p) \in \mathbb{H}^n. \tag{3.62}$$

Here, $A_{jk}(z)$ is a smooth function of z with values in $OP\Sigma^0$, $\alpha \in C^\infty(\mathbb{H}^n)$, and $B \in OPS^1$ has principal symbol vanishing on Λ^+. If we work microlocally on Λ^+, which is a ray bundle over $\mathbb{H}^n$, we can write

$$A_{jk}(z) = a_{jk}(z) + B_{jk}(z) \tag{3.63}$$

where $a_{jk} \in C^\infty(\mathbb{H}^n)$ and $B_{jk}(z)$ is a smooth function of z with values in right invariant operators in OPS^1 whose principal symbols vanish on Λ^+. It is easy to show that, for $(z,\zeta) \in \Lambda^+ \subset T^*\mathbb{H}^n \setminus 0$,

$$\mathrm{sub}\sigma(P) = \mathrm{sub}\sigma(\widetilde{P})(z,\zeta), \tag{3.64}$$

where $\widetilde{P} \in OPS^2$ is defined by

$$\widetilde{P}u(z) = \sum a_{jk}(z) X_j X_k u + i\alpha(z) T u.$$

In fact, we have the stronger statement: at $(z_0,\zeta_0) \in \Lambda^+$,

$$\mathrm{sub}\sigma(P)(z_0,\zeta_0) = \mathrm{sub}\sigma(\widetilde{P}_{z_0})(z_0,\zeta_0), \tag{3.65}$$

where $\widetilde{P}_{z_0}$ is the right invariant differential operator

$$\widetilde{P}_{z_0} u(z) = \sum a_{jk}(z_0) X_j X_k u(z) + i\alpha(z_0) T u(z). \tag{3.66}$$

Furthermore, the Hamilton matrices for P and $\widetilde{P}_{z_0}$ have the same eigenvalues at $(z_0,\zeta_0) \in \Lambda^+$. Consequently, Theorem 3.2 implies the following result.

Theorem 3.3. *Let $P \in OPS^2$ have principal symbol $p_2 \geq 0$, vanishing to second order on the variety $\Lambda^+ \subset T^*\mathbb{H}^n \setminus 0$ defining the contact structure for $\mathbb{H}^n$. Let $(z_0,\zeta_0) \in \Lambda^+$ and suppose that, for all nonnegative integers α_ν, the quantity (3.60) is nonvanishing at (z_0,ζ_0). Then P is microlocally hypoelliptic on a conis neighborhood of (z_0,ζ_0), with microlocal parametrix in $OP\widetilde{\Omega}^{-1,-2}$.*

The microlocal hypoellipticity of P, with loss of one derivative, is a special case of a more general result proved by Sjöstrand [S6] and Boutet de Monvel [B7], and Boutet de Monvel and Treves [B13]:

Theorem 3.4. *Let $P \in OPS^2$ have principal symbol $p_2 \geq 0$ on $T^*\Omega \setminus 0$, vanishing to second order on a conic set Σ, assumed to be a symplectic submanifold of $T^*\Omega \setminus 0$. Assume a nondegenerate transverse Hessian. Then P is hypoelliptic with loss of one derivative provided that, at each point of Σ, for all nonnegative integers α_ν, the quantity (3.60) is nonvanishing.*

These authors also prove that (3.60) is necessary for hypoellipticity with loss of one derivative, as well as sufficient, granted the other hypotheses of the theorem. In [B7], a parametrix E is produced with

$$E \in OPS^{-2,-2}(\Omega, \Sigma) \subset OPS^{-1}_{1/2,1/2}(\Omega).$$

As mentioned in Chapter II, §3, if $(\Omega, \Sigma) = (\mathbb{H}^n, \Lambda)$, we have

$$OP\widetilde{\Omega}^{-1,-2} \subset OPS^{-2,-2}(\mathbb{H}^n, \Lambda).$$

Hypoellipticity of P with loss of one derivative, under more general conditions, when Σ need not be symplectic, is studied in detail in [B10], H8], and other places, but we will not go into it here.

We do want to show that "half" of Theorem 3.4 follows from Theorem 3.3.

Proposition 3.5. *Suppose $2n$ is the codimension of Σ in $T^*\Omega \setminus 0$. Then, if*

$$k = dim\ \Omega \leq 2n + 1,$$

Theorem 3.3 implies Theorem 3.4.

Proof. The proof will utilize the following result in symplectic geometry. If Σ_j are conic submanifolds of $T^*\Omega_j \setminus 0$, if $\dim \Omega_1 = \dim \Omega_2$ and $\dim \Sigma_1 = \dim \Sigma_2$, and if Σ_j are both symplectic submanifolds, then, given $p_j \in \Sigma_j$, there exists a homogeneous symplectic diffeomorphism from a conic neighborhood U_1 of p_1 to a conic neighborhood U_2 of p_2, taking $\Sigma_1 \cap U_1$ onto $\Sigma_2 \cap U_2$. First consider the case

$$\dim \Omega = 2n + 1 = \operatorname{codim} \Sigma + 1. \tag{3.67}$$

In this case, with $\Omega_1 = \Omega,\ \Sigma_1 = \Sigma$, we can take $\Omega_2 = \mathbb{H}^n,\ \Sigma_2 = \Lambda$, and get Σ locally symplectically equivalent to Λ. If we implement this local transformation by an elliptic Fourier integral operator J, we see that JPJ^{-1} satisfies the hypotheses of Theorem 3.3 on its principal symbol. Since the quantity (3.60) is invariant under conjugation by J, it follows that the condition for hypoellipticity of P in Theorem 3.4 is equivalent to the condition for hypoellipticity of JPJ^{-1} given in Theorem 3.3.

This establishes Proposition 3.5 in case $\dim \Omega = \operatorname{codim} \Sigma + 1$. Now suppose that

$$\operatorname{codim} \Sigma + 1 = \dim \Omega + \ell, \quad \ell > 0. \tag{3.68}$$

Let us form $\widetilde{\Omega} = \Omega \times \mathbb{R}^\ell$, and let $\widetilde{P} = P \otimes I$. If

$$Pu = f \tag{3.69}$$

we have

$$\widetilde{P}\tilde{u} = \tilde{f}, \tag{3.70}$$

where $\tilde{u}(x, y) = u(x),\ x \in \Omega,\ y \in \mathbb{R}^\ell$, and $\tilde{f}$ is similarly defined. Note that $WF(\tilde{u})$ and $WF(\tilde{f})$ are contained in $\{\eta = 0\}$, if (ξ, η) are dual to $(x, y) \in \Omega \times \mathbb{R}^\ell$. $\widetilde{P}$ is not quite a pseudodifferential operator, but its symbol is smooth near $\{\eta = 0\}$, so we can treat it as a pseudodifferential operator, when analyzing (3.70). Note that, if $\widetilde{\Sigma}$ is the characteristic variety of P, then near $\{\eta = 0\}$ we have

$$\widetilde{\Sigma} = \{(x, y, \xi, \eta) : (x, \xi) \in \Sigma\}.$$

Thus $\operatorname{codim} \widetilde{\Sigma} = \operatorname{codim} \Sigma = \dim \Omega + \ell - 1 = \dim \widetilde{\Omega} - 1$. Note that, if (3.60) is nonvanishing for P at $(x_0, \xi_0) \in \Sigma$, then it is nonvanishing for $\widetilde{P}$ at $(x_0, y_0, \xi_0, 0)$. Thus the previous argument applies, to give hypoellipticity for $\widetilde{P}$, which implies hypoellipticity for P. This proves Proposition 3.5.

Note that, generally, $2 \le \operatorname{codim} \Sigma \le 2 \dim \Omega$. If $2 \le \operatorname{codim} \Sigma \le \dim \Omega - 2$, then Theorem 3.4 can be proved using analysis on the group $\mathbb{H}^n \times \mathbb{R}^\ell$, the Cartesian product of a Heisenberg group and an abelian group. It has been my intention to give the details of this in a future publication, but the reader is not advised to wait with baited breath for this.

4. The Neumann operator for the $\overline{\partial}$-Neumann problem

There is a pseudodifferential operator $\Box^+ \in OPS^1(M)$ that arises on the boundary $M = \partial\Omega$ of a strongly pseudoconvex domain $\Omega \subset \mathbb{C}^{n+1}$, which is doubly characteristic on half the contact line bundle, $\Lambda^+ \subset T^*M \setminus 0$, and which arises in the study of the $\overline{\partial}$-Neumann problem. We discuss this here. First we briefly indicate how $\Box^+$ arises and how it is related to the Kohn Laplacian $\Box_b$. We will be sketchy on this, referring to Greiner and Stein [G9] and to Chapter 12, §9 of [[T]] for further details. Then we show that the construction of a parametrix for $\Box^+$ is much simpler than the general construction used for Theorem 3.2, and the resulting parametrix has a simpler structure; it lies in $OP\widetilde{\mathfrak{H}}^{-2,1}_{\alpha,\delta}$, which is smaller than $OP\widetilde{\Omega}^{0,-2}$.

We consider the following $\overline{\partial}$-Neumann problem:

$$\Box u = 0 \ \text{ on } \ \Omega; \quad \sigma_{\overline{\partial}^*}(x,\nu)u = 0, \quad \sigma_{\overline{\partial}^*}(x,\nu)\overline{\partial}u = f \ \text{ on } \partial\Omega. \tag{4.1}$$

Here $\Box u = \overline{\partial}\,\overline{\partial}^* u + \overline{\partial}^*\overline{\partial}u = -(1/2)\Delta u$. We want to produce u in terms of a solution to the Dirichlet problem

$$\Box u = 0 \ \text{ on } \ \Omega, \quad u\big|_{\partial\Omega} = g, \tag{4.2}$$

i.e., $u = \mathrm{PI}\, g$.

One ingredient involves the Neumann operator $\mathcal{N}$ for the Dirichlet problem:

$$\mathcal{N} g = \frac{\partial u}{\partial \nu}, \quad u = \mathrm{PI}\, g. \tag{4.3}$$

The operator $\mathcal{N}$ is an elliptic operator in $OPS^1(M)$, whose structure is well understood. One has

$$\mathcal{N} = -\sqrt{-\Delta_M} + B, \quad B \in OPS^0(M), \tag{4.4}$$

and the principal symbol of B is given in terms of the second fundamental form of $M \subset \mathbb{R}^{2n+2}$. (Cf. (9.31) in [[T]], Chapter 12.)

After some computation (cf. (9.8)–(9.24) of [[T]], Chapter 12) one obtains

$$8f = (\mathcal{N} + iY)g + Cg. \tag{4.5}$$

Here Y is a certain vector field tangent to $M = \partial\Omega$, namely

$$Y = J(\nabla\rho) \tag{4.6}$$

where M is defined by $\{\rho = 0\}$ and $|\nabla\rho| = 1$ on M, and J gives the complex structure on $\mathbb{R}^{2n+2} \approx \mathbb{C}^{n+1}$. Furthermore $C \in OPS^0(M)$, and its principal symbol on Λ^+ involves the Levi form. (Cf. (9.25)–(9.29) in [[T]], Chapter 12.) We define $\Box^+ g$ to be the right side of (4.5). The operator $\Box^+$ is the Neumann operator for the $\overline{\partial}$-Neumann problem. Its principal symbol is given by

$$\sigma_{\Box^+}(x,\xi) = -|\xi| + \langle Y,\xi\rangle, \tag{4.7}$$

which vanishes to second order on Λ^+ and is elliptic elsewhere. We can also write

$$-\Box^+ = \sqrt{-\Delta_M} - iY + B_1, \quad B_1 = -B - C \in OPS^0(M). \tag{4.8}$$

One can express the principal symbol of B_1 on Λ^+ in terms of the Levi form.

Now, if we define $\Box^- \in OPS^1(M)$ by

$$-\Box^- = \sqrt{-\Delta_N} + iY + B_2, \tag{4.9}$$

then a certain choice of $B_2 \in OPS^0(M)$ yields

$$\Box^-\Box^+ = -\Delta_M + Y^2 - i\alpha(x)Y + R = \Box_b + R, \tag{4.10}$$

with $\alpha \in C^\infty(M)$ given in terms of the Levi form and

$$R \in OPS^1(M), \quad \sigma_R(x,\xi) = 0 \text{ on } \Lambda. \tag{4.11}$$

Cf. (9.36)–(9.44) of [[T]], Chapter 12. The operator

$$\Box_b = -\Delta_M + Y^2 - i\alpha(x)Y \tag{4.12}$$

is the Kohn Laplacian on $M = \partial\Omega$. The hypothesis that Ω is strongly pseudoconvex implies that $\Box_b$ satisfies the condition for hypoellipticity given in Theorem 3.1, except for the case of $(0,1)$-forms on $\Omega \subset \mathbb{C}^2$. In this case the Kohn Laplacian fails to satisfy the condition for hypoellipticity with loss of one derivative. However, microlocally this failure occurs only on the component Λ^- of Λ, where $\Box^+$ is elliptic, so the construction described below will still provide a microlocal parametrix for $\Box^+$ even in this case.

So let $\Box_b^{-1} \in OP\widetilde{\Psi}^{-2}$ denote a parametrix for $\Box_b$ (appropriately modified away from Λ^+ in case $\Omega \subset \mathbb{C}^2$). In view of (4.11), write

$$R = \sum X_j R_j + R_0, \quad R_j \in OPS^0. \tag{4.13}$$

We have

$$\begin{gathered}(\Box_b^{-1}\Box^-)\Box^+ = I + S, \\ S = \sum \Box_b^{-1} X_j R_j + \Box_b^{-1} R_0.\end{gathered} \tag{4.14}$$

Note that $\Box_b^{-1} X_j \in OP\widetilde{\Psi}^{-1}$, so

$$S \in OP\widetilde{\mathfrak{H}}^{-1,0}_{\alpha,\delta}. \tag{4.15}$$

Hence a left parametrix for $\Box^+$ is given by

$$E \sim (1 - S + S^2 - \cdots)\Box_b^{-1}\Box^-. \tag{4.16}$$

We are in a position to establish the following result.

Theorem 4.1. *If $\Omega \subset \mathbb{C}^{n+1}$ is a strongly pseudoconvex domain, the Neumann operator $\square^+$ for the $\overline{\partial}$-Neumann problem on $(0,1)$-forms is hypoelliptic, with parametrix*

$$E \in OP\widetilde{\mathfrak{H}}^{-2,1}_{\alpha,\delta}. \tag{4.17}$$

More precisely, we can write $E = E_1 + E_2 + E_3$ with (microlocally near Λ^+)

$$E_1 \in OP\widetilde{\Psi}^{0,1}, \quad E_2 \in OP\widetilde{\Psi}^{-1}, \quad E_3 \in OP\widetilde{\mathfrak{H}}^{0,-1}_{\alpha,\delta}. \tag{4.18}$$

Proof. The result (4.17) is an immediate consequence of (4.16) and the operator properties of $OP\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$, given in Proposition 2.11 of Chapter I. To obtain (4.18) we argue as follows. Let us set $\square^- = \square_0^- + \square_1^-$, with $\square_0^-$ having essential support near Λ^+ and $\square_1^-$ having order $-\infty$ near Λ^+. Then

$$(I - S + S^2 - \cdots)\square_b^{-1}\square_1^- \in OPS^{-1}(M). \tag{4.19}$$

To analyze the rest of E, let us write (microlocally near Λ^+)

$$\square_0^- = TM_\varphi + \sum X_j A_j + A_0, \quad A_j \in OPS^0, \ \varphi \in C^\infty(M), \tag{4.20}$$

and expand this further by setting (microlocally near Λ^+)

$$A_j = M_{\varphi_j} + \sum X_k B_{jk} + C_j, \quad 0 \le j \le 2n, \tag{4.21}$$

with

$$\varphi_j \in C^\infty(M), \quad B_{jk}, \ C_j \in OPS^{-1}, \tag{4.22}$$

so

$$\begin{aligned} \square_0^- = TM_\varphi &+ \sum X_j M_{\varphi_j} + \sum X_j X_k B_{jk} \\ &+ \sum X_j C_j + M_{\varphi_0} + \sum X_k B_{0k} + C_0. \end{aligned} \tag{4.23}$$

We have

$$\square_b^{-1}\square_0^- = F_1 + F_2 + F_3, \tag{4.24}$$

with

$$\begin{aligned} F_1 &= \square_b^{-1} T M_\varphi && \in OP\widetilde{\Psi}^{0,1}, \\ F_2 &= \sum \square_b^{-1} X_j M_{\varphi_j} + \square_b^{-1} M_{\varphi_0} && \in OP\widetilde{\Psi}^{-1}, \\ F_3 &= \sum (\square_b^{-1} X_j X_k) B_{jk} + \sum \square_b^{-1} X_j (C_j + B_{0j}) + \square_b^{-1} C_0 && \in OP\widetilde{\mathfrak{H}}^{0,-1}_{\alpha,\delta}. \end{aligned} \tag{4.25}$$

Similarly the operator S given in (4.14) can be written (microlocally near Λ^+)

$$S = S_0 + S_1, \quad S_0 \in OP\widetilde{\Psi}^{-1}, \ S_1 \in OP\widetilde{\mathfrak{H}}^{0,-1}_{\alpha,\delta}. \tag{4.26}$$

Putting together (4.19), (4.25), and (4.26), we obtain (4.18).

Note that the operators E_j are equally strong away from Λ^+, each being microlocally in OPS^{-1}, but E_1 carries the strongest singularity and E_2 the second strongest singularity on Λ^+. Note that (parallel to (3.47)–(3.48))

$$E_1 \in OPS^0_{1/2,1/2}, \quad E_2 \in OPS^{-1/2}_{1/2,1/2}, \quad E_3 \in OPS^{-1}_{1/2,1/2}. \tag{4.27}$$

5. Heat equations and spectral asymptotics for subelliptic operators

Suppose M is a compact contact manifold, of dimension $2n+1$. Let $-P$ be a positive self adjoint second order differential operator, with principal symbol $p_2 \geq 0$, vanishing to exactly second order on $\Lambda \subset T^*M \setminus 0$, the span of the contact form on M. Assume

$$|\mathrm{sub}\sigma(P)| < \mathrm{Tr}^+ F \quad \text{on} \quad \Lambda, \tag{5.1}$$

where F is the Hamilton map of p_2, and $\mathrm{Tr}^+ F$ is the sum of the positive eigenvalues of F/i. It follows from Theorem 3.4 that P is hypoelliptic. Note that $P \subset OP\widetilde{\Psi}^2$; its parametrix belongs to $OP\widetilde{\Psi}^{-2}$. In particular, P has compact resolvent, since

$$(\lambda - P)^{-1} : L^2(M) \longrightarrow H^1(M), \tag{5.2}$$

so P has discrete spectrum. We aim to study the spectral asymptotics of P, by means of a study of the "heat semigroup" e^{tP} and an asymptotic analysis of $\mathrm{Tr}\, e^{tP}$ as $t \to \infty$. Menikoff and Sjöstrand [M8] have made such studies in more general contexts. However, the method used here is shorter and simpler. It also has the advantage that we produce a complete asymptotic expansion for this heat trace, whereas [M8] produces only the principal term. We note that Metivier [M9] has results on spectral asymptotics for P of the form $\sum X_j^2$, X_j vector fields, using analysis on nilpotent Lie groups. Also Fefferman and Phong [F3] have estimates on the spectrum of subelliptic operators in very general contexts. We also mention Iwasaki and Iwasaki [I2].

