DYNAMICS AND BIFURCATIONS OF A FAMILY OF RATIONAL MAPS WITH PARABOLIC FIXED POINTS

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ABSTRACT. We study a family of rational maps of the sphere with the property that each map has two fixed points with multiplier -1; moreover each map has no period 2 orbits. The family we analyze is $R_a(z) = \frac{z^3-z}{-z^2+az+1}$, where a varies over all nonzero complex numbers. We discuss many dynamical properties of R_a including bifurcations of critical orbit behavior as a varies, connectivity of the Julia set $J(R_a)$, and we give estimates on the Hausdorff dimension of $J(R_a)$.

1. INTRODUCTION

There is great interest in rational maps of the Riemann sphere of degree ≥ 2, with periodic orbits whose multiplier is a root of unity [1, 2, 4, 5, 7, 8, 15, 19, 21]. By this we mean that letting $\mathbb{C}_\infty$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$, suppose $R : \mathbb{C}_\infty \to \mathbb{C}_\infty$, and for some $z_0 \in \mathbb{C}_\infty$, $k \in \mathbb{N}$, and $q \in \mathbb{N}$, we have: $R^k(z_0) = z_0$ and $\lambda = R'(z_0) \cdot R'(Rz_0) \cdots R'(R^{k-1}z_0) = (R^k)'(z_0)$, satisfies $\lambda^q = 1$. We call the orbit of such a point a *parabolic cycle*. Each parabolic cycle attracts at least one critical point, and therefore the closure of the critical orbit contains points from the Julia set [3]. If all other critical points stay a bounded distance away from the Julia set under iteration by R, then it has been shown that R is expansive and shares many properties with the well-studied hyperbolic maps [7, 15, 21]. However in the presence of other types of Fatou components, which arise when there are other critical points, the expansiveness gives way to interesting bifurcations. In this paper we study a holomorphic family of maps with the property that every map in the family has two rationally neutral fixed points, and show that as the parameter varies over $\mathbb{C}^* \equiv \mathbb{C} \setminus \{0\}$, many dynamical bifurcations occur.

One of the unique aspects of this family of mappings (of degree 3) is that each map has at least two parabolic fixed points, one at 0 and one at ∞, and no period two orbits at all except for the fixed points. This eliminates the period 2 limb of the analog of the Mandelbrot set when looking at the parameter space; though with one free critical point shown to occur, all other quadratic bifurcations, plus some new ones, arise in this family. For example in certain regions of parameter space, the free critical point tends to 0 under iteration, in others it iterates to ∞, and then there are bifurcations that mark the transition from one to another. Clearly if there

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is an attracting cycle, the free critical point must iterate to that cycle; we show this can occur.

Our starting point is a family of rational maps of the sphere with no period 2 orbits, one of several such families that illustrates a theorem of Baker [2]. He proved that if there is a rational map of degree d with no periodic orbit of period k, then the only (d, k) pairs that can occur are: $(2, 2)$, $(2, 3)$, $(3, 2)$, or $(4, 2)$. That is, rational maps with no period 2 orbits occur only for rational maps of degrees 2, 3, or 4, and maps with no period 3 orbits must be of degree 2. The theorem was refined by a complete classification by Kisaka [14], and a proof of the classification with additional analysis appears in the Ph.D. thesis of the first author, written under the supervision of the second author [9, 10]. In the degree 3 case there are three parametrized families of maps, each having no 2-cycles, that can occur; but in this paper we focus only on one of these; many of the results here can be carried over to the other families.

A common property of all rational maps of missing period 2 orbits, in all degrees ≥ 2, is that they have one or more parabolic fixed points; the period 2 cycles only appear as fixed points of higher multiplicity. In the family of maps studied here, there is always a fixed point at 0 and another at ∞, and each of these fixed points has derivative -1, where we use local coordinates to define the derivative at ∞ (see the discussion below).

The outline of the paper is as follows. In Section 1.1 we give the basic definitions for the setting of iterated rational maps; Section 2 begins the discussion of the specific family of interest. We start with the algebraic properties and then turn to the dynamical properties of the maps in Section 3. We show that the Julia set is connected for many maps in the family, in particular for maps with the property that each Fatou component contains at most one critical value. In Section 4 we give a brief discussion of measure theoretic properties and the connection to Hausdorff dimension of the Julia set. We end with a few numerical estimates on the Hausdorff dimension of the Julia sets in the family.

1.1. Definitions and background material.

We let $\mathbb{C}_\infty$ denote the Riemann sphere.

Example 1.1. Main Example. Throughout this paper we study the family of rational maps $R_a : \mathbb{C}_\infty \to \mathbb{C}_\infty$, defined for each $a \in \mathbb{C}^*$:

$$R_a(z) = \frac{z^3 - z}{-z^2 + az + 1}. \tag{1}$$

Since R_a is of degree $d = 3$, it has $d + 1 = 4$ fixed points and $2d - 2 = 4$ critical points in $\mathbb{C}_\infty$, counting multiplicity. Because this family arises naturally as one of the families of degree 3 rational maps with no period 2 orbits that are not fixed points, a useful tool is the rational map $R_a^2 \equiv R_a \circ R_a$, which has degree 9. Hence R_a^2 has 10 fixed points and 16 critical points, counting multiplicity.

For a general rational map R of degree $d \geq 2$, we let R^n denote the n-fold composition of R with itself. The *Fatou set*, $F(R)$, is the maximal open set in $\mathbb{C}_\infty$ on which the family $\{R^n\}$ is normal; the *Julia set*, $J(R)$, is its complement. It is well-known that for any $k \in \mathbb{N}$, $J(R^k) = J(R)$ (see, e.g., [3], [6], [17]).