Locally, on an open set $U \subset M$, mapped diffeomorphically to an open set in $\mathbb{H}^n$, preserving the contact form, we can write

$$Pu(x) = P(x)u(x), \tag{5.3}$$

where, for each y, $P(y)$ is a right invariant second order differential operator on $\mathbb{H}^n$. For y fixed, we have analyzed the semigroup $e^{tP(y)}$ on $\mathbb{H}^n$ in Chapter II, §4. We have, for $y \in U$, $(s,z) \in \mathbb{H}^n$,

$$e^{tP(y)}\, \delta_0(s,z) = e^{tb(y)}\, k_t(y;s,z), \quad t > 0, \tag{5.4}$$

where

$$k_t(y;s,z) = k_t^{Q(y)}(s + it\alpha(y), z), \tag{5.5}$$

the right side of (5.5) being given by (4.61)–(4.62) of Chapter II. Here we suppose

$$P(y) = \sum a_{jk} X_j X_k + i\alpha(y)T + b(y),$$

as in (2.70) of Chapter II, and $Q(y)$ is the quadratic form determined by $(a_{jk}(y))$. Note that

$$k_t(y;s,z) = t^{-n-1}\, k_1\Big(y;\frac{s}{t},\frac{z}{\sqrt{t}}\Big), \tag{5.6}$$

and $k_1(y;\cdot,\cdot)$ is a smooth function of y with values in $\mathcal{S}(\mathbb{H}^n)$. Thus, for $(s,z)\neq(0,0)$, $k_t(y;s,z)$ vanishes to infinite order as $t\searrow 0$. Define $k^{\#}(y;t,s,z)$ by

$$\begin{aligned} k^{\#}(y;t,s,z) &= k_t(y;s,z) \quad \text{for } t\ge 0,\\ &\qquad\quad 0 \qquad\quad \text{for } t<0. \end{aligned} \tag{5.7}$$

Note that (5.6) yields

$$k^{\#}(y;rt,rs,r^{1/2}z) = r^{-n-1}\, k^{\#}(y;t,s,z). \tag{5.8}$$

Also, $k^{\#}$ is smooth for $(t,s,z)\neq(0,0,0)$. Let us set

$$k_{\#}(y;t,s,z) = \chi(t,s,z)k^{\#}(y;t,s,z), \tag{5.9}$$

where χ is a compactly supported cut-off, equal to 1 near $(0,0,0)$. Then $k_{\#}$ is a smooth function of y with values in $\widehat{\mathfrak{H}}(G,\alpha,-2)$, where

$$G=\mathbb{R}\times\mathbb{H}^n,\quad \alpha(\rho)=(r^2t,r^2s,rz),\quad r=e^{\rho},\ t\in\mathbb{R},\ (s,z)\in\mathbb{H}^n. \tag{5.10}$$

Note that $\alpha(\rho)$ expands volumes by a factor of $r^{2n+4}=e^{(2n+4)\rho}$. Let us denote

$$\mathfrak{X}_0^m = \mathfrak{H}(G,\alpha,m), \tag{5.11}$$

with G and α given by (5.10), and let $\mathfrak{X}^m$ consist of asymptotic sums of elements of $\mathfrak{X}_0^{m-j}$, $j=0,1,2,\dots$. We see that, if $\mathfrak{K}_0$ is defined by

$$\mathfrak{K}_0u(t,x) = k_b(t,x;\cdot)*u(t,x),\quad (t,x)\in\mathbb{R}\times\mathbb{H}^n, \tag{5.12}$$

where $k_b=e^{tb(y)}k_{\#}$, then

$$\mathfrak{K}_0\in OP\widetilde{\mathfrak{X}}^{-2}. \tag{5.13}$$

In view of results of Chapter I, we have

$$\Big(\frac{\partial}{\partial t}-P\Big)\mathfrak{K}_0u(t,x) = (I-\mathfrak{R})u,\quad \mathfrak{R}\in OP\widetilde{\mathfrak{X}}^{-1}. \tag{5.14}$$

Note that both $\mathfrak{K}_0$ and $\mathfrak{R}$ have the *evolution property*: if $v(t,x)$ is defined on $\mathbb{R}\times\mathbb{H}^n$ and vanishes for $t<T$, so do $\mathfrak{K}_0v$ and $\mathfrak{R}v$. Also, so does any iterate $\mathfrak{R}^k$. Proposition 1.8 of Chapter I implies

$$\mathfrak{R}^k\in OP\widetilde{\mathfrak{X}}^{-k}, \tag{5.15}$$

and

$$\mathfrak{K}_0\mathfrak{R}^k \in OP\widetilde{\mathfrak{X}}^{-2-k}. \tag{5.16}$$

Thus $(\partial/\partial t - P)$ has a left parametrix

$$\mathfrak{K} \sim \mathfrak{K}_0 + \mathfrak{K}_0\mathfrak{R} + \mathfrak{K}_0\mathfrak{R}^2 + \cdots = \mathfrak{K}_0 + \mathfrak{K}_1 + \mathfrak{K}_2 + \cdots, \tag{5.17}$$

satisfying the evolution property, and similar considerations show $\mathfrak{K}$ is also a right parametrix. Now if

$$\mathfrak{K}v(t,x) = K(t,x;\cdot) * v(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{H}^n, \tag{5.18}$$

the convolution being on $\mathbb{R}\times\mathbb{H}^n$, we have $K(t,x;t',x')$ independent of t and

$$e^{tP}\delta_p(x) = K(0,x;x^{-1}p), \quad t > 0, \tag{5.19}$$

modulo an error smooth in $\overline{\mathbb{R}}^+ \times M$. Thus

$$\operatorname{Tr} e^{tP} = \int\limits_M K(0,x;t,0)\, d\mathrm{Vol}(x) + A(t), \tag{5.20}$$

with $A \in C^\infty(\overline{\mathbb{R}}^+)$. Now, by (5.17), upon rearrangement, we can write

$$K(t,x;t',x') \sim \sum_{j\ge 0} K_j(t,x;t',x'), \tag{5.21}$$

where $K_j(t,x;t',x')$ is smooth in (t,x) and approximately homogeneous of degree $-2n-2+j$ in (t',x'), with respect to the group of dilations given in (5.10). According to Proposition 1.9 of Chapter I, subtracting a smooth function from K_j, we can suppose

$$K_j(t,x;r^2t',r^2s,rz) = r^{-2n-2+j}\, K_j(t,x;t',x'), \tag{5.22}$$

provided

$$j < 2n+2. \tag{5.23}$$

In particular, modulo a smooth function of t,

$$K_j(0,x;t,0) = t^{-n-1+j/2}\, K_j(0,x;1,0), \quad \text{for } j < 2n+2. \tag{5.24}$$

For $j \ge 2n+2$, say $j = 2n+1+k,\ k \ge 1$, we can apply Proposition 1.9 of Chapter I to $D^k_{t'}K_j$, the kernel of an element of $OP\widetilde{\mathfrak{X}}^{-2-j+2k}$, and conclude

$$D^k_{t'}K_j(0,x;rt',0) = r^{-1/2-k/2}\, D^k_{t'}K_j(0,x;t',0). \tag{5.25}$$

Integrating, we see that K_j must be given by a smooth function plus

$$K_j(0,x;t,0) = C_j t^{(k-1)/2}\, K_j(0,x;1,0), \quad \text{if } k \ge 0 \text{ is even}, \tag{5.26}$$

and

$$C'_j t^{(k-1)/2}\, K_j(0,x;1,0) + C''_j(x) t^{(k-1)/2} \log t, \quad \text{if } k \ge 1 \text{ is odd}, \tag{5.27}$$

where

$$k = j - 2n - 1 \ge 0. \tag{5.28}$$

From (5.20), we have our conclusion:

Theorem 5.1. *If P is a second order differential operator on a compact contact manifold, satisfying the hypotheses of the first paragraph of this section, then*

$$\begin{aligned} Tr\, e^{tP} \sim\ & t^{-n-1}(A_0 + B_0 t^{1/2} + A_1 t + B_1 t^{3/2} + \cdots + A_n t^n + B_n t^{n+1/2} + \cdots) \\ & + (A'_{n+1} \log t + A'_{n+2} t \log t + \cdots). \end{aligned} \tag{5.29}$$

The asymptotic expansion (5.29) together with Karamata's Tauberian theorem (for a proof see p. 341 of [T2]) implies the following result on the eigenvalue asymptotics for P. Suppose

$$\operatorname{spec} P = \{-\lambda_j : j \geq 1\}, \quad \lambda_j \nearrow, \tag{5.30}$$

and

$$N(\lambda) = \#\{-\lambda_j \in \operatorname{spec} P : \lambda_j < \lambda\}. \tag{5.31}$$

Corollary 5.2. *Under the hypotheses of Theorem 3.1, we have*

$$\lim_{\lambda \to \infty} \lambda^{-n-1} N(\lambda) = \frac{A_0}{\Gamma(n+2)}. \tag{5.32}$$

Our construction of a parametrix for $\partial/\partial t - P$ on $\mathbb{R} \times M$ could have proceeded along lines more directly analogous to the construction of a parametrix for P in §3, utilizing a symbol calculus on $\mathbb{R} \times \mathbb{H}^n$. The infinite dimensional irreducible unitary representations of this group are of the form

$$\pi_{\sigma,\pm\lambda}(t, s, q, p) = e^{i\sigma t} \pi_{\pm\lambda}(s, q, p), \quad \sigma \in \mathbb{R},\ \lambda \in (0, \infty), \tag{5.33}$$

where $t \in \mathbb{R},\ (s, q, p) \in \mathbb{H}^n$. If $k \in \mathcal{E}'(\mathbb{R} \times \mathbb{H}^n),\ Ku = k * u$,

$$\begin{aligned} \pi_{\sigma,\pm\lambda}(k) &= \hat{k}(\sigma, \pm\lambda, \pm\lambda^{1/2} X, \lambda^{1/2} D) \\ &= \sigma_K(\sigma, \pm\lambda)(X, D), \end{aligned} \tag{5.34}$$

which implies

$$\hat{k}(\sigma, \pm\tau, y, \eta) = \sigma_K(\sigma, \pm\tau)(\pm\tau^{-1/2} y, \tau^{-1/2} \eta). \tag{5.35}$$

We are interested in $\hat{k}$, smooth on $(\sigma, \lambda, y, \eta) \neq 0$ and homogeneous:

$$\hat{k}(\tau\sigma, \tau\lambda, \tau^{1/2} y, \tau^{1/2} \eta) = \tau^{m/2} \hat{k}(\sigma, \lambda, y, \eta), \quad \tau > 0. \tag{5.36}$$

This homogeneity is equivalent to

$$\sigma_K(\tau\sigma, \pm\tau\lambda)(x, \xi) = \tau^{m/2}\, \sigma_K(\sigma, \pm\lambda)(x, \xi), \quad \tau > 0. \tag{5.37}$$

It would be convenient to have a result parallel to Proposition 2.2 of Chapter II, characterizing when $\sigma_K(\sigma, \pm\lambda)$ given such that (5.37) holds gives $\hat{k}$, via (5.35), which is C^∞ on the complement of the origin. In setting down (5.4)–(5.5), to construct the parametrix $\mathfrak{K}$ of $\partial/\partial t - P$, we have exploited the explicit calculation from §4 of Chapter II, rather than such a general theory of harmonic analysis and pseudodifferential operator calculus on $\mathbb{R} \times \mathbb{H}^n$, whose construction we will not pursue here, but merely mention to the reader as one sort of direction in which to extend the theory developed in this chapter and the preceding one.

We point out that Melin [M4] has developed an operator calculus and parametrix construction (but not a symbol calculus) in a context that contains $\mathbb{R} \times M$ here as a special case.

Remark. Work of R. Beals, P. Greiner, and N. Stanton [[BGS]] has produced an expansion more precise than (5.39). They are able to show that all the B_j are zero and all the log terms are absent. See also a related study of Stanton and Tartakoff [[ST]].

6. Szegö operators, Toeplitz operators, and hypoellipticity with loss of two derivatives

Let M be a compact contact manifold, with contact line bundle Λ, and let Λ_+ denote one of the two connected components of $\Lambda \setminus 0$. As in §5, let P be a self-adjoint, second order differential operator (or pseudodiferential operator) whose principal symbol p_2 is ≥ 0, vanishing on Λ but with Hessian nondegenerate transverse to $\Lambda \setminus 0$.

We have seen that P is hypoelliptic with loss of one derivative provided that, for all nonnegative integers α_ν, the quantity

$$\text{sub}\sigma(P) + \sum_\nu (2\alpha_\nu + 1)\mu_\nu \tag{6.1}$$

is nowhere vanishing on $\Lambda \setminus 0$. See Theorem 3.3. Here we shall consider a case where this condition is violated uniformly on Λ_+; we will suppose it holds on the other connected component Λ_-. More precisely, we will suppose (6.1) vanishes when $\alpha_\nu = 0$:

$$\text{sub}\sigma(P) + \operatorname{Tr}^+ F = 0 \quad \text{on} \quad \Lambda_+. \tag{6.2}$$

The best known case when this happens is when P is the Kohn Laplacian $\square_b^{(0)}$ on functions (0-forms) on the boundary of a strongly pseudoconvex domain $\mathcal{O} \subset \mathbb{C}^k$. In this case, if S is the Szegö projector of $L^2(\partial\mathcal{O})$ onto the subspace of boundary values of functions holomorphic on $\mathcal{O}$, then $\square_b^{(0)} S = S \square_b^{(0)} = 0$. Now there are holomorphic functions on $\mathcal{O}$ with singularities at any given point of $\partial\mathcal{O}$, hence singular elements of the range of S, so $\square_b^{(0)}$ is certainly not hypoelliptic. However, many zero order perturbations of $\square_b^{(0)}$ turn out to be hypoelliptic.

One of our goals in this section is to analyze the structure of the Szegö projector for $\square_b^{(0)}$ mentioned above. Also we will examine the relationship between an operator $P \in OPS^2$ having a Szegö operator and certain zero order perturbations of P being hypoelliptic with loss of two derivatives. First, some definitions. An operator $P \in OPS^2$ will be said to have a Szegö operator S and complementary parametrix E provided the following three conditions hold:

$$E \in OPS^{-1}_{1/2,1/2}, \quad S \in OPS^0_{1/2,1/2}, \tag{6.3}$$

$$EP + S \sim I, \tag{6.4}$$

$$SP \sim 0, \tag{6.5}$$

where we say

$$A \sim B \Longleftrightarrow A - B \in OPS^{-\infty}. \tag{6.6}$$

If these conditions hold microlocally in some conic open set $\Gamma \subset T^*M \setminus 0$, we say P has a Szegö operator S and complementary parametrix E microlocally on Γ. One might want to weaken the hypotheses in (6.3); however when we show such operators exist we will want to obtain them in smaller operator classes, such as $OP\widetilde{\Psi}^{-2}$ and $OP\widetilde{\Psi}^{0,\infty}$, respectively, in favorable cases. It is also useful to weaken hypotheses (6.4)–(6.5), to

$$EP + S = I + R_1, \quad SP = R_2, \quad R_j \in OPS^{-\varepsilon}_{1/2,1/2}, \tag{6.7}$$

for some $\varepsilon > 0$. A pair (S, E) satisfying (6.3) and (6.7) will be called a weak Szegö operator with complementary parametrix.

The following result generalizes the argument of Stein [S7], which we discussed in (2.63)–(2.67) of Chapter II.

Proposition 6.1. *Let $P \in OPS^2$ satisfy the hypotheses of the first paragraph of this section, and assume (6.2) holds, while (6.1) is always nonzero on Λ_-. Suppose P has a weak Szegö operator S with complementary parametrix E. Consider $P_1 = P + A$, with $A \in OPS^0$. Then P_1 is hypoelliptic, with loss of two derivatives, provided the principal symbol of A is nonvanishing on Λ_+.*

Proof. Since P is assumed to satisfy the condition for hypoellipticity with loss of one derivative away from Λ_+, we can suppose S is essentially supported in any given conic neighborhood of Λ_+. Let $B \in OPS^0$ be a parametrix for A, microlocally near Λ_+. Then BS is well defined, mod $OPS^{-\infty}$, and we have

$$\begin{aligned} (E + BS)P_1 &= (E + BS)(P + A) \\ &= I - S + R_1 + BR_2 + BSA + EA \\ &= I - R_3, \end{aligned} \tag{6.8}$$

where

$$R_3 = S - BSA - R_1 - BR_2 - EA \in OPS^{-\varepsilon}_{1/2,1/2}. \tag{6.9}$$

It follows that the Neumann expansion yields a parametrix for $I - R_3$:

$$F \sim I + R_3 + R_3^2 + \cdots \in OPS^0_{1/2,1/2}, \tag{6.10}$$

so

$$F(E + BS)P_1 \sim I. \tag{6.11}$$

Thus P_1 has a left parametrix in $OPS^0_{1/2,1/2}$, and the proposition is proven.

We will next show that, when a Szegö operator S and complementary parametrix E exist, microlocally, they are microlocally unique. One use of this is to reduce the global construction of Szegö operators to microlocal constructions.

Proposition 6.2. *Let $P \in OPS^2$ be self-adjoint. Suppose (6.3)–(6.5) hold on an open set $\Gamma \subset T^*M \setminus 0$. Then, on Γ,*

$$S \sim S^* \sim S^2, \tag{6.12}$$

and, if $S_1 \in OPS^0_{1/2,1/2}$ satisfies

$$S_1 P \sim 0, \tag{6.13}$$

then

$$S_1 \sim S_1 S. \tag{6.14}$$

Furthermore, if there exists $E_1 \in OPS^{-1}_{1/2,1/2}$ such that

$$E_1 P + S_1 \sim I, \tag{6.15}$$

then

$$S_1 \sim S, \tag{6.16}$$

and, provided E and E_1 are normalized so

$$ES \sim 0 \sim E_1 S_1, \tag{6.17}$$

then

$$E \sim E_1. \tag{6.18}$$

Finally, granted the normalization (6.17), we have

$$E \sim E^*. \tag{6.19}$$

Proof. If $P = P^*$, then (6.4) implies

$$PE^* + S^* \sim I. \tag{6.20}$$

If we multiply on the left by S, we get

$$S \sim SS^*.$$

Since this implies $S^* \sim SS^*$, this gives (6.12). If we multiply (6.20) on the left by S_1, hypothesis (6.13) implies

$$S_1 \sim S_1 S^*,$$

and, in light of (6.12), (6.14) follows. Now, if (6.15) holds, then, as with S, we have $S_1 \sim S_1^* \sim S_1^2$, and $S \sim SS_1$. Combining with (6.14), we have

$$S_1 \sim S_1^* \sim S^* S_1^* \sim SS_1 \sim S,$$

so (6.16) is proven.

Note that, if (6.4) holds, it also holds with E replaced by $E(I - S)$ (assuming (6.5) holds), so the normalization (6.17) may be assumed for E. If E_1 enjoys the same sort of normalization, since $S \sim S_1$, we have

$$(E - E_1)P \sim 0,$$

and hence, by (6.14), in conjunction with (6.17) and its analogue for E_1,

$$E - E_1 \sim (E - E_1)S \sim 0,$$

so (6.18) is proven. Finally

$$(I - S)E^* \sim EPE^* \sim E(I - S),$$

and hence, if $ES \sim SE^* \sim 0$, we obtain (6.19).

Suppose $P \in OPS^2$ is a self-adjoint operator on a compact manifold M, satisfying the conditions of this section, such that one can construct microlocally on a conic neighborhood of each point of $T^*M \setminus 0$ a Szegö operator S with complementary paramterix E. By the uniqueness result of Proposition 6.2 it follows that there exist global operators $S \in OPS^0_{1/2,1/2}$, $E \in OPS^{-1}_{1/2,1/2}$ on M such that (6.4)–(6.5) hold. It then follows that (6.12), (6.17), and (6.19) hold globally on M. One can replace S by $(S + S^*)/2$, altering S by a smoothing operator, and achieve $S = S^*$. From the relation $S^2 - S \in OPS^{-\infty}$ we deduce that S can be altered further by an element of $OPS^{-\infty}$, so that

$$S^* = S \quad \text{and} \quad S = S^2, \tag{6.21}$$

i.e., S is an orthogonal projection. We still have $SP \sim PS \sim 0$, of course. However, it is not necessarily true that S can be taken to be the orthogonal projection onto the kernel of P. If the orthogonal projection of $L^2(M)$ onto ker P does satisfy the condition to be a Szegö operator, we say it is the Szegö projector (associated with P).

The orthogonal projection of $L^2(\partial\mathcal{O})$ onto the space of boundary values of functions holomorphic on a strongly pseudoconvex domain $\mathcal{O} \subset \mathbb{C}^k$ is indeed a Szegö projector associated to the Kohn Laplacian $\Box_b^{(0)}$. We will discuss this in a moment. On the other hand, it follows from an example of Nirenberg [N5] that there is a P on a three dimensional M such that ker P consists of constants, and the orthogonal projection fails to satisfy (6.3)–(6.5)

We turn now to the analysis of the orthogonal projection S of $L^2(\partial\mathcal{O})$ onto the space of boundary values of functions holomorphic in $\mathcal{O}$, given $\mathcal{O} \subset \mathbb{C}^k$ strongly pseudoconvex. We begin with some observations of Boutet de Monvel and Sjöstrand [B12]. We first consider the case dim $\mathcal{O} \ge 5$, so $k \ge 3$. We have the $\overline{\partial}_b$-complex; see, e.g., Krantz [K9]; and S is the orthogonal projection of $L^2(\partial\mathcal{O})$ onto ker $\Box_b^{(0)}$, where $\Box_b^{(0)} = \overline{\partial}_b^*\overline{\partial}_b$ on 0-forms. More generally, we have $\Box_b^{(j)} = \overline{\partial}_b^*\overline{\partial}_b + \overline{\partial}_b\overline{\partial}_b^*$ on sections of certain vector bundles E_j; $\overline{\partial}_b$ maps sections of E_j to sections of E_{j+1}. Note that

$$\overline{\partial}_b \,\Box_b^{(0)} = \overline{\partial}_b\overline{\partial}_b^*\overline{\partial}_b = \Box_b^{(1)}\overline{\partial}_b. \tag{6.22}$$

As long as $k \ge 3$, the Kohn Laplacian $\Box_b^{(1)}$ on sections of E_1 satisfies the criterion for hypoellipticity with loss of one derivative, as in Theorem 3.1 and Theorem 3.3 of this chapter; cf. [F4], [G9], or [[T]], Chapter 12. Consequently, there exists

$$L_1 \in OP\widetilde{\Psi}^{-2} \tag{6.23}$$

such that

$$L_1\Box_b^{(1)} = \Box_b^{(1)}L_1 = I - S_1, \quad S_1L_1 = L_1S_1 = 0, \quad L_1 = L_1^*, \tag{6.24}$$

where S_1 is the orthogonal projection onto the kernel of $\Box_b^{(1)}$, which is a finite dimensional space of smooth sections of E_1. Thus $\overline{\partial}_b\overline{\partial}_b^*L_1$ is the orthogonal projection onto the image of $\overline{\partial}_b$ and $\overline{\partial}_b^*L_1\overline{\partial}_b$ is the orthogonal projection onto the orthogonal complement of ker $\overline{\partial}_b$. Hence S is given by the following formula:

$$S = I - \overline{\partial}_b^*L_1\overline{\partial}_b \in OP\widetilde{\Psi}^0. \tag{6.25}$$

This formula for S was derived in [B12]. If we set

$$E = \overline{\partial}_b^*L_1^2\overline{\partial}_b(I - S) \in OP\widetilde{\Psi}^{-2}, \tag{6.26}$$

we see that

$$S\Box_b^{(0)} = 0, \quad E\Box_b^{(0)} = I - S, \quad ES = 0, \tag{6.27}$$

so S is a Szegö operator for $\Box_b^{(0)}$ with complementary parametrix E.

The formula (6.25) is not wholly satisfactory. For example it hides the fact that S belongs to $OPS^{-\infty}$ outside any conic neighborhood of Λ_+, which is an automatic consequence of the fact that $S\Box_b^{(0)} = 0$ together with the hypoellipticity of $\Box_b^{(0)}$ away from Λ_+. We will proceed to analyze S as an element of the smaller operator class $OP\widetilde{\Psi}^{0,\infty}$, and produce an explicit formula for its leading term. This will produce an analysis of the leading singularity of S alternative to, and in some respects simpler than, that of Boutet de Monvel and Sjöstrand [B12].

In this case, in local coordinates identifying a patch of M with a neighborhood of the identity in $\mathbb{H}^n$ ($\dim M = 2n+1$), we have $\square_b^{(0)}$ of the form

$$(6.28)\quad Pu = P(x)u(x) = \sum a_{jk}(x)X_jX_ku + i\alpha(x)Tu + \sum b_j(x)X_ju + c(x)u, \quad x \in \mathbb{H}^n.$$

We have shown that there exists a Szegö operator $S \in OP\widetilde{\Psi}^0$ and complementary parametrix $E \in OP\widetilde{\Psi}^{-2}$. Say

$$(6.29)\qquad Su(x) = S(x)u(x), \quad Eu(x) = E(x)u(x),$$

where $S(x)$ is a smooth function on $\mathbb{H}^n$ with values in $OP\Psi^0$, etc. According to Proposition 6.2, the complete symbols of S and E are uniquely determined by the properties $SP \sim 0,\ EP + S \sim I$. Let us proceed to derive their principal symbols.

Now the hypothesis (6.2), which is satisfied by $P = \square_b^{(0)}$, implies that, for each $x_0 \in \mathbb{H}^n$, the right invariant operator

$$(6.30)\qquad P_2(x_0) = \sum a_{jk}(x_0)X_jX_k + i\alpha(x_0)T$$

fails to satisfy the condition for hypoellipticity that $\sigma_{P_2(x_0)}(+1)(X,D)$ be invertible. Indeed, this self-adjoint operator has a one dimensional kernel, varying smoothly with x_0, and consisting of elements of $\mathcal{S}(\mathbb{R}^n)$. Let $\pi_0(x_0)$ denote the L^2-orthogonal projection onto this kernel. Now define

$$(6.31)\qquad S_0 \in OP\widetilde{\Psi}^{0,\infty}$$

by

$$(6.32)\qquad \begin{aligned} \sigma_{S_0}(x,\lambda)(X,D) = \pi_0(x) &\quad \text{for } \lambda > 0,\\ 0 &\quad \text{for } \lambda < 0. \end{aligned}$$

Similarly define

$$(6.33)\qquad E_0 \in OP\widetilde{\Psi}^{-2}$$

by the requirement that $\sigma_{E_0}(x,\lambda)(X,D)$ be the inverse of $\sigma_{P_2}(x,\lambda)(X,D)$ for $\lambda < 0$, and for $\lambda > 0$ it should annihilate the range of $\pi_0(x)$ and should invert σ_{P_2} on the kernel of $\pi_0(x)$. Now since

$$(6.34)\qquad P(x) = P_2(x) + \sum b_j(x)X_j + c(x) = P_2(x) + B(x),$$

with $B(x)$ smooth in $OP\Psi^1$, we see from the symbol calculus of §1 that S_0 and E_0 must be the principal parts of S and E:

$$(6.35)\qquad S - S_0 \in OP\widetilde{\Psi}^{-1}, \quad E - E_0 \in OP\widetilde{\Psi}^{-3}.$$

That S_0 belongs to $OP\widetilde{\Psi}^{0,\infty}$ and not merely to $OP\widetilde{\Psi}^0$ is a hint of the following:

$$(6.36)\qquad S \in OP\widetilde{\Psi}^{0,\infty}.$$

To see this, note that the relation $S\square_b^{(0)} = 0$ together with the fact that $\square_b^{(0)}$ is hypoelliptic off Λ_+ implies S belongs to $OPS^{-\infty}$ microlocally outside any conic neighborhood of Λ_+. Thus (6.36) is a consequence of the following.

Lemma 6.3. *Let $T \in OP\widetilde{\Psi}^m$, and suppose that, away from any conic neighborhood of $\Lambda_+ \subset T^*M \setminus 0$, T belongs to $OPS^{-\infty}$. Then $T \in OP\widetilde{\Psi}^{m,\infty}$.*

Proof. If $Tu(z) = T(z)u(z)$, the hypothesis implies each right invariant $T(z_0) \in OP\Psi^m$ belongs to $OPS^{-\infty}$ outside Λ_+. Set

$$T(z) \sim \sum_{j\geq 0} T_j(z)$$

with

$$\sigma_{T_j}(z, \pm\lambda)(X, D) = \lambda^{(m-j)/2}\, \tau_j^{\pm}(z, X, D).$$

We have

$$\sigma_{T_j}(z, -\lambda)(X, D) = 0 \quad \text{for} \quad \lambda > 0,$$

for all j. Now, the characterization of $OP\Psi_0^{m-j}$ given by Proposition 2.2 of Chapter II implies $\sigma_{T_j}(z, +\lambda)(X, D) \in OP\mathcal{S}_1^{-\infty}$, for $\lambda > 0$, for each j, and this completes the proof.

A general element of $OP\widetilde{\Psi}^{m,\infty}$ belongs to $OPS^{-\infty}$ away from $\Lambda \setminus 0 = \Lambda_+ \setminus 0 \cup \Lambda_- \setminus 0$. If $T \in OP\widetilde{\Psi}^{m,\infty}$ actually belongs to $OPS^{-\infty}$ away from Λ_+, we will say

$$T \in OP\widetilde{\Psi}_+^{m,\infty}. \tag{6.37}$$

We now tackle the task of specifying the complete asymptotic expansion of S and E for $\Box_b^{(0)}$. We have $S_0 \in OP\widetilde{\Psi}_+^{0,\infty}$ and $E_0 \in OP\widetilde{\Psi}^{-2}$ such that

$$S_0 P = R_0 \in OP\widetilde{\Psi}_+^{1,\infty}, \quad E_0 P + S_0 - I = Q_0 \in OP\widetilde{\Psi}^{-1}. \tag{6.39}$$

We will produce by induction

$$S_j \in OP\widetilde{\Psi}_+^{-j,\infty}, \quad E_j \in OP\widetilde{\Psi}^{-2-j}, \tag{6.40}$$

such that

$$(S_0 + \cdots + S_k)P = R_k \in OP\widetilde{\Psi}_+^{1-k,\infty} \tag{6.41}$$

and

$$(E_0 + E_1 + \cdots + E_k)P + (S_0 + \cdots + S_k) - I = Q_k \in OP\widetilde{\Psi}^{-1-k}. \tag{6.42}$$

We also want $ES \sim 0$, which is equivalent to

$$(E_0 + \cdots + E_k)(S_0 + \cdots + S_k) = W_k \in OP\widetilde{\Psi}^{-3-k}. \tag{6.43}$$

Suppose we have $S_0, \dots, S_k$ and $E_0, \dots, E_k$. We will specify S_{k+1} and E_{k+1}. Let $R_k^{\#}$ and $Q_k^{\#}$ denote the principal parts of R_k and Q_k, belonging, respectively, to $OP\widetilde{\Psi}_0^{1-k,\infty}$ and $OP\widetilde{\Psi}_0^{-1-k}$. Similarly let $W_k^{\#}$ denote the principal part of W_k. We have

$$R_k^{\#}u(x) = R_k^{\#}(x)u(x), \quad x \in \mathbb{H}^n,$$

where $R_k^{\#}(y)$ is smooth in y with values in $OP\Psi_0^{1-k,\infty}$, and similar formulas for $Q_k^{\#}$, etc. Now for (6.41) to hold with k replaced by $k+1$, it is necessary and sufficient that, for each $y \in \mathbb{H}^n$,

$$S_{k+1}(y)P_2(y) = -R_k^{\#}(y). \tag{6.44}$$

We note that a necessary and sufficient condition for there to exist $S_{k+1} \in OP\widetilde{\Psi}_0^{-k-1,\infty}$ such that this holds is that the quantity

$$\pi_1\big(R_k^{\#}(y)S_0(y)\big) = \sigma_{R_k^{\#}(y)S_0(y)}(+1)(X,D) \tag{6.45}$$

vanish for each y. For a general P of the form (6.48) this might be expected to be a nontrivial condition. However, for $P = \square_b^{(0)}$, we know that S and E exist, in the operator classes $OP\widetilde{\Psi}_+^{0,\infty}$ and $OP\widetilde{\Psi}^{-2}$, so the vanishing of (6.45) is automatic. It follows that

$$S_{k+1}(y) = -R_k^{\#}(y)E_0(y) + \alpha_{k+1}(y)S_0(y), \tag{6.46}$$

where $\alpha_{k+1}(y)$ is a smooth function of y with values in $OP\Psi_0^{-k-1}$, which remains to be determined. In view of the formulas of Chapter I, §1, applied to the symbol of a product S_0P_2, we have

$$R_0^{\#}(y) = \sum_{|\gamma_1|+|\gamma_2|=1} S_0^{[(0,\gamma_1,\gamma_2)]}(y)\, P_{2[(0,\gamma_1,\gamma_2)]} + \sum b_j(y)S_0(y)X_j. \tag{6.47}$$

Specifying $\alpha_{k+1}(y)$ will be a byproduct of the study of E_{k+1}, to which we turn. For (6.42) to hold with k replaced by $k+1$, it is necessary and sufficient that, for each y,

$$E_{k+1}(y)P_2(y) = -S_{k+1}(y) - Q_k^{\#}(y). \tag{6.48}$$

Now, for (6.48) to be solvable for $E_{k+1}(y)$, it is necessary and sufficient that

$$\pi_1\big([S_{k+1}(y) + Q_k^{\#}(y)]S_0(y)\big) \tag{6.49}$$

vanish. This vanishing is equivalent to

$$\pi_1\big(\alpha_{k+1}(y)\big)\pi_0(y) = -\pi_1(Q_k^{\#}(y))\pi_0(y),$$

so we can replace (6.46) by

$$S_{k+1}(y) = -R_k^{\#}(y)E_0(y) - Q_k^{\#}(y)S_0(y). \tag{6.50}$$

With this granted, we have

$$\begin{aligned} E_{k+1}(y) &= -[S_{k+1}(y) + Q_k^{\#}(y)]E_0(y) + \beta_{k+1}(y)S_0(y) \\ &= [R_k^{\#}(y)E_0(y) - Q_k^{\#}(y)]E_0(y) + \beta_{k+1}(y)S_0(y). \end{aligned} \tag{6.51}$$

Here, the factor $\beta_{k+1}(y)$ is a smooth function of y with values in $OP\Psi_0^{-k-3}$ that remains to be specified. This specification is a result of (6.43), with k replaced by $k+1$, which requires

$$E_0(y)S_{k+1}(y) + E_{k+1}(y)S_0(y) + W_k^{\#}(y) = 0.$$

If we plug in (6.50) and (6.51), we see this is equivalent to

$$\beta_{k+1}(y)S_0(y) = -W_k^{\#}(y) + E_0(y)R_k^{\#}(y)E_0(y) + E_0(y)Q_k^{\#}(y)S_0(y). \tag{6.52}$$

Again this apparently nontrivial identity is a consequence of the existence of S and E in $OP\widetilde{\Psi}^0$ and $OP\widetilde{\Psi}^{-2}$, and it uniquely specifies $\beta_{k+1}(y)S_0(y)$, since the other terms in (6.52) have already been specified. We can rewrite (6.51) as

$$\begin{aligned} E_{k+1}(y) = {} & [R_k^{\#}(y)E_0(y) - Q_k^{\#}(y)]E_0(y) \\ & + E_0(y)[R_k^{\#}(y)E_0(y) + Q_k^{\#}(y)S_0(y)] - W_k^{\#}(y). \end{aligned} \tag{6.53}$$

To summarize, for $k \ge 3$, we have proven the following result. We note that a result along these lines was suggested by Dynin [D2].

Theorem 6.4. *Let $\mathcal{O} \subset \mathbb{C}^k$ be a strongly pseudoconvex domain, with smooth boundary $M = \partial\mathcal{O}$. Then the Szegö projector S, the orthogonal projection of $L^2(M)$ onto the space of boundary values of holomorphic functions, is an operator belonging to $OP\widetilde{\Psi}_+^{0,\infty}$, whose complete symbol is given by (4.32) and (4.50).*

The construction of the Szegö projector for $\square_b^{(0)}$ based on (6.25) does not quite work for $k = 2$, i.e., dim $\partial\mathcal{O} = 3$. In that case, the hypoellipticity of $\square_b^{(1)}$ fails on Λ_-, though not on Λ_+. Hence we can pick $L_1 \in OP\widetilde{\Psi}^{-2}$ to be a parametrix for $\square_b^{(1)}$ microlocally away from Λ_-, and if we define S_a and E_a by (6.25)–(6.26), we see that

$$S_a \in OP\widetilde{\Psi}^0, \quad E_a \in OP\widetilde{\Psi}^{-2},$$

and

$$S_a\square_b^{(0)} = A, \quad E_a\square_b^{(0)} + S_a = I - B, \quad E_aS_a = C,$$

where $A \in OP\widetilde{\Psi}^{2,\infty}$, $B \in OP\widetilde{\Psi}^0$, and $C \in OP\widetilde{\Psi}^{-2,\infty}$ belong to $OPS^{-\infty}$ outside any conic neighborhood of Λ_-. In analogy with (6.37), and with Lemma 6.3 in mind, we write

$$A \in OP\widetilde{\Psi}_-^{2,\infty}, \quad B \in OP\widetilde{\Psi}_-^{0,\infty}, \quad C \in OP\widetilde{\Psi}_-^{-2,\infty}.$$

The important point is that, microlocaly on a conic neighborhood of Λ_+, S_a and E_a furnish a Szegö operator and complementary parametrix for $\square_b^{(0)}$. Since $\square_b^{(0)}$ has a parametrix in $OP\widetilde{\Psi}^{-2}$ away from Λ_+, this shows that in the case dim $\partial\mathcal{O} = 3$, we have a Szegö operator S' and complementary parametrix E' for $\square_b^{(0)}$ on $\partial\mathcal{O}$, and the complete symbol of the Szegö operator is still specified by (6.32) and (6.50). Once one has this result, it is possible to deduce that the Szegö projector in this case is equal to S' mod $OPS^{-\infty}$, using Kohn's estimate for the $\overline{\partial}$-Neumann problem. This argument is carried out in detail in Boutet de Monvel and Sjöstrand [B12]. It relies heavily on $M = \partial\mathcal{O}$ being embedded in $\mathbb{C}^k$ as the boundary of a strongly pseudoconvex domain. The Szegö projector for more general strongly pseudoconvex CR-manifolds has been studied by Kohn [K6]. We also mention another approach to the analysis of the Szegö projector, of Kerzman and Stein [K3]. The Szegö projector is related to the Bergman kernel function; see [B12], and also the paper [B1] of Beals, Fefferman and Grossman for a description of this and of Fefferman's original analysis of the Bergman kernel function.

Let us take a closer look at what sort of operator S was produced by the preceding construction. First look at S_0, defined by (6.32). Take the case where

$$P_2(y) = \mathcal{L}_0 + inT, \tag{6.54}$$

on $\mathbb{H}^n$. Then $\pi_0 = \pi_0(y)$ is the orthogonal projection onto the lowest eigenspace of $H = -\Delta + |x|^2$, which is spanned by $e^{-|x|^2/2}$. From the formula (1.13) of Chapter II relating an operator to its Weyl symbol, we readily obtain

$$\pi_0 = \pi_0(X, D), \quad \pi_0(x,\xi) = c_n e^{-|x|^2 - |\xi|^2}. \tag{6.55}$$

Now if we write

$$S_0 u(x) = s_0 * u(x) \tag{6.56}$$

in this case, the formulas

$$\begin{aligned} \hat{s}_0(\tau, y, \eta) &= \pi_0(\tau^{-1/2} y, \tau^{-1/2}\eta) \\ &= c_n\, e^{-(|y|^2 + |\eta|^2)/\tau}, \quad \tau > 0, \\ \hat{s}_0(\tau, y, \eta) &= 0 \qquad\qquad\qquad\quad \tau < 0, \end{aligned} \tag{6.57}$$

give

$$s_0(t, q, p) = c_n' \, \frac{\partial^n}{\partial t^n} \Big(\frac{1}{4}|q|^2 + \frac{1}{4}|p|^2 + it \Big)^{-1}, \tag{6.58}$$

a formula derived in Folland and Stein [F4].

In the more general case

$$P_2(y) = \sum a_{jk}(y) X_j X_k + i\alpha(y) T, \tag{6.59}$$

under the hypothesis on α that follows from (6.2), we see that $\pi_0(y)$ is the orthogonal projection onto the lowest eigenspace of $Q(X, D) = Q(y, X, D)$, where

$$Q(x, \xi) = \sum a_{jk}(y) \chi_j \chi_k, \tag{6.60}$$

and

$$\chi_j = x_j, \quad \chi_{j+n} = \xi_j, \quad 1 \le j \le n. \tag{6.61}$$

If we choose a symplectic basis of $\mathbb{R}^{2n}$ to diagonalize $Q(x, \xi)$ and exploit metaplectic covariance, we see that $\pi_0(y) = \pi_0(X, D)$ with

$$\pi_0(x, \xi) = C(Q)\, e^{-\psi(Q,x,\xi)}, \tag{6.62}$$

where $\psi(Q, x, \xi)$ is a positive quadratic form in (x, ξ), given explicitly by

$$\psi(Q, x, \xi) = Q(A_Q^{-1} \zeta, \zeta), \quad \zeta = (x, \xi). \tag{6.63}$$

Compare formula (A.31) of Appendix A, or formulas (4.51) and (4.60) of Chapter II. Now, if we write $S_0(y)$ as a convolution operator:

$$S_0(y) u(x) = s_{0y} * u(x), \tag{6.64}$$

we have

$$\hat{s}_{0y}(\tau, y, \eta) = \begin{cases} C_n(Q)\, e^{-\psi(Q,y,\eta)/\tau}, & \tau > 0, \\ 0, & \tau < 0, \end{cases} \tag{6.65}$$

and hence

$$s_{0y}(t, q, p) = \int_0^\infty \iint e^{-\psi(Q,y,\eta)/\tau}\, e^{iy\cdot q + i\eta \cdot p + i\tau t}\, dy\, d\eta\, d\tau. \tag{6.66}$$

Now the Fourier transform of a Gaussian is another Gaussian:

$$\iint e^{-\psi(Q,y,\eta)/\tau}\, e^{iy\cdot q + i\eta\cdot p}\, dy\, d\eta = C'(Q)\, e^{-\psi^{\#}(Q,q,p)\tau}, \tag{6.67}$$

where, if we use the natural coordinates on $\mathbb{R}^{2n}$ to represent the quadratic form $\psi(Q, x, \xi)$ as a $2n \times 2n$ matrix,

$$\psi(Q, x, \xi) = \zeta \cdot \Psi_Q \zeta,$$

we have

$$\psi^{\#}(Q,q,p)=\frac{1}{4}z\cdot\Psi_Q^{-1}z,\quad z=(q,p). \tag{6.68}$$

Hence the distribution s_{0y} in (6.64) is given by

$$s_{0y}(t,q,p)=C''(Q)\,\frac{\partial^n}{\partial t^n}\big[\psi^{\#}(Q,q,p)+it\big]^{-1}. \tag{6.69}$$

Using (6.65) or (6.69), it is possible to analyze S_0 as a Fourier integral operator with complex phase, associated with a certain positive almost complex canonical relation $\mathfrak{C}$, and of order zero:

$$S_0\in I^0(M,M;\mathfrak{C}). \tag{6.70}$$

See Melin and Sjöstrand [M5] for the theory of Fourier integral operators with complex phase. Inductively from (6.50) one obtains

$$S_j\in I^{-j/2}(M,M;\mathfrak{C}). \tag{6.71}$$

Thus S is a Fourier integral operator "adapted" to the contact structure of $M=\partial\mathcal{O}$, in the sense of [B11], Appendix A. We omit the details of passing from (6.65) to (6.70)–(6.71), which do not differ essentially from the model case analysis given in [B11] and [B12].

It may be tempting to ask if the order of the elements in the expansion of S actually drops by integers rather than by half-integers. To see that this does not necessarily happen, consider

$$P=\mathcal{L}_0+inT+iX_1-1. \tag{6.72}$$

We will construct the Szegö operator for this right invariant operator on $\mathbb{H}^n$. With $S\sim S_0+S_1+\cdots,\ S_j\in OP\Psi_0^{-j,\infty}$, we see that S_0 must be the operator defined by (6.55)–(6.58), the Szegö operator for $\mathcal{L}_0+inT=P_2$. If S_1 exists, it must satisfy the identity (6.44), which in this case becomes

$$S_1P_2=-iS_0X_1. \tag{6.73}$$

The condition for solvability,

$$\pi_1(S_0X_1S_0)=0, \tag{6.74}$$

is seen to hold; all elements of the (one-dimensional) range of $\pi_1(S_0)$ are even; applying X_1 makes them odd, hence orthogonal to the range of $\pi_1(S_0)$. In this case, we have

$$S_1=-iS_0X_1E_0-iE_0X_1S_0. \tag{6.75}$$

If S_2 exists, it must satisfy

$$S_2 P_2 = -iS_1 X_1 - S_0 = -S_0 X_1 E_0 X_1 - E_0 X_1 S_0 X_1 - S_0. \tag{6.76}$$

Note that $\pi_1(\mathcal{L}_0 + inT)$ acts as the identity operator on the range of $\pi_1(X_1 S_0)$, so $S_0 X_1 E_0 X_1 S_0 = S_0 X_1^2 S_0$. Now

$$\pi_1(S_0 X_1^2 S_0) = -\gamma\, \pi_1(S_0)$$

with

$$\gamma = \frac{\int x_1^2 e^{-|x|^2}\, dx}{\int e^{-|x|^2}\, dx} = 1.$$

This guarantees that then the right side of (6.76) is multiplied on the right by S_0, the product vanishes. Indeed, the exact Szegö operator for (6.72) is given by

$$S = S_0 + S_1 + S_2, \quad S_j \in OP\widetilde{\Psi}^{-j,\infty}\ (S_j \in I^{-j/2}(M, M; \mathfrak{C})), \tag{6.77}$$

and the term S_1, given by (6.75), is not zero.

We note that the operator P in (6.72) is unitarily equivalent to $\mathcal{L}_0 + inT$; a simple calculation shows

$$\mathcal{L}_0 + inT + iX_1 - 1 = e^{(i/2)T^{-1}X_{n+1}}\, (\mathcal{L}_0 + inT)\, e^{-(i/2)T^{-1}X_{n+1}}. \tag{6.78}$$

Microlocally near Λ, $cT^{-1}X_{n+1}$ is a pseudodifferential operator in OPS^0, and so is its exponential. Thus the operator (6.72) is microlocally conjugate to an operator of the form $\Box_b^{(0)}$ near the characteristic set. In particular, the Szegö projector (6.77) is microlocally conjugate to the Szegö projector S_0, given by (6.56)–(6.58).

As shown by Boutet de Monvel and Sjöstrand [B12], any two Szegö operators are microlocally conjugate, via a Fourier integral operator.

Given a Szegö operator S acting on $\mathcal{D}'(M)$, one defines a Toeplitz operator (of order m) as an operator of the form

$$T = SQS, \quad Q \in OPS^m(M). \tag{6.79}$$

As noted above, any two Szegö operators are microlocally conjugate, via a Fourier integral operator, so microlocally one is reduced to studying (6.79) when $S = S_0$ is the Szegö projector for $\mathcal{L}_0 + inT$ on $\mathbb{H}^n$ (or any other local model). We will not develop the basic theory of Toeplitz operators as a consequence of the operator calculus for $OP\widetilde{\Psi}_+^{2m,\infty}$. Instead we refer to Boutet de Monvel and Guillemin [B11], where the basic theory is worked out from a different point of view. We mention the following three results, proven in the first chapters of [B11].

Theorem A. *If $Q \in OPS^m$ has principal symbol vanishing on Λ_+, then SQS is a Toeplitz operator of order $m-1$.*

Theorem B. *Any Toeplitz operator (6.79) can be written in the form $T = SQ'S$, where $Q' \in OPS^m$ commutes with S. If T is self-adjoint, then Q' can be taken self-adjoint.*

Theorem C. *If T_1 and T_2 are Toeplitz operators, of order m_1 and m_2, respectively, then T_1T_2 is a Toeplitz operator of order $m_1 + m_2$.*

We remark that Theorem C follows from Theorem B and that Theorem A is used to prove Theorem B. Theorem A might seem surprising, in light of (6.77), and we will say a little more about it below. This result is strongly related to a result of R. Howe, which we will discuss in Appendix B, which sets up a natural isomorphism between the space $OP\mathcal{H}_b^{2m}$ of pseudodifferential operators on $\mathbb{R}^n$ and the space of Toeplitz operators of order m on the unit sphere in $\mathbb{C}^n$. Our discussion of this in Appendix B will also make use of Theorem B above. Another use of Theorem B is to set one up to study spectral asymptotics for $T_Q = SQS$ in case Q is elliptic and self-adjoint, by studying $e^{isT_Q} = S\, e^{isQ'}\, S$. Much of [B11] is devoted to studying the composition of Fourier integral operators like $e^{isQ'}$ and operators like S, and applying this to obtain deep information on the behavior of the spectrum of T_Q.

Let us take a look at the symbol of a Toeplitz operator (6.79), as an element of $OP\widetilde{\Psi}_+^{2m,\infty}$, in the case $S = S_0$, the Szegö projector for $\mathcal{L}_0 + inT$ on $\mathbb{H}^n$. Recall that we have

$$\sigma_{S_0}(\lambda)(X, D) = \pi_0(X, D), \quad \lambda > 0, \tag{6.80}$$

with $\pi_0(x, \xi)$ given by (6.55). If $Q \in OP\widetilde{\Sigma}_0^m,$. then, from our operator calculus, we have, for $w \in \mathbb{H}^n$,

$$\sigma_{QS_0}(w, \lambda)(X, D) \sim \lambda^m \sum_{k \ge 0} \frac{1}{k!} \lambda^{-k/2}\, (X \cdot \nabla_x + D \cdot \nabla_\xi)^k q(w; 0, 0)\, \pi_0(X, D), \tag{6.81}$$

where

$$\sigma_Q(w, \lambda)(x, \xi) = \lambda^m q(w; \lambda^{-1/2}x, \lambda^{-1/2}\xi), \quad \lambda > 0. \tag{6.82}$$

Consequently, we can obtain the symbol of $S_0QS_0 = T_Q$, since $T_Q u(w) = T_Q(w)u(w)$ with

$$T_Q(y) \sim \sum_{\gamma \ge 0} S_0^{[\gamma]} (QS_0)_{[\gamma]}. \tag{6.83}$$

Note that, after the principal symbol, which is

$$\lambda^m\, q(w; 0, 0)\pi_0(X, D), \tag{6.84}$$

the next highest order symbol (defining an element of $OP\widetilde{\Psi}^{2m-1,\infty}$) is

$$\lambda^{m-1/2}\pi_0(X, D)(X \cdot \nabla_x + D \cdot \nabla_\xi)q(w; 0, 0)\pi_0(X, D) \tag{6.85}$$

$$+ \lambda^{m-1/2} \sum_{|\gamma_1| + |\gamma_2| = 1} \pi_0^{[(0,\gamma_1,\gamma_2)]}(X, D)\, q_{[(0,\gamma_1,\gamma_2)]}(w; 0, 0)\, \pi_0(X, D). \tag{6.86}$$

Now (6.85) always vanishes since X_j and D_j map even functions to odd functions. As for (6.86), if the principal symbol of Q vanishes everywhere on Λ_+, then $q(w; 0, 0) = 0$ on $\mathbb{H}^n$ and its w-derivatives $q_{[(0,\gamma_1,\gamma_2)]}(w; 0, 0)$ also vanish on $\mathbb{H}^n$. Hence T_Q belongs to $OP\widetilde{\Psi}_+^{2m-2,\infty}$ in this case, which is consistent with Theorem A.

Appendix A. The Weyl calculus

In our development of harmonic analysis on the Heisenberg group, we made considerable use of the Weyl calculus, which is defined as follows. Given $a(x,\xi)$, we set

$$a(X,D) = (2\pi)^{-n} \iint \hat{a}(q,p) e^{i(q\cdot X + p\cdot D)}\, dq\, dp, \tag{A.1}$$

where $e^{i(q\cdot X+p\cdot D)}$ is given by (1.5)–(1.7) of Chapter II and $\hat{a}(q,p)$ is the Fourier transform of $a(x,\xi)$. A few manipulations of integrals give

$$a(X,D)u(x) = (2\pi)^{-n} \iint a\bigl(\tfrac{1}{2}(x+y),\xi\bigr) e^{i(x-y)\cdot\xi} u(y)\, dy\, d\xi. \tag{A.2}$$

This defines a kernel in $\mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ and hence a map $a(X,D) : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, for any $a \in \mathcal{S}'(\mathbb{R}^{2n})$. We will touch on only a few properties of the Weyl calculus here, referring to the papers [H10], [G11] for a complete treatment. See also [[T]], Chapter 7.

One important class of symbols $a(x,\xi)$ is $\mathcal{S}^m_\rho$, defined to consist of smooth $a(x,\xi)$ such that

$$|D_x^\beta D_\xi^\alpha a(x,\xi)| \le C_{\alpha\beta}(1+|x|+|\xi|)^{m-\rho(|\alpha|+|\beta|)}. \tag{A.3}$$

If $a(x,\xi)$ satisfies (A.3), we say $a(X,D) \in OP\mathcal{S}^m_\rho$; generally $\rho \in [0,1]$. If $\rho > 0$, we have the following:

$$a_j(X,D) \in OP\mathcal{S}^{m_j}_\rho \Longrightarrow a_1(X,D)a_2(X,D) \in OP\mathcal{S}^{m_1+m_2}_\rho,$$

and the the symbol $b(x,\xi)$ of this product has the asymptotic expansion

$$b(x,\xi) \sim \sum_{j\ge 0} \frac{1}{j!}\,\{a_1,a_2\}_j(x,\xi), \tag{A.4}$$

where

$$\begin{aligned}&\{a_1,a_2\}_j(x,\xi)\\ &= \Bigl(\frac{1}{2i}\Bigr)^j \sum_{k=1}^n \Bigl(\frac{\partial^2}{\partial y_k \partial x_k} - \frac{\partial^2}{\partial x_k \partial \eta_k}\Bigr)^j a_1(x,\xi)a_2(y,\eta)\Big|_{y=x,\eta=\xi}.\end{aligned} \tag{A.5}$$

In particular $\{a_1,a_2\}_1 = \{a_1,a_2\}$ is $1/2i$ times the Poisson bracket. The meaning of (A.4) is that the difference between $b(x,\xi)$ and the sum of the right side of (A.4) over $0 \le j < N$ belongs to $\mathcal{S}^{m_1+m_2-2N\rho}_\rho$. We refer to [H10] for a proof of results containing (A.4).

Note that, if $a_1(x,\xi)$ or $a_2(x,\xi)$ is a polynomial in (x,ξ), of degree d, then all the terms in (A.5) are zero for $j>d$. In fact, in this case, (A.4) is not merely asymptotic; we have the identity

$$b(x,\xi)=\sum_{j=0}^{d}\frac{1}{j!}\{a_1,a_2\}_j(x,\xi), \tag{A.6}$$

if either $a_1(x,\xi)$ or $a_2(x,\xi)$ is a polynomial of degree d in (x,ξ). The proof of (A.6) is elementary, since one of the factors $a_j(X,D)$ is a differential operator. In particular, if either $a_1(x,\xi)$ or $a_2(x,\xi)$ is a quadratic polynomial, we have

$$\begin{aligned} b(x,\xi)&=a_1(x,\xi)a_2(x,\xi)-\frac{i}{2}\{a_1,a_2\}(x,\xi)\\ &\quad-\frac{1}{8}\sum_{k=1}^{n}\Big(\frac{\partial^2}{\partial y_k\partial\xi_k}-\frac{\partial^2}{\partial x_k\partial\eta_k}\Big)^2 a_1(x,\xi)a_2(y,\eta)\Big|_{y=x,\eta=\xi}. \end{aligned} \tag{A.7}$$

If we compare the symbol of $a_2(X,D)a_1(X,D)$, we see that only the middle term changes (by a sign), and so, whenever $Q(x,\xi)$ is a quadratic polynomial, the commutator

$$[Q(X,D),a(X,D)]=C(X,D)$$

has the symbol

$$C(x,\xi)=-\{Q,a\}(x,\xi). \tag{A.8}$$

We can use (A.8) to analyze the conjugate of an operator $a(X,D)$ by $e^{iQ(X,D)}$, with $Q(x,\xi)$ quadratic. We claim

$$e^{-isQ(X,D)}a(X,D)e^{isQ(X,D)}=(a\circ g_s)(X,D), \tag{A.9}$$

where $g_s\in Sp(n,\mathbb{R})$ is the one parameter group of symplectic linear maps on $\mathbb{R}^n$ generated by the Hamilton vector field

$$H_Q=\sum\Big(\frac{\partial Q}{\partial x_j}\frac{\partial}{\partial\xi_j}-\frac{\partial Q}{\partial\xi_j}\frac{\partial}{\partial x_j}\Big), \tag{A.10}$$

i.e.,

$$g_s=\exp(sH_Q). \tag{A.11}$$

To prove (A.9), we note that it is equivalent to the operator differential equation

$$\frac{d}{ds}(a\circ g_s)(X,D)=-i[Q(X,D),a\circ g_s(X,D)]. \tag{A.12}$$

Now the left side of (A.12) is clearly equal to $\{Q, a \circ g_s\}(X, D)$, so (A.12) is equivalent to

$$\{Q, a\}(X, D) = -i[Q(X, D), a(X, D)]. \tag{A.13}$$

But this follows from (A.8), so we have proved (A.9).

The set of quadratic polynomials $Q(x, \xi)$, with the Poisson bracket, is naturally isomorphic to the Lie algebra $\mathrm{sp}(n, \mathbb{R})$ of the symplectic group $Sp(n, \mathbb{R})$. The map $Q \mapsto Q(X, D)$ gives a Lie algebra representation of $\mathrm{sp}(n, \mathbb{R})$ by skew-adjoint operators. This generates a representation of the universal cover of $Sp(n, \mathbb{R})$:

$$\omega : \widetilde{Sp}(n, \mathbb{R}) \longrightarrow \mathfrak{U}(L^2(\mathbb{R}^n)). \tag{A.14}$$

We deduce from (A.9) that, if $j : \widetilde{Sp}(n, \mathbb{R}) \to Sp(n, \mathbb{R})$ is the natural projection, then

$$\big(a \circ j(g)\big)(X, D) = \omega(g)^{-1} a(X, D) \omega(g). \tag{A.15}$$

As a matter of fact, one can reduce (A.14) to a representation of $Mp(n, \mathbb{R})$, the 2-fold cover of $Sp(n, \mathbb{R})$, known as the metaplectic group. See Chapter 10 of [T5] for a discussion of this.

We now want to use the identity (A.7) to derive a formula for the Weyl symbol of

$$e^{-tH} = h_t(X, D), \quad H = -\Delta + |x|^2 = Q(X, D), \tag{A.16}$$

where

$$Q(x, \xi) = |x|^2 + |\xi|^2. \tag{A.17}$$

We will show that

$$h_t(x, \xi) = (\cosh t)^{-n} e^{-(\tanh t)(|x|^2 + |\xi|^2)}, \tag{A.18}$$

thus providing an alternative proof of Proposition 3.1 of Chapter II. First, we can see by symmetry that $h_t(x, \xi)$ is a function of $|x|^2 + |\xi|^2$. Indeed, the unitary group $U(n)$ is naturally contained in $Sp(n, \mathbb{R})$, and it acts transitively on spheres, so this is a consequence of (A.15), since cetainly $\omega(g)$ for $j(g) \in U(n)$ commutes with all functions of H. Thus, if we set

$$b_t(X, D) = H\, h_t(X, D), \tag{A.19}$$

then (A.7) implies

$$b_t(x, \xi) = Q(x, \xi) h_t(x, \xi) - \frac{1}{4} \sum_k \Big(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial \xi_k^2} \Big) h_t(x, \xi). \tag{A.20}$$

Note that the Poisson bracket vanishes in this case. Hence, $h_t(x,\xi)$ must satisfy the equation

$$\frac{\partial h_t}{\partial t}(x,\xi) = -\big(|x|^2+|\xi|^2\big)h_t(x,\xi) + \frac{1}{4}\sum_k \Big(\frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial \xi_k^2}\Big)h_t(x,\xi). \tag{A.21}$$

If we write

$$h_t(x,\xi) = g(t,Q), \quad Q = |x|^2+|\xi|^2, \tag{A.22}$$

then (A.21) becomes

$$\frac{\partial g}{\partial t} = -Qg + Q\frac{\partial^2 g}{\partial Q^2} + n\frac{\partial g}{\partial Q}. \tag{A.23}$$

Now (having peeked at (A.18)) we will make the "inspired guess" that

$$h_t(x,\xi) = a(t)\, e^{b(t)(|x|^2+|\xi|^2)}, \quad \text{i.e.,} \;\; g(t,Q) = a(t)\, e^{b(t)Q}. \tag{A.24}$$

Then the left side of (A.23) is $(a'/a + b'Q)g$ and the right side is $(-Q+Qb^2+nb)g$, so the identity (A.23) is equivalent to

$$\frac{a'(t)}{a(t)} = n\,b(t), \quad \text{and} \;\; b'(t) = 1 - b(t)^2. \tag{A.25}$$

We can solve the second equation for $b(t)$ by separation of variables. Since $h_0(x,\xi)=1$, we need $b(0)=0$, so the unique solution is easily seen to be

$$b(t) = -\tanh t.$$

Then the equation $a'/a = -n\tanh t$, with $a(0)=1$, immediately gives

$$a(t) = (\cosh t)^{-n}.$$

This completes the proof of the identity (A.18).

We can also analyze the Weyl symbol of

$$e^{-tQ(X,D)} = h_t^Q(X,D) \tag{A.26}$$

for a general positive-definite quadratic form $Q(x,\xi)$. If

$$Q(x,\xi) = \sum \mu_j(x_j^2+\xi_j^2), \tag{A.27}$$

with $\mu_j > 0$, then the proof of (A.18) also gives

$$h_t^Q(x,\xi) = \prod_{j=1}^n \frac{1}{\cosh t\mu_j}\, \exp\Bigl(-\sum_j (\tanh t\mu_j)(x_j^2+\xi_j^2)\Bigr). \tag{A.28}$$

We can express this in invariant form, as follows. Use the Hamilton map F_Q associated with Q, given by

$$\sigma(u, F_Q v) = Q(u,v), \tag{A.29}$$

where σ is the symplectic form on $\mathbb{R}^{2n}$ and $Q(u,v)$ is the symmetric bilinear form polarizing the quadratic form $Q(u),\ u=(x,\xi)$. For Q positive definite, a basic result of symplectic algebra is that F_Q has spectrum $\{\pm i\mu_j\},\ 1\le j\le n,\ \mu_j>0$; see, e.g., Chapter 1, §6 of [T5]. Given this, we have

$$\det\,\cosh i^{-1}tF_Q = \Bigl(\prod_{j=1}^n \cosh t\mu_j\Bigr)^2.$$

Also, if we set

$$A_Q = (-F_Q^2)^{1/2}, \tag{A.30}$$

the unique square root of $-F_Q^2$ with positive spectrum, we obtain (with $\zeta=(x,\xi)$)

$$\sum (\tanh t\mu_j)(x_j^2+\xi_j^2) = Q(A_Q^{-1}\tanh tA_Q\,\zeta,\zeta),$$

and hence

$$h_t^Q(x,\xi) = \frac{e^{-Q(A_Q^{-1}\tanh tA_Q\,\zeta,\zeta)}}{\bigl(\det\,\cosh\,(t/i)F_Q\bigr)^{1/2}}. \tag{A.31}$$

Since any positive-definite form $Q(x,\xi)$ can be put in the form (A.27) via a symplectic change of coordinates, the identity (A.15) implies that (A.31) is valid for a general positive-definite quadratic form $Q(x,\xi)$, as the formula for the Weyl symbol of $e^{-tQ(X,D)}$.

We turn to a study of L^2 boundedness of operators $a(X,D)$. We will show that $a(X,D)$ is bounded on $L^2(\mathbb{R}^n)$ if $a(x,\xi)\in\mathcal{S}_0^0$. This is a result of Calderon and Vaillancourt [C1]; see also [H10]. The approach we take is due to Cordes [C6] and Kato [K2], in a slightly different context. A related approach is given in Howe [H11]. The proof starts with the following two simple operator-theoretic results, whose proofs can be found in Chapter 13 of [T2], or the reader can try them as exercises.

Lemma A.1. *Let Y be any σ-finite measure space and $U(y)$ a weakly measurable family of bounded operators on a Hilbert space H such that*

$$\int_Y |(U(y)f,g)|^2\,dy \le C\|f\|^2\|g\|^2, \quad f,g\in H. \tag{A.31}$$

If $b\in L^\infty(Y)$, then for any trace-class operator G on H,

$$B = b\{G\} = \int_Y b(y)U(y)^*GU(y)\,dy \tag{A.33}$$

satisfies the operator bound

$$\|B\| \le C\|b\|_{L^\infty}\|G\|_{\mathrm{tr}}. \tag{A.34}$$

In our case, we take $Y = \mathbb{R}^{2n}$ with Lebesgue measure, $y=(x,\xi)$. We let $H = L^2(\mathbb{R}^n)$, $X_ju(x)=x_ju(x)$, $D_ju(x) = -i\partial u/\partial x_j$, and

$$U(y) = U(x,\xi) = e^{i\xi\cdot X + ix\cdot D}. \tag{A.35}$$

Lemma A.2. *If $U(y)$ is given by (A.35), then (A.32) is satisfied.*

The way in which (A.33)–(A.34) will be implemented is described as follows.

Proposition A.3. *Let $a = b * g$, $G = g(X,D)$. Then*

$$a(X,D) = \iint b(x,\xi)e^{i\xi\cdot X - ix\cdot D}\,G\,e^{-i\xi\cdot X + ix\cdot D}\,dx\,d\xi. \tag{A.36}$$

Proof. Clearly the superposition principle gives

$$a(X,D) = \iint b(x,\xi)g(X-x,D-\xi)\,dx\,d\xi.$$

Thus (A.36) is equivalent to the observation that

$$g(X-x,D-\xi) = e^{i\xi\cdot X - ix\cdot D}\,g(X,D)\,e^{-i\xi\cdot X + ix\cdot D}, \tag{A.37}$$

which is elementary (and which complements (A.9)).

We shall apply (A.36) to the case

$$b(x,\xi) = (1-\Delta_x-\Delta_\xi)^k a(x,\xi), \quad g(x,\xi) = (1-\Delta_x-\Delta_\xi)^{-k}\delta, \tag{A.38}$$

so $\hat{g}(q,p) = (1+|q|^2+|p|^2)^{-k}$. We will pick $k = k(n)$ sufficiently large below. In such a case, the hypothesis

$$|D^\alpha_{x,\xi} a(x,\xi)| \le A, \quad |\alpha| \le K(n), \tag{A.39}$$

implies $b(x,\xi) \in L^\infty(\mathbb{R}^{2n})$, provided $K(n) = 2k(n)$. It remains to ensure that $G = g(X,D)$ is trace class. Note that

$$Gu(x) = \int K(x,y)u(y)\,dy, \tag{A.40}$$

where

$$K(x,y) = \int e^{i(x-y)\cdot\xi}\, g\big(\tfrac{1}{2}(x+y),\xi\big)\,d\xi = L\big(x-y, \tfrac{1}{2}(x+y)\big). \tag{A.41}$$

Note that

$$x^\gamma y^\delta D^\alpha_x D^\beta_y L(x,y) = \int e^{ix\cdot\xi}\, y^\delta \xi^\alpha D^\gamma_\xi D^\beta_y g(y,\xi)\,d\xi. \tag{A.42}$$

Now the definition of $g(x,\xi)$ given in (A.38) implies that $g \in C^\infty(\mathbb{R}^{2n}\setminus 0)$ and $g(x,\xi)$ is rapidly decreasing together with all its derivatives, as $|x|+|\xi| \to \infty$. Also g is C^ℓ in a neighborhood of the origin, if $k(n) > \ell + n/2$. Thus (A.42) is bounded and continuous for $|\alpha|, |\beta|, |\gamma|, |\delta| \le \mu$ if $\ell \ge 2\mu$. On the other hand, it is easy to verify that, for $\mu = \mu(n)$ large enough, such a condition implies K is the integral kernel of a trace-class operator. Thus for $G = g(X,D)$ defined by (A.38) with $k = k(n)$ sufficiently large, G is of trace class.

Now the first two lemmas and Proposition A.3 yield the following Calderon-Vaillancourt theorem, which was stated in Chapter II, §1; see (1.26), (1.27).

Theorem A.4. *If the estimate (A.39) holds, with $K(n)$ sufficiently large, then*

$$a(X,D) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \quad \|a(X,D)\| \le C(n)\,A. \tag{A.43}$$

In particular, A is bounded on L^2 if $A \in OP\mathcal{S}^0_0$. The class $\mathcal{S}^0_0$ coincides with the class $S^0_{0,0}$, where we say $a(x,\xi)$ belongs to $S^m_{\rho,\delta}$ if and only if

$$|D^\beta_x D^\alpha_\xi a(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}. \tag{A.44}$$

A more general version of the Calderon-Vaillancourt theorem is that $a(X,D)$ is continuous on $L^2(\mathbb{R}^n)$ provided that, for some $\rho \in [0,1)$, $a(x,\xi) \in S^0_{\rho,\rho}$. This result is particularly important in the case $\rho = 1/2$. We will not give the proof of this, but note that it can be deduced from Theorem A.4 via the Cotlar-Stein lemma; see Beals [B2] for this argument, in a more general context.

We now show how the identity (A.18) for the Weyl symbol of e^{-tH}, with $H = -\Delta + |x|^2$, helps one to prove a sharp Gårding inequality. That one could do this was pointed out by

Unterberger [U1], which concerned itself not with sharp Gårding inequalities, but rather with further generalizations of the Calderon-Vaillancourt theorem. We seek conditions under which a symbol $p(x,\xi) \geq 0$ defines an operator which, if not positive, is at least semi-bounded:

$$(p(X,D)u,u) \geq -K\|u\|_{L^2}^2. \tag{A.45}$$

Now what the identity (A.18) implies is that

$$\alpha_t(x,\xi) = \Big(\frac{t}{\pi}\Big)^n e^{-t(|x|^2+|\xi|^2)} \Rightarrow \alpha_t(X,D) \geq 0, \quad 0 < t \leq 1, \tag{A.46}$$

since clearly e^{-sH} is a positive operator for any $s \geq 0$. Now operators can be synthesized from $\alpha_t(X,D)$ and various other operators unitarily equivalent to it, and these will provide a sufficiently rich class of positive operators to enable us to obtain the following sharp Gårding inequality.

Proposition A.5. *The semiboundedness (A.45) holds for all $p(x,\xi)$ such that*

$$p \in \mathcal{S}_1^2. \tag{A.47}$$

Proof. Consider the operator $p_1(X,D)$, where $p_1 = p * \alpha_1$. Then, as in (A.36),

$$p_1(X,D) = \iint p(x,\xi)\, e^{i\xi\cdot X - ix\cdot D}\, \alpha_1(X,D)\, e^{-i\xi\cdot X + ix\cdot D}\, dx\, d\xi \geq 0. \tag{A.48}$$

Consequently (A.45) follows from the fact that

$$p_1 - p \in \mathcal{S}_1^0, \tag{A.49}$$

in view of Theorem A.4. The result (A.49) in turn is straightforward, and we leave it as an exercise.

A somewhat more complicated variation of the argument given above can be used to show that the semiboundedness (A.45) holds for all $p(x,\xi) \geq 0$ when $p \in S^1_{1,0}$. Proposition A.5 has the appearance of a weak form of the Fefferman-Phong inequality, to the effect that the semiboundedness (A.45) holds when $p(x,\xi) \geq 0$ and $p \in S^2_{1,0}$, if p is scalar; cf. [F1]. This result is more powerful than Proposition A.5, in the scalar case. However, Proposition A.5 holds for systems.

Appendix B. Toeplitz operators and more general Weyl calculus

A principal goal of this appendix is to discuss the following result of R. Howe: there is a natural unitary map from $L^2(\mathbb{R}^n)$ to $L^2_{\mathcal{H}}(B^n)$, the space of holomorphic L^2 functions on the unit ball $B \subset \mathbb{C}^n$, which sets up an isomorphism between the algebra $OP\mathcal{H}^m_b$ of pseudodifferential operators on $\mathbb{R}^n$ and the algebra of Toeplitz operators on B^n of order $m/2$. This elegant result was given in [H11] and also discussed in [G15]. Since these references confine the proof to a sketch, we shall take the space to provide a full proof here. The proof will arise as an application of a more general sort of Weyl calculus than that dealt with in Appendix A.

The unitary isomorphism of $L^2(\mathbb{R}^n)$ with $L^2_{\mathcal{H}}(B^n)$ starts with the unitary transformation intertwining the representation π_1 of the Heisenberg group $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$ (given in Chapter II, §1) with the Bargman-Fok representation, which we will now define. The representation β acts on the Hilbert space

(B.1) $$\mathcal{H} = \Big\{ u(z) \text{ holomorphic on } \mathbb{C}^n : \int |u(z)|^2 e^{-|z|^2/2}\, dV(z) < \infty \Big\}.$$

Then, for $(t, q, p) \in \mathbb{H}^n$, the unitary operator $\beta(t, q, p)$ on $\mathcal{H}$ is defined by

(B.2) $$\beta(t, q, p)u(z) = e^{it + i(q+ip)\cdot z\sqrt{2} - (|q|^2+|p|^2)}\, u\big(z + i(q - ip)/\sqrt{2}\big).$$

Note that, on the Lie algebra level,

(B.3) $$\beta(T) = iI, \quad \beta(L_j) = \sqrt{2}\Big(\frac{\partial}{\partial z_j} + z_j\Big), \quad \beta(M_j) = \frac{1}{i\sqrt{2}}\Big(\frac{\partial}{\partial z_j} - z_j\Big).$$

Recall that, for the Schrödinger representation π_1 of $\mathbb{H}^n$ on $L^2(\mathbb{R}^n)$,

(B.4) $$\pi_1(T) = iI, \quad \pi_1(L_j) = \frac{\partial}{\partial x_j}, \quad \pi_1(M_j) = ix_j.$$

The unitary operator $W : L^2(\mathbb{R}^n) \to \mathcal{H}$ intertwining π_1 and β is given by

(B.5) $$\begin{aligned} Wf(z) &= \int f(x)K(x, z)\, dx, \\ K(x, z) &= \exp\Big(\sqrt{2}z \cdot x - \frac{z \cdot z + |x|^2}{2}\Big). \end{aligned}$$

Now there is a natural unitary isomorphism

(B.6) $$V : \mathcal{H} \longrightarrow L^2_{\mathcal{H}}(B^n),$$

which arises as follows. An orthonormal basis of $\mathcal{H}$ is

$$u_\alpha = a_\alpha z^\alpha, \quad a_\alpha = \sqrt{\frac{2}{\alpha!}}, \tag{B.7}$$

(see, e.g., [T5], Chapter 1), while an orthonormal basis of $L^2_{\mathcal{H}}(B^n)$ is given by

$$v_\alpha = b_\alpha z^\alpha, \quad b_\alpha = \sqrt{\frac{(n+|\alpha|)!}{\alpha!}}. \tag{B.8}$$

See Rudin [R7]. Now we define V by $Vu_\alpha = v_\alpha$, i.e.,

$$Vz^\alpha = \gamma_\alpha z^\alpha, \quad \gamma_\alpha = \frac{b_\alpha}{a_\alpha} = \sqrt{\frac{(n+|\alpha|)!}{2}} = \gamma_{|\alpha|}. \tag{B.9}$$

Now we want to analyze the operators $\mathcal{Z}_j$ and $\mathcal{L}_j$, defined by

$$\mathcal{Z}_j = Vz_jV^{-1}, \quad \mathcal{L}_j = V\,\frac{\partial}{\partial z_j}\,V^{-1}. \tag{B.10}$$

In fact, (B.9) shows that

$$\mathcal{Z}_j z^\alpha = \frac{\gamma_{|\alpha|+1}}{\gamma_{|\alpha|}} z_j z^\alpha, \quad \mathcal{L}_j z^\alpha = \frac{\gamma_{|\alpha|-1}}{\gamma_{|\alpha|}} \frac{\partial}{\partial z_j} z^\alpha.$$

In particular,

$$\mathcal{Z}_j z^\alpha = \sqrt{|\alpha|+n+1}\, z_j z^\alpha. \tag{B.11}$$

The operator $z^\alpha \mapsto |\alpha| z^\alpha$ can be extended to $C^\infty(B^n)$ as $u(z) \mapsto i^{-1}Xu$, where X is the *real* vector field on $\mathbb{R}^{2n} = \mathbb{C}^n$ generated by the flow $z \mapsto e^{it}z$ on $\mathbb{C}^n$. If we set $D = i^{-1}X$, a first order differential operator, we get

$$\mathcal{Z}_j = z_j\sqrt{|D|+n+1}. \tag{B.12}$$

Since z_j and $\partial/\partial z_j$ are adjoints of each other on $\mathcal{H}$, the adjoint of $\mathcal{Z}_j$ on $L^2_{\mathcal{H}}(B^n)$ is $\mathcal{L}_j$. Thus, if ν is the unitary representation of $\mathbb{H}^n$ on $L^2_{\mathcal{H}}(B^n)$ defined by

$$\nu(g) = V\beta(g)V^{-1}, \quad g \in \mathbb{H}^n, \tag{B.13}$$

then, with π denoting the orthogonal projection of $L^2(B^n)$ onto $L^2_{\mathcal{H}}(B^n)$, we have

$$\begin{gathered}\nu(T) = iI, \quad \nu(L_j) = i\pi\big(z_j(|D|+n+1)^{1/2} + (|D|+n+1)^{1/2}\overline{z}_j\big)\pi, \\ \nu(M_j) = \pi\big(z_j(|D|+n+1)^{1/2} - (|D|+n+1)^{1/2}\overline{z}_j\big)\pi,\end{gathered} \tag{B.14}$$

on $L^2_{\mathcal{H}}(B^n)$. This was derived in [H11] and [G15]. Here π is the orthogonal projection of $L^2(B^n)$ onto $L^2_{\mathcal{H}}(B^n)$.

Alternatively, we can regard π as the orthogonal projection of $H^{-1/2}(S^{2n-1})$ on $L^2_{\mathcal{H}}(B^n)$, which we identify with a closed linear subspace of the Sobolev space $H^{-1/2}(S^{2n-1})$ via the Poisson integral. The projection π is then essentially the Szegö projector. We can write

$$\nu(L_j) = i\pi\mathfrak{D}_j\pi, \quad \nu(M_j) = i\pi\mathfrak{X}_j\pi, \tag{B.15}$$

where

$$\begin{aligned} \mathfrak{D}_j &= z_j(|D| + n + 1)^{1/2} + (|D| + n + 1)^{1/2}\overline{z}_j, \\ i\mathfrak{X}_j &= z_j(|D| + n + 1)^{1/2} - (|D| + n + 1)^{1/2}\overline{z}_j. \end{aligned} \tag{B.16}$$

Now $\mathfrak{D}_j$ and $\mathfrak{X}_j$ are not quite pseudodifferential operators. But note that, for all $u \in \mathcal{D}'(S^{2n-1})$, $\pi\, u$ has wave front set in Λ_+, a ray bundle in $T^*S^{2n-1} \setminus 0$ defining the contact structure, and the differential operator D is noncharacteristic on Λ_+, and so elliptic on a conic neighborhood of Λ_+. Thus, when analyzing $\mathfrak{D}_j$ and $\mathfrak{X}_j$, we can treat these operators as pseudodifferential operators on S^{2n-1}, belonging to $OPS^{1/2}$. Note that, on Λ_+,

$$\sigma_{\mathfrak{X}_j}(\omega, \xi) = (\operatorname{Re}\omega_j)|\xi|^{1/2}, \quad \sigma_{\mathfrak{D}_j}(\omega, \xi) = (\operatorname{Im}\omega_j)|\xi|^{1/2}, \ \ \text{mod} \ \ S^{-1/2}, \tag{B.17}$$

where $\omega = (\omega_1, \dots, \omega_n) \in S^{2n-1} \subset \mathbb{C}^n$.

We will now alter $\mathfrak{X}_j$ and $\mathfrak{D}_j$, making use of Theorem B from Chapter III, §6, to operators $\mathfrak{X}_j$ and $\mathfrak{D}_j$ that *commute* with π. The map

$$U = VW : L^2(\mathbb{R}^n) \longrightarrow L^2_{\mathcal{H}}(B^n)$$

is unitary, and under it, a pseudodifferential operator $a(X, D)$ is taken to

$$Ua(X, D)U^{-1} = \pi a(\mathfrak{X}, \mathfrak{D})\pi, \tag{B.18}$$

where $a(\mathfrak{X}, \mathfrak{D})$ is a function of the self-adjoint operators $\mathfrak{X}_1, \dots, \mathfrak{X}_n, \mathfrak{D}_1, \dots, \mathfrak{D}_n$, defined by the Weyl calculus:

$$a(\mathfrak{X}, \mathfrak{D}) = (2\pi)^{-n} \iint \hat{a}(q, p) e^{iq\cdot\mathfrak{X} + ip\cdot\mathfrak{D}}\, dq\, dp, \tag{B.19}$$

where $q \cdot \mathfrak{X} = \sum q_j \mathfrak{X}_j$ and $p \cdot \mathfrak{D} = \sum p_j \mathfrak{D}_j$.

With this in mind, we will consider the Weyl calculus in the following more general context. On a compact manifold, take $L_1, \dots, L_k$ self-adjoint operators,

$$L_j \in OPS^a, \quad 0 < a \le 1. \tag{B.20}$$

We suppose L_j have scalar principal symbol. We also suppose

$$\mathcal{L} = L_1^2 + \cdots + L_k^2 \in OPS^{2a} \ \ \text{is elliptic.} \tag{B.21}$$

We want to analyze the operator $a(L)$, defined by the formula

$$a(L) = \int \hat{a}(s)\, e^{is\cdot L}\, ds, \tag{B.22}$$

as a pseudodifferential operator, given $a(\xi_1, \dots, \xi_k) \in \mathcal{S}^m_\rho$, i.e.,

$$|D^\alpha a(\xi)| \le C_\alpha (1 + |\xi|)^{m-\rho|\alpha|}. \tag{B.23}$$

In (B.22), we take $s \cdot L = s_1 L_1 + \cdots + s_k L_k,\ s \in \mathbb{R}^k$. Note that (B.23) implies $\hat{a}(s)$ is C^∞ outside the origin and rapidly decreasing, with all its derivatives, as $|s| \to \infty$. We can localize the analysis of $a(L)$ under the following hypothesis:

$$a \in \mathcal{S}(\mathbb{R}^k) \Longrightarrow a(L) \in OPS^{-\infty}. \tag{B.24}$$

We will go into cases where (B.24) is sure to be valid shortly. Granted this hypothesis, in (B.22), we can suppose $a(s)$ is supported on a small set $|s| \le \varepsilon$. We will use the method of geometrical optics to obtain a parametrix for $e^{-s\cdot L}$, for $|s| \le \varepsilon$. In fact, let $A \in OPS^a$ be some real linear combination of $L_1, \dots, L_k$, and we represent e^{itA}, for $|t|$ small, in the form (in local coordinates)

$$e^{itA} u(x) = \int a(t, x, \xi)\, e^{i\varphi(t,x,\xi) + ix\cdot\xi}\, \hat{u}(\xi)\, d\xi.$$

Here, the phase φ satisfies the eikonal equation

$$\frac{\partial \varphi}{\partial t} = A_1(x, \xi + \nabla_x \varphi), \quad \varphi(0, x, \xi) = 0,$$

where $A_1(x,\xi) \in S^a$ is the principal symbol of $A \in OPS^a$. Thus, for small t we can solve for φ, real valued in S^a. We get a sequence of transport equations for $a \sim \sum a_j$, yielding $a(t,x,\xi) \in S^0$ with $a(0,x,\xi) = 1$. Writing $A = \sum s_j L_j$, the functions φ and a depend smoothly on their parameters on some set $|s| \le s_0,\ |t| \le t_0$. We can freeze t at t_0 and rescale s, to write, for $|s| \le \varepsilon$,

$$e^{is\cdot L} u(x) = \int b(s, x, \xi) e^{i\psi(s,x,\xi) + ix\cdot\xi}\, \hat{u}(\xi)\, d\xi. \tag{B.25}$$

Here $b(0, x, \xi) = 1,\ b \in S^0$, and $\psi \in S^a$ is real valued and satisfies

$$\frac{\partial \psi}{\partial s_j}(0, x, \xi) = \sigma_{L_j}(x, \xi), \quad \psi(0, x, \xi) = 0. \tag{B.26}$$

Returning to (B.22), we have, for $a \in \mathcal{S}^m_\rho$ such that $\hat{a}(s)$ is supported on $|s| \le \varepsilon$,

$$\begin{aligned} a(L) &= \iint \hat{a}(s) b(s, x, \xi) e^{i\psi(s,x,\xi)} e^{ix\cdot\xi}\, \hat{u}(\xi)\, d\xi\, ds \\ &= \int a(D_s)(b e^{i\psi})\big|_{s=0}\, e^{ix\cdot\xi}\, d\xi. \end{aligned} \tag{B.27}$$

Now we can use the stationary phase method to evaluate $a(D_s)(be^{i\psi})$ provided we know $\nabla_s\psi \neq 0$ at $s = 0$. In view of (B.26), this is equivalent to the hypothesis (B.21). Thus we obtain, for $a \in \mathcal{S}^m_\rho$, provided $1/2 < \rho \le 1$,

$$a(D_s)(be^{i\psi}) = B\, e^{i\psi}, \tag{B.28}$$

where

$$B \in S^{ma}_{\rho',1-\rho'} \tag{B.29}$$

has an asymptotic expansion

$$B(s,x,\xi) \sim b(s,x,\xi)a(\nabla_s\psi) + \cdots . \tag{B.30}$$

In particular,

$$B(0,x,\xi) \sim a\big(\sigma_{L_1}(x,\xi),\dots,\sigma_{L_k}(x,\xi)\big) + \cdots .$$

Now (B.27) gives

$$a(L)u = \int B(0,x,\xi)e^{ix\cdot\xi}\hat{u}(\xi)\, d\xi, \tag{B.31}$$

and this implies $a(L)$ is a pseudodifferential operator. In particular, for $a \in \mathcal{S}^m_1$, we have $a(L) \subset OPS^{ma}_{1,0}$, and

$$\sigma_{a(L)}(x,\xi) = a\big(L_1(x,\xi),\dots,L_k(x,\xi)\big) \mod S^{ma-a}_{1,0}, \tag{B.32}$$

where $L_j(x,\xi)$ is the principal symbol of L_j.

In general, one might not have (B.24), but the operator calculus (B.31)–(B.32) still holds for $a \in \mathcal{S}^m_\rho$ such that $\hat{a}(s)$ is supported in a small neighborhood of 0. This can still be quite useful, as we will see below. One case where we can guarantee (B.24) is when $L_1,\dots,L_k$ all commute. In that case, one has

$$L_j^2 a(L) = \int D^2_{s_j}\hat{a}(a)\, e^{is\cdot L}\, ds,$$

and hence, if $a \in \mathcal{S}(\mathbb{R}^k)$,

$$\mathcal{L}^j a(L) = \int \Delta^j \hat{a}(a)\, e^{is\cdot L}\, ds$$

is bounded on each Sobolev space $H^s(M)$ for all j. Since $\mathcal{L}$ is elliptic, this implies $A(L) \in OPS^{-\infty}$, if $a \in \mathcal{S}(\mathbb{R}^k)$. In this commutatice case, the analysis of the functional calculus given here is equivalent to that given by the author in [T3], and in Chapter 12 of [T2]. Other approaches to functional calculi in the commutative case were given by Strichartz [S9] and Colin de Verdiere [C5].

This argument can be pushed to yield (B.24) as long as

$$D_{s_j} e^{is\cdot L} = B_j(s) e^{is\cdot L} + L_j e^{is\cdot L}, \tag{B.33}$$

where each $B_j(s)$ is a polynomial in s with values in $OPS^{a-\varepsilon}$, for some $\varepsilon > 0$. A favorable case is $L_j \in OPS^{1/2}$ and $B_j(s)$ polynomials in s with values in OPS^0. Since $\pi\mathfrak{X}_j\pi$, $\pi\mathfrak{D}_j\pi$ satisfy the same commutation relations as X, D, we see that such an identity holds in this case, so

$$a \in \mathcal{S} \Longrightarrow \pi a(\mathfrak{X}, \mathfrak{D})\pi \ \text{ is smoothing.} \tag{B.34}$$

Thus the geometrical optics analysis applies, to give

$$\pi a(\mathfrak{X}, \mathfrak{D})\pi = \pi P_a \pi, \tag{B.35}$$

where

$$a \in \mathcal{S}_1^m \Longrightarrow P_a \in OPS^{m/2}_{1,0}(S^{2n-1}). \tag{B.36}$$

Suppose $a \in \mathcal{S}^m$, i.e., $a(x,\xi) \sim \sum a_j(x,\xi)$ with a_j homogeneous of degree $m-j$ in (x,ξ). In this case, since $\mathfrak{X}_j, \mathfrak{D}_j \in OPS^{1/2}$ (microlocally near Λ_+), we see that the phase function $\psi(s,x,\xi)$, obtained as a solution to an eikonal equation, is asymptotic to $\psi_1 + \psi_2 + \cdots$, with ψ_{1+k} homogeneous of degree $1 - k/2$. Also the amplitude $B(s,x,\xi)$ is asymptotic to $b_0 + b_1 + b_2 + \cdots$, with b_j homogeneous of degree $-j/2$ in ξ. It follows that (B.35) holds, with

$$a \in \mathcal{S}_m \Longrightarrow P_a \sim P_0 + P_1 + P_2 + \cdots, \quad P_j \in OPS^{m/2-j/2}. \tag{B.37}$$

Now, if $a(x,\xi)$ is equal to a function homogeneous of degree m for $|x| + |\xi|$ large, the expansion above for P_a can be improved. In fact, if we use our class of symbols

$$\mathcal{H}_b^m = \Big\{ a(x,\xi) : a \sim \sum_{j\ge 0} a_j,\ a_j \text{ homogeneous of degree } m - 2j \Big\}, \tag{B.38}$$

then we will prove the following.

Proposition B.1. *If $a(x,\xi) \in \mathcal{H}_b^m$, then (B.35) holds with $P_a \in OPS^{m/2}$, so*

$$P_a \sim P_0 + P_1 + P_2 + \cdots, \quad P_j \in OPS^{m/2-j}. \tag{B.39}$$

The content of this proposition is that for $a \in \mathcal{H}_b^m$, the terms in the expansion (B.37) vanish for j odd. The amplitude $B(0,x,\xi)$ in (B.31) arises by an inductive construction, and such a construction does not lend itself easily to proving this infinite sequence of identities. We will take a different approach, one involving the Weyl functional calculus in a context not quite covered by hypothesis (B.20).

Namely, we need a functional calculus for the quadratic forms in X and D occurring in the metaplectic representation (on the Lie algebra level). Thus, with $\mathrm{sp}(n,\mathbb{R}) \approx \mathfrak{P}_2$, the linear space of second-order homogeneous polynomials $Q(x,\xi)$, we pick a basis Q_j for this vector space and set

$$p(Q_1(X,D),\dots,Q_N(X,D)) = \int \hat{p}(s)\, e^{is_1Q_1(X,D)+\cdots+is_NQ_N(X,D)}\, ds, \tag{B.40}$$

where $N = \dim\, \mathrm{sp}(n,\mathbb{R}) = n(n+1)/2$. To write it in a more invariant fashion, let $\mathfrak{Q}$ denote the linear map from $\mathrm{sp}(n,\mathbb{R})$ to self-adjoint operators on $L^2(\mathbb{R}^n)$ given by

$$\mathfrak{Q}(Q) = Q(X,D), \tag{B.41}$$

and set

$$p(\mathfrak{Q}) = \int \hat{p}(Q) e^{i\mathfrak{Q}(Q)}\, dQ = \int \hat{p}(Q) e^{iQ(X,D)}\, dQ. \tag{B.42}$$

We will parallel the analysis of (B.22), via a representation of $e^{iQ(X,D)}$ as a Fourier integral operator and use of stationary phase, as in (B.27). Our next order of business is hence to represent $e^{iQ(X,D)}$ as a Fourier integral operator for $Q \in \mathfrak{P}_2$.

That this can be done is related to the fact that commutators of operators in $OP\mathcal{S}^{m_j}$ belong to $OP\mathcal{S}^{m_1+m_2-2}$, so operators in $OP\mathcal{S}^2$ behave for many purposes like pseudodifferential operators of order 1 on compact manifolds. The Fourier integral representation of $e^{iQ(X,D)}$ can be derived by the method of geometrical optics (see, e.g., Helffer and Robert [H4]), but we will derive it from the identity (A.31) of Appendix A, which says that, for $Q(x,\xi) \in \mathfrak{P}_2$, positive definite,

$$e^{-tQ(X,D)} = h_t^Q(X,D), \tag{B.43}$$

with

$$h_t^Q(x,\xi) = \frac{e^{-Q(A_Q^{-1}\tanh tA_Q\,\zeta,\zeta)}}{\big(\det\, \cosh\,(t/i)F_Q\big)^{1/2}}. \tag{B.44}$$

Here, $\zeta = (x,\xi)$, and $Q(u,v)$ is the bilinear form such that $Q(\zeta,\zeta) = Q(\zeta)$. Recall that F_Q is the linear transformation on $\mathbb{R}^{2n}$ defined by

$$Q(u,v) = \sigma(F_Q u, v), \tag{B.45}$$

and $A_Q = (-F_Q^2)^{1/2}$. Compare (4.55)–(4.60) of Chapter II. If we set

$$\theta(t) = t^{-1}\tanh t, \tag{B.46}$$

which is an even function of t, we can write (B.44) as

$$h_t^Q(x,\xi) = \frac{e^{-tQ(\theta(tF_Q)\zeta,\zeta)}}{\big(\det\,\cosh\,(t/i)F_Q\big)^{1/2}}. \tag{B.47}$$

We can analytically continue to imaginary time, and, with

$$\vartheta(t) = t^{-1}\tan t, \tag{B.48}$$

we get

$$h_{it}^Q(x,\xi) = \frac{e^{itQ(\vartheta(tF_Q)\zeta,\zeta)}}{\big(\det\,\cos\,(t/i)F_Q\big)^{1/2}}. \tag{B.49}$$

Consequently we have the Fourier integral representation

$$e^{iQ(X,D)}u(x) = \int a(Q)e^{i\varphi(Q,\frac{1}{2}(x+y),\xi)+i(x-y)\cdot\xi}\,u(y)\,dy\,d\xi, \tag{B.50}$$

where

$$\varphi(Q,x,\xi) = -Q(\vartheta(F_Q)\zeta,\zeta), \quad \zeta = (x,\xi), \tag{B.51}$$

and

$$a(Q) = (\det\,\cos F_Q/i)^{-1/2}. \tag{B.52}$$

Now (B.50) was derived under the hypothesis that $Q(x,\xi)$ was positive definite, but it is clear that the resulting formula is valid for all $Q \in \mathfrak{P}_2$. The amplitude $a(Q)$ and phase φ have singularities at $Q \in \mathfrak{P}_2$ for which F_Q/i has $(k+1/2)\pi$ as an eigenvalue for some integer k, but formula (B.50) is valid, with $a(Q)$ and φ smooth, for Q in a neighborhood $\mathcal{O}$ of the origin.

If $p \in \mathcal{S}^m$ and $\hat{p}$ is supported in $\mathcal{O}$, then (B.42) and (B.50) give

$$\begin{aligned} p(\mathfrak{Q})u(x) &= \iiint \hat{p}(Q)\big(a(Q)e^{i\varphi(Q,\frac{1}{2}(x+y),\xi)}\big)e^{i(x-y)\cdot\xi}\,u(y)\,dy\,d\xi\,dQ \\ &= \iint p(D_Q)\big(a(Q)e^{i\varphi(Q,\frac{1}{2}(x+y),\xi)}\big)\big|_{Q=0}e^{i(x-y)\cdot\xi}\,u(y)\,dy\,d\xi. \end{aligned} \tag{B.53}$$

Now, in view of (B.51), we see that $\nabla_Q\varphi(0,x,\xi) = Q(x,\xi)$, so the Q-gradient vector of φ is nonvanishing for $(x,\xi) \neq (0,0)$, and hence the stationary phase method yields

$$p(\mathfrak{Q})u(x) = \iint b\big(0,\tfrac{1}{2}(x+y),\xi\big)e^{i(x-y)\cdot\xi}\,u(y)\,dy\,d\xi = b(0,X,D)u(x), \tag{B.54}$$

where

$$b\big(Q, \tfrac{1}{2}(x+y), \xi\big) = e^{-i\varphi} p(D_Q)\big(a(Q)e^{i\varphi}\big). \tag{B.55}$$

If $p \in \mathcal{S}^m$, we obtain the asymptotic expansion

$$b(Q,x,\xi) = \sum_{j\ge 0} b_j(Q,x,\xi), \tag{B.56}$$

with $b_j(Q,x,\xi)$ homogeneous of degree $2(m-j)$ in (x,ξ). (Note that $b_j(Q,x,\xi) = b_j(Q,-x,-\xi)$.) Thus

$$p \in \mathcal{S}^m \Longrightarrow p(\mathfrak{Q}) \in OP\mathcal{H}_b^{2m}, \tag{B.57}$$

if $\hat{p}$ is supported sufficiently near the origin, and we have

$$\sigma_{p(\mathfrak{Q})}(x,\xi) = p(\widetilde{Q}(x,\xi)), \quad \bmod\ \mathcal{H}_b^{2m-2}. \tag{B.58}$$

Here we define $\widetilde{Q} : \mathbb{R}^{2n} \to \mathfrak{P}_2'$, the linear dual of $\mathfrak{P}_2$, by

$$\langle \widetilde{Q}(x,\xi), Q\rangle = Q(x,\xi). \tag{B.59}$$

Using (B.57) and (B.58), we will establish the following.

Lemma B.2. *Let $q(x,\xi) \in \mathcal{H}_b^m$ be even, i.e., $q(x,\xi) = q(-x,-\xi)$. Then there exists $p \in \mathcal{S}^{m/2}$ such that*

$$p(\mathfrak{Q}) - q(X,D) \in OP\mathcal{S}_1^{-\infty}. \tag{B.60}$$

Proof. We begin with the observation that if $q(x,\xi) \in \mathcal{H}_b^m$ is even, there is a smooth function $p_0 \in \mathcal{S}^{m/2}$ such that $p_0(\widetilde{Q}(x,\xi))$ is equal to $q(x,\xi)$, outside some neighborhood of the origin. This follows from the result of A. Schwartz [S2]; see also Mather [M1]. This argument was noted in another context in [G15]; see the proof of Lemma 6.14 there. One can truncate $\hat{p}_0$, preserving evenness, so $\hat{p}_0$ is supported near the origin. This alters p_0 by an element of $\mathcal{S}_1^{-\infty}$. For such p_0, (B.57)–(B.58) give

$$p_0(\mathfrak{Q}) - q(X,D) \in OP\mathcal{H}_b^{m-2}. \tag{B.61}$$

Now an inductive argument yields $p_j(\mathfrak{Q}) \in OP\mathcal{H}_b^{m-2j}$ such that, if $p \sim \sum p_j$, (B.60) holds.

We return to our analysis of

$$\pi a(\mathfrak{X},\mathfrak{D})\pi = U a(X,D) U^{-1} \tag{B.62}$$

for $a(x,\xi)\in\mathcal{H}_b^m$. Recall that we have

$$\pi a(\mathfrak{X},\mathfrak{D})\pi=\pi P_a\pi \tag{B.63}$$

with $P_a\in OPS_{1,0}^{m/2}$ being asymptotic to $\sum P_j,\ P_j\in OPS^{m/2-j/2}$. Now suppose $a(x,\xi)\in\mathcal{H}_b^m$ is *even.* Then, by Lemma B.2, we can write

$$a(X,D)=p(\mathfrak{Q}),\quad \text{mod}\ \ OP\mathcal{S}_1^{-\infty}, \tag{B.64}$$

where $p\in\mathcal{S}^{m/2}$. Now if we define a linear map Σ from $\mathfrak{P}_2$ to self-adjoint operators by

$$\Sigma(Q)=Q(\mathfrak{X},\mathfrak{D}), \tag{B.65}$$

we have, modulo a smoothing operator,

$$\pi a(\mathfrak{X},\mathfrak{D})\pi=\pi p(\Sigma)\pi. \tag{B.66}$$

Note that, by (B.16)–(B.17), for each $Q\in\mathfrak{P}_2,\ Q(\mathfrak{X},\mathfrak{D})\in OPS^1$ microlocally near Λ_+, and the functional calculus for $p(L),\ L_j\in OPS^1$ developed above applies to $p(\Sigma)$. In other words, on a conic neighborhood of $\Lambda_+\subset T^*(S^{2n-1})\setminus 0$,

$$p\in\mathcal{S}^{m/2}\Longrightarrow p(\Sigma)\in OPS^{m/2}, \tag{B.67}$$

at least as long as $\hat{p}$ is supported near the origin. Consequently we have proved:

$$a(x,\xi)\in\mathcal{H}_b^m\ \text{ even }\Longrightarrow \pi a(\mathfrak{X},\mathfrak{D})\pi=\pi P_a\pi,\quad P_a\in OPS^{m/2}. \tag{B.68}$$

We are now in a position to prove Proposition B.1. In fact, the weak result (B.37) implies that the correspondence $a(X,D)\mapsto P_a$ is "microlocal," and in analyzing P_a on a microlocal level there is no loss of generality in supposing $a(x,\xi)$ is even. From Proposition B.1 easily follows the more rounded out statement of affairs whish we record here:

Theorem B.3. *With the unitary map U given above, the correspondence*

$$a(X,D)\mapsto Ua(X,D)U^{-1}=\pi a(\mathfrak{X},\mathfrak{D})\pi \tag{B.69}$$

sets up an isomorphism between $OP\mathcal{H}_b^m$ and $\mathcal{T}^{m/2}$, where

$$\mathcal{T}^{m/2}=\{\pi P\pi : P\in OPS^{m/2}(S^{2n-1})\}. \tag{B.70}$$

Proof. We need only check the surjectivity. If $\pi P\pi\in\mathcal{T}^{m/2}$, the construction above produces $a_0(X,D)\in OP\mathcal{H}_b^m$ whose image in $\mathcal{T}^{m/2}$ is $\pi P_1\pi$, where $P_1\in OPS^{m/2}$ has the same principal symbol as that of P on Λ_+. Now we appeal to Theorem A, stated near the end of §6, Chapter III. The difference $\pi(P-P_1)\pi$ is a Toeplitz operator of order $m/2-1$. Inductively we obtain $a_j(X,D)\in OP\mathcal{H}_b^{m-2j}$ which asymptotically sum to $a(X,D)$, whose image differs from $\pi P \pi$ by a smoothing operator, which in turn is readily absorbed. The proof is complete.

Note that Theorem B.3 implies an equivalence between index theorems for elliptic operators in $OP\mathcal{H}_b^m$ (contained, e.g., in [H10]) and the index theorem of Venugopalkrishna [V2] for Toeplitz operators on B^n. Boutet de Monvel [B8] has index theorems for general strongly pseudoconvex domains. Also Theorem B.3 implies an equivalence between results on spectral asymptotics for elliptic operators obtained by Helffer and Robert [H4], Shubin [S5], Chazarin [C2], and others, and results on spectral asymptotics for elliptic Toeplitz operators on B^n. Such results on spectral asymptotics, for elliptic Toeplitz operators on general strongly pseudoconvex domains, have been proved by Boutet de Monvel and Guillemin [B11]. Also, Guillemin and Sternberg [G15] analyze spectral asymptotics for $a(X, D) \in OP\mathcal{H}_b^m$, in the special case when $a(x, \xi)$ is even, using a different "compactification."

Index of symbol classes

For pseudodifferential operators on $\mathbb{R}^n$, we have an operator $p(x, D)$ associated to a symbol $p(x,\xi)$, via formula (1.1) of Chapter I. In this case, symbol classes of particular importance include $S^m_{\rho,\delta}$, consisting of $p(x,\xi)$ such that

$$|D_x^\beta D_\xi^\alpha p(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|},$$

and the subclass S^m of $S^m_{1,0}$, consisting of $p(x,\xi)$ with an asymptotic expansion

$$p(x,\xi) \sim \sum_{j\geq 0} p_{m-j}(x,\xi)$$

where, for $|\xi| \geq 1$, $p_{m-j}(x,\xi)$ is homogeneous of degree $m-j$ in ξ. A large number of other classes of "symbols" has arisen in this paper, to which operators have been associated, by various rules. We provide a list of these classes here, for convenience. We list the chapter and numbered formula at (or near) which each listed symbol class is defined.

Symbol Class	Defined at:	Symbol Class	Defined at:
$S^m_{\rho\#}$	I (1.4)	$\Omega^{m,k}_b$	II (3.41)
$\mathfrak{H}(G,\alpha,m)$	I (2.6)	$\Omega^{m,k}$	II (3.51)-
$\mathfrak{H}^{m,\mu}_{\alpha,\delta}$	I (2.40)	$S^{m,k}(\mathbb{H}^n,\Lambda)$	II (3.51)
Ψ^m_0	II (2.4)	$\Omega^{m,k}_{1/2}$	II (5.13)
$\mathcal{H}^m_b$	II (2.13)	$\widetilde{\Psi}^m$	III (1.1)
$\mathcal{H}^m$	II (2.14)	$\widetilde{\Psi}^{m,k}$	III (1.1)
$\mathcal{S}^m_1$	II (2.16)	$\widetilde{\Sigma}^m$	III (1.1)
$\mathcal{H}^{m,k}$	II (2.23)	$\widetilde{\Omega}^{m,k}$	III (1.1)
$\Psi^{m,k}_0$	II (2.36)	$\widetilde{\mathfrak{H}}^{m,\mu}_{\alpha,\delta}$	III (1.1)+
Ψ^m	II (2.56)	$\widetilde{\Psi}^{m,\infty}_+$	III (6.37)
Σ^m_0	II (3.1)	$\mathcal{S}^m_\rho$	(A.3)
Σ^m	II (3.2)	$\mathcal{S}^m$	(B.36)+

Index of other notations

The following is a list of other special symbols, and the chapter and formula number at (or near) which each is introduced.

References

[A1] R. Anderson, "The Weyl functional calculus," J. Funct. Anal. 4 (1969), 240–267.

[B1] M. Beals, C. Fefferman, and R. Grossman, "Strictly pseudoconvex domains in $\mathbb{C}^n$," Bull. AMS 8 (1983), 125–322.

[B2] R. Beals, "A general calculus of pseudodifferential operators," Duke Math. J. 42 (1979), 1–42.

[B3] R. Beals, "Spatially inhomogeneous pseudodifferential operators II," Comm. Pure Appl. Math. 27 (1974), 161–205.

[B4] R. Beals, "Weighted distribution spaces and pseudodifferential operators," J. d'Anal. Math. 39 (1981), 131–187.

[B5] R. Beals and C. Fefferman, "Spatially inhomogeneous pseudodifferential operators," Comm. Pure Appl. Math. 27 (1974), 1–24.

[B6] R. Beals and P. Greiner, "Pseudodifferential operators associated to hyperplane bundles," Rend. Sem. Mat. di Torino (1983), 7–40.

[B7] L. Boutet de Monvel, "Hypoelliptic operators with double characteristics and related pseudodifferential operators," Comm. Pure Appl. Math. 27 (1974), 585–639.

[B8] L. Boutet de Monvel, "On the index of Toeplitz operators in several complex variables," Invent. Math. 50 (1979), 249–272.

[B9] L. Boutet de Monvel, "Operateurs a coefficients polynomiaux, espace de Bargman et operateurs de Toeplitz," Sem. Goulaouic-Meyer-Schwartz 1980, Expose no. II bis.

[B10] L. Boutet de Monvel, A. Grigis, and B. Helffer, "Parametrices d'operateurs pseudodifferentiels a caracteristiques multiple," Asterisque 34–35 (1976), 93–121.

[B11] L. Boutet de Monvel and V. Guillemin, *The Spectral Theory of Toeplitz Operators*, Princeton Univ. Press, Princeton, N.J., 1981.

[B12] L. Boutet de Monvel and J. Sjöstrand, "Sur la singularité des noyaux de Bergman et de Szegö," Asterisque 34–36 (1976), 123–164.

[B13] L. Boutet de Monvel and F. Treves, "On a class of pseudodifferential operators with double characteristics," Comm. Pure Appl. Math. 27 (1974), 59–89.

[C1] A. Calderon and R. Vaillancourt, "A class of bounded pseudodifferential operators," Proc. NAS USA 69 (1972), 1185–1187.

[C2] J. Chazarin, "Spectre d'un hamiltonien quantique et mecanique classique," Comm. PDE 5 (1980), 595–644.

[C3] M. Christ and D. Geller, "Singular integral characterization of Hardy spaces on homogeneous groups," Preprint, 1983.

[C4] R. Coifman and G. Weiss, *Analyze Harmonique Non-commutative sur Certains Espaces Homogenes*, LNM #242, Springer-Verlag, New York, 1971.

[C5] Y. Colin de Verdiere, "Spectre conjoint d'operateurs qui commutent," Duke Math. J. 46 (1979), 169–182.

[C6] H. Cordes, "On compactness of commutators of multiplications and convolutions, and boundedness of pseudo-differential operators," J. Funct. Anal. 18 (1975), 115–131.

[C7] L. Corwin, "Tempered distributions on Heisenberg groups whose convolution with Schwartz class functions is Schwartz class," J. Funct. Anal. 44 (1981), 328–347.

[C8] L. Corwin and L. Rothschild, "Necessary conditions for local solvability of homogeneous left invariant differential operators on nilpotent Lie groups," Acta Math. 147 (1981), 265–288.

[D1] J. Duistermaat and V. Guillemin, "The spectrum of positive elliptic operators and periodic bicharacteristics," Invent. Math. 29 (1975), 39–79.

[D2] A. Dynin, "An algebra of pseudodifferential operators on the Heisenberg group; symbolic calculus," Soviet Math. Dokl. 17 (1976), 508–512.

[D3] A. Dynin, "Pseudodifferential operators on Heisenberg groups," Sem. CIME, Bressanone, Italy, June 1977.

[E1] Yu. V. Egorov, "On canonical transformations of pseudo-differential operators," Uspehi Mat. Nauk. 24 (1969), 235–236.

[F1] C. Fefferman and D. Phong, "On positivity of pseudo-differential operators," Proc. NAS USA 75 (1978), 4673–4674.

[F2] C. Fefferman and D. Phong, "The uncertainty principle and sharp Gårding inequailites," Comm. Pure Appl. Math. 34 (1981), 285–331.

[F3] C. Fefferman and D. Phong, "On the eigenvalue distribution of a pseudodifferential operator," Proc. NAS USA 77 (1980), 5622–5625.

[F4] G. Folland and E. Stein, "Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group," Comm. Pure Appl. Math. 17 (1974), 429–522.

[F5] G. Folland and E. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, Princeton, N.J., 1982.

[G1] B. Gaveau, "Principe de moindre action, propagation de chaleur, et estimees sous elliptiques sur certains groups nilpotents," Acta Math. 139 (1977), 95–153.

[G2] D. Geller, "Local solvability and homogeneous distributions on the Heisenberg group," Comm. PDE 5 (1980), 475–560.

[G3] D. Geller, "Fourier analysis on the Heisenberg group, I. Schwartz space," J. Funct. Anal. 36 (1980), 205–254.

[G4] D. Geller, Ph. D. Thesis, Princeton Univ., 1977.

[G5] D. Geller, "Spherical harmonics, the Weyl transform, and the Fourier transform on the Heisenberg group," Preprint, 1983.

[G6] D. Geller and E. Stein, "Estimates for singular convolution operators on the Heisenberg group," Preprint, 1983.

[G7] R. Goodman, *Nilpotent Lie Groups, Structure and Applications to Analysis*, LNM #562, Springer-Verlag, New York, 1976.

[G8] P. Greiner, "On the Laguerre calculus of left-invariant convolution (pseudodifferential) operators on the Heisenberg group," Sem. Goulaouic-Meyer-Schwartz, 1980–81, expose no. XI.

[G9] P. Greiner and E. Stein, *Estimates for the $\overline{\partial}$-Neumann problem*, Math. Notes #19, Princeton Univ. Press, Princeton, N.J., 1977.

[G10] A. Grigis, "Hypoellipticite pour une class d'operateurs pseudodifferentiels a caracteristiques doubles et parametrix associes," C.R. Acad. Sci. Paris.

[G11] A. Grossman, G. Loupias, and E. Stein, "An algebra of pseudodifferential operators and quantum mechanics in phase space," Ann. Inst. Fourier 18 (1969), 343–368.

[G12] V. Grusin, "Pseudo-differential operators on $\mathbb{R}^n$ with bounded symbols," Funct. Anal. and Appl. (1970).

[G13] V. Grusin, "On a class of hypoelliptic operators," Math. USSR Sb. 12 (1970), 458–476.

[G14] V. Guillemin and S. Sternberg, *Geometric Asymptotics*, AMS, Providence, R.I., 1977.

[G15] V. Guillemin and S. Sternberg, "The metaplectic representation, Weyl operators, and spectral theory," J. Funct. Anal. 42 (1981), 128–225.

[H1] B. Helffer, "Sur une classe d'operateurs hypoelliptiques a caracteristiques multiple," J. Math. Pure Appl. 55 (1976), 207–215.

[H2] B. Helffer, *Theorie Spectrale Pour des Operateurs Globalement Elliptiques*, Notas de Curso, Universidade Federal de Pernambuco, 1981.

[H3] B. Helffer and J. Nourrigat, "Caracterization des operateurs hypoelliptiques homogenes invariant a gauche sur un groupe de Lie nilpotent gradue," Comm. PDE 4 (1979), 899–958.

[H4] B. Helffer and D. Robert, "Proprietes asymptotiques du spectre d'operateurs pseudodifferentiels sur $\mathbb{R}^n$," Comm. PDE.

[H5] L. Hörmander, "Pseudodifferential operators and non-elliptic boundary problems," Ann. of Math. 83 (1966), 129–209.

[H6] L. Hörmander, "Pseudodifferential operators and hypoelliptic equations," Proc. Symp. Pure Math. X (Singular Integrals), pp. 138–183, AMS, Providence, R.I., 1967.

[H7] L. Hörmander, "Fourier integral operators I," Acta Math. 127 (1971), 79–183.

[H8] L. Hörmander, "A class of pseudodifferential operators with double characteristics," Math. Annalen 217 (1975), 165–188.

[H9] L. Hörmander, "The Cauchy problem for differential equations with double characteristics," J. Anal. Math. 32 (1977), 118–196.

[H10] L. Hörmander, "The Weyl calculus of pseudodifferential operators," Comm. Pure Appl. Math. 32 (1979), 355–443.

[H11] R. Howe, "Quantum mechanics and partial differential equations," J. Funct. Anal. 38 (1980), 188–254.

[H12] R. Howe, "On the role of the Heisenberg group in harmonic analysis," Bull. AMS 3 (1980), 821–843.

[H13] R. Howe, "A symbolic calculus for nilpotent groups," Preprint.

[H14] A. Hulanicki, "The distribution of energy in the Brownian motion in the Gaussian field and analytic-hypoellipticity of certain subelliptic operators on the Heisenberg group," Studia Math. 51 (1976), 165–173.

[I1] V. Ivrii and V. Petkov, "Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed," Uspehi Mat. Nauk. 29 (1974), 5.

[I2] C. Iwasaki and N. Iwasaki, "Parametrix for a degenerate parabolic equation and its application to the asymptotic behavior of spectral functions for stationary problems," Publ. RIMS, Kyoto Univ. 17 (1981), 577–655.

[K1] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1980.

[K2] T. Kato, "Boundedness of some pseudo-differential operators," Osaka J. Math. 13 (1976), 1–9.

[K3] N. Kerzman and E. Stein, "The Szegö kernel in terms of Cauchy-Fantappie kernels," Duke Math. J. 45 (1978), 197–224.

[K4] J. Kohn, "Harmonic integrals on strongly pseudoconvex manifolds," I and II, Ann. of Math. 78 (1963), 112–147; 79 (1964), 450–472.

[K5] J. Kohn, "Pseudo-differential operators and hypoellipticity," Proc. Symp. Pure Math. 23, pp. 61–69, AMS, Providence, R.I., 1973.

[K6] J. Kohn, "Microlocalization of CR structures," Abstracts AMS 4 (Jan. 1983), p. 107.

[K7] J. Kohn and L. Nirenberg, "An algebra of pseudo-differential operators," Comm. Pure Appl. Math. 18 (1965), 269–305.

[K8] A. Koranyi and S. Vagi, "Singular integrals in homogeneous spaces and some problems of classical analysis," Ann. Scuola Norm. Sup. Pisa 25 (1971), 575–648.

[K9] S. Krantz, *Function Theory of Several Complex Variables*, Wiley-Interscience, New York, 1982.

[K10] H. Kumano-go, *Pseudo-differential Operators*, MIT Press, Cambridge, Mass., 1981.

[L1] B. Lascar, "Propagation des singularities pour les equations hyperboliques a caracteristiques de multiplicite au plus double et singularities Masloviennes," Amer. J. Math. 104 (1982), 227–285.

[L2] B. Lascar and R. Lascar, "Propagation des singularities pour des equations hyperboliques a caracteristiques de multiplicite au plus double et singularities Masloviennes II," Jour. d'Anal. Math. 41 (1982), 1–38.

[L3] P. Lax and L. Nirenberg, "On stability for difference schemes; a sharp form of Garding's inequality," Comm. Pure Appl. Math. 19 (1966), 473–492.

[L4] N. Lebedev, *Special Functions and Applications*, Dover, New York, 1972.

[M1] J. Mather, "Differential invariants," Topology 16 (1977), 145–155.

[M2] G. Mauceri, "The Weyl transform and bounded operators on $L^p(\mathbb{R}^n)$, J. Funct. Anal. 39 (1980), 408–429.

[M3] A. Melin, "Parametrix constructions for right invariant differential operators on nilpotent groups," Comm. PDE 6 (1981), 1363–1405.

[M4] A. Melin, "Lie filtrations and pseudo-differential operators," Preprint, 1982.

[M5] A. Melin and J. Sjöstrand, "Fourier integral operators with complex phase functions," LNM #459, pp. 120–223, Springer-Verlag, New York, 1975.

[M6] R. Melrose, "Hypoelliptic operators with characteristic variety of codimension two and the wave equation," Sem. Goulaouic-Schwartz 1979–80, Expose XI.

[M7] R. Melrose, "The wave equation for a hypoelliptic operator with symplectic characteristics of codimension two," Unpublished manuscript, 1980.

[M8] A. Menikoff and J. Sjöstrand, "On the eigenvalues of a class of hypoelliptic operators," Math. Ann. 235 (1978), 55–85.

[M9] G. Metivier, "Fonction spectrale et valeurs propres d'une classe d'operateurs non elliptiques," Comm. PDE 1 (1976), 467–519.

[M10] K. Miller, "Hypoellipticity on the Heisenberg group," J. Funct. Anal. 31 (1979), 306–320.

[M11] K. Miller, "Parametrices for hypoelliptic operators on step two nilpotent Lie groups," Comm. PDE 5 (1980), 1153–1184.

[M12] K. Miller, "Invariant pseudodifferential operators on two step nilpotent Lie groups," Michigan Math. J. 29 (1982), 315–328.

[N1] A. Nachman, "The wave equation on the Heisenberg group," Ph.D. Thesis, Princeton Univ., 1980.

[N2] M. Nagase, "The L^p boundedness of pseudodifferential operators with non-regular symbols," Comm. PDE 2 (1977), 1045–1061.

[N3] A. Nagel and E. Stein, *Lectures on Pseudo-differential Operators: Regularity Theorems and Applications to Non-elliptic Problems*, Princeton Univ. Press, Princeton, N.J., 1980.

[N4] E. Nelson, "Operants, a functional calculus for non-commuting operators," pp. 172–187 in *Functional Analysis and Related Topics*, Springer-Verlag, New York, 1970.

[N5] L. Nirenberg, "On a problem of Hans Lewy," Uspehi Mat. Nauk. 292 (1974), 241–251.

[P1] J. Peetre, "The Weyl transform and Laguerre polynomials," La Mathematische, Univ. de Catania Sem. 27, 301–323.

[P2] D. Phong, "On integral representations for the Neumann operator," Proc. NAS USA 76 (1979), 1554–1558.

[P3] D. Phong and E. Stein, "Some further classes of pseudo-differential and singular-integral operators arising in boundary-value problems I. Composition of operators," Amer. J. Math. 104 (1982), 141–172.

[P4] D. Phong and E. Stein, "Estimates for the Bergman and Szegö projections on strongly pseudoconvex domains," Duke Math. J. 44 (1977), 695–704.

[R1] C. Rockland, *Hypoellipticity and Eigenvalue Asymptotics*, LNM #464, Springer-Verlag, New York, 1975.

[R2] C. Rockland, "Hypoellipticity on the Heisenberg group: representation-theoretic criteria," Trans. AMS 240 (1978), 1–52.

[R3] L. Rothschild, "Local solvability of second order differential operators on nilpotent Lie groups," Preprint.

[R4] L. Rothschild, " C^∞ regularity for solutions of invariant differential equations on the Heisenberg group," Preprint, 1982.

[R5] L. Rothschild and E. Stein, "Hypoelliptic differential operators and nilpotent groups," Acta Math. 137 (1976), 247–320.

[R6] L. Rothschild and D. Tartakoff, "Inversion of analytic matrices and local solvability of some invariant differential operators on nilpotent Lie groups," Comm. PDE 6 (1981), 625–650.

[R7] W. Rudin, *Function Theory on the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[S1] M. Sato, T. Kawai, and M. Kashiwara, "Microfunctions and pseudo-differential equations," pp. 265–529 in *Hyperfunctions and Pseudo-Differential Equations*, LNM #287, Springer-Verlag, New York, 1973.

[S2] A. Schwartz, "Smooth functions invariant under the action of a compact group," Topology 14 (1975), 63–68.

[S3] R. Seeley, "Complex powers of an elliptic operator," Proc. Symp. Pure Math. X, pp. 288–307, AMS, Providence, R.I., 1967.

[S4] I. Segal, "Transforms for operators and symplectic automorphisms over a locally compact abelian group," Math. Scand. 13 (1963), 31–43.

[S5] M. Shubin, *Pseudodifferential Operators and Spectral Theory*, Moscow, 1978 (In Russian).

[S6] J. Sjöstrand, "Parametrices for pseudodifferential operators with multiple characteristics," Ark. för Math. 12 (1974), 85–130.

[S7] E. Stein, "An example on the Heisenberg group related to the Lewy operator," Invent. Math. 69 (1982), 209–216.

[S8] E. Stein, *Lectures on Pseudodifferential Operators*, Course Notes, Princeton Univ., 1977.

[S9] R. Strichartz, "A functional calculus for elliptic pseudodifferential operators," Amer. J. Math. 94 (1972), 711–722.

[S10] R. Strichartz, "Invariant pseudo-differential operators on a Lie group," Ann. Scuola Norm. Sup. Pisa 26 (1972), 587–611.

[T1] D. Tartakoff, "Local analytic hypoellipticity of $\Box_b$ on nondegenerate Cauchy Riemann manifolds," Proc. NAS USA 75 (1978), 3027–3028.

[T2] M. Taylor, *Pseudodifferential Operators*, Princeton Univ. Press, Princeton, N.J., 1981.

[T3] M. Taylor, "Fourier integral operators and harmonic analysis on compact manifolds," Proc. Symp. Pure Math., Vol. 35, Part 2, pp. 115–136, AMS, Providence, R.I., 1979.

[T4] M. Taylor, "Pseudodifferential operators on contact manifolds, preliminary notes," Graduate lecture notes, Stony Brook, Spring 1981.

[T5] M. Taylor, *Noncommutative Harmonic Analysis*, Math. Surveys and Monogr., #22, AMS, Providence, R.I., 1986.

[T6] F. Treves, "Analytic hypoellipticity for a class of pseudodifferential operators," Comm. PDE 3 (1978), 475–642.

[T7] F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators*, Vols. 1–2, Plenum, New York, 1980.

[U1] A. Unterberger, "Oscillator harmonique et operateurs pseudodifferentiels," Ann. Inst. Fourier, Grenoble 29 (1979), 201–221.

[U2] A. Unterberger, "Les operateurs metadifferentiels," in *Complex Analysis, Microlocal Calculus, and Relativistic Quantum Theory*, pp. 205–241, Lecture Notes in Physics #126, Springer-Verlag, New York, 1980.

[V1] V. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

[V2] M. Venugopalkrishna, "Fredholm operators associated with strongly pseudoconvex domains," J. Funct. Anal. 9 (1972), 349–373.

[V3] A. Voros, "An algebra of pseudodifferential operators and asymptotics of quantum mechanics," J. Funct. Anal. 29 (1978), 104–132.

[W1] N. Wallach, *Symplectic Geometry and Fourier Analysis*, Math. Sci. Press, Brookline, Mass., 1977.

[W2] G. Warner, *Harmonic Analysis on Semisimple Lie Groups*, Springer-Verlag, New York, 1972.

[W3] A. Weil, "Sur certaines groupes d'operateurs unitaires," Acta Math. 111 (1964), 143–211.

[W4] H. Weyl, *Group Theory and Quantum Mechanics*, Dover, New York.

More Recent References

[[BG]] R. Beals and P. Greiner, *Calculus on Heisenberg Manifolds*, Princeton Univ. Press, Princeton, N.J., 1988.

[[BGS]] R. Beals, P. Greiner, and N. Stanton, "The heat equation on a CR manifold," J. Diff. Geom. 20 (1984), 343–387.

[[BS]] R. Beals and N. Stanton, "Estimates on kernels for the $\overline{\partial}$-equation and the $\overline{\partial}$-Neumann problem," Math. Ann. 289 (1991), 73–83.

[[Ca]] D. Catlin, "Subelliptic estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains," Ann. of Math. 105 (1987), 131–191.

[[CNS]] D. Chang, A. Nagel, and E. Stein, "Estimates for the $\overline{\partial}$-Neumann problem on pseudoconvex domains of finite type in $\mathbb{C}^2$," Acta Math. 169 (1992), 153–228.

[[Chr]] M. Christ, "Regularity properties of the $\overline{\partial}_b$-equation on weakly pseudoconvex CR manifolds of dimension three," JAMS 1 (1988), 587–646.

[[EMM]] C. Epstein, R. Melrose, and G. Mendoza, "Resolvent of the Laplacian on strictly pseudoconvex domains," Acta Math. 167 (1991), 1–106.

[[FeK]] C. Fefferman and J.J. Kohn, "Hölder estimates on domains of complex dimension 2 and on 3 dimensional CR manifolds," Advances in Math. 69 (1988), 223–303.

[[Gel]] D. Geller, *Analytic Pseudodifferential Operators on the Heisenberg Group and Local Solvability*, Princeton Univ. Press, Princeton, N.J., 1990.

[[NW]] A. Nagel, J. Rosay, E. Stein, and S. Wainger, "Estimates for the Bergman and Szegö kernels in $\mathbb{C}^2$," Ann. of Math. 129 (1989), 113–149.

[[PS]] D. Phong and E. Stein, "Hilbert integrals, singular integrals, and Radon transforms, II," Invent. Math. 86 (1986), 75–113.

[[Sm]] H. Smith, "A calculus for three-dimensional CR manifolds of finite type," J. Funct. Anal. 120 (1994), 135–162.

[[ST]] N. Stanton and D. Tartakoff, "The heat equation for the $\overline{\partial}_b$-Laplacian," Comm. PDE 9 (1984), 597–686.

[[T]] M. Taylor, *Partial Differential Equations, Vols. 1–3*, Springer-Verlag, New York, 1996.