A point $z_0 \in \mathbb{C}_\infty$ is a *periodic point* of R of period k if $R^k(z_0) = z_0$ and $k \in \mathbb{N}$ is minimal. When $k = 1$, we call z_0 a *fixed point.* A periodic point z_0 of period k forms part of a cycle of length k, namely $\{z_0, R(z_0), \ldots, R^{k-1}(z_0)\}$, and each element of the cycle is fixed under R^k.

For each fixed point of R, we define the *multiplier* of z_0 by $R'(z_0)$ if $z_0 \in \mathbb{C}$; if $z_0 = \infty$, we define it to be $\lim_{z\to 0} 1/R'(1/z)$. Using a slight abuse of notation, we denote the multiplier by $R'(z_0)$ for all $z_0 \in \mathbb{C}_\infty$.

Periodic points are classified according to their multipliers as follows.

Definition 1.2. Assume R is a rational map of degree $d \geq 2$ and z_0 is a periodic point of period $k \geq 1$. Then:

(i) z_0 is *attracting (superattracting)* if $|(R^k)'(z_0)| < 1(= 0)$;
(ii) z_0 is *repelling* if $|(R^k)'(z_0)| > 1$;
(iii) z_0 is *neutral* if $|(R^k)'(z_0)| = 1$;
  (a) a neutral periodic point is *rationally neutral* or *parabolic* if $(R^k)'(z_0)$ is an mth root of unity.
  (b) a neutral periodic point is *irrationally neutral* if it is not parabolic.

There are some well-known facts connecting periodic points and Julia and Fatou sets and we summarize a few here. Detailed expositions, along with proofs of the following result, can be found for example in books by Beardon [3], Carleson and Gamelin [6], and Milnor [17].

Theorem 1.3. Assume R is a rational map of degree $d \geq 2$ and U is a connected component of the Fatou set $F(R)$. Then some forward iterate of U is periodic, i.e., there exists $m \in \mathbb{N}$ such that $V = R^m(U)$ is periodic, and V is one of the following types:

(i) an *attracting component* and V contains an attracting (superattracting) periodic point z_0;
(ii) a *parabolic component* and ∂V contains a parabolic periodic point z_0;
(iii) if V contains an irrationally neutral periodic point z_0, then V is simply connected and called a *Siegel disk*;
(iv) otherwise V is doubly connected and called a *Herman ring.*

In the first two cases, if $R^k(z_0) = z_0$, then $(R^k)^n \to z_0$ locally uniformly on V as $n \to \infty$. In the last two cases, if $R^k : V \to V$, then R^k is conformally conjugate on V to an irrational rotation, either of a disk (Siegel disk case) or of an annulus (Herman ring case).

If z_0 is an attracting or superattracting periodic point, then $z_0 \in F(R)$ and the *immediate basin of attraction* is the connected component of $F(R)$ containing z_0. The *basin of attraction* is the open set $B \subset F(R)$ consisting of all points $z \in \mathbb{C}_\infty$ such that $R^n(z)$ converges to the cycle containing z_0.

1.1.1. *Parabolic periodic points.* For a parabolic periodic point z_0 of R we have analogous but slightly more complicated definitions for the attracting and immediate attracting basin, and we refer to expositions for detailed explanations (see, e.g., [3], [6], [17]).

Suppose that a simple parabolic fixed point z_0 of R has multiplicity $mp+1$ $(p > 0)$ as a fixed point of R^m, where m is the smallest positive integer such that $(R'(z_0))^m = 1$. Then z_0 as a fixed point of R has p distinct immediate basins of attraction, each of which consists of m disjoint Fatou components forming a period m cycle. We call each component of the immediate basin of attraction a *Leau domain* $L_1, L_2, \ldots, L_{mp}$. As above, the attracting basin of z_0 is the open set of points in $\mathbb{C}_\infty$ converging to the fixed point. Each Leau domain in turn contains an *attracting petal* $P_j \subset L_j$, $j = 1, \ldots, mp$, such that each P_j is a domain on which the map R^m is conformally conjuagate to the translation: $\zeta \mapsto \zeta + 1$ on the right half-plane in $\mathbb{C}$. We have that $R^n|_{L_j} \to z_0 \in \partial P_j$ for each j and the convergence is locally uniform.

If z_0 is periodic with period greater than 1, in an analogous way we obtain a cycle of Leau domains. The following result was shown by Fatou.

Theorem 1.4. Every cycle of Leau domains contains a critical point of R.

The next result gives the rays of symmetry of the attracting petals of a rational map with a fixed point z_0 such that $R'(z_0) = 1$ (see, e.g., [17], Lemma 10.1 and its proof). For the statement of the next proposition we assume $z_0 = 0$ and $m = 1$ for simplicity.

Theorem 1.5. Suppose $R(z) = z(1 + \alpha z^p + (\text{higher order terms}))$ for some $\alpha \neq 0$, $p \in \mathbb{N}$, near the origin. Then there exist p attracting petals, or equivalently p evenly spaced attracting directions at the origin and any point that approaches the rationally neutral fixed point at 0 must approach it in one of these p directions. The rays v_j, $j = 1, \ldots, p$, that give the attracting directions are determined by the solutions to the equation: $p\alpha v^p = -1$ or equivalently, $(-1/(p\alpha))^{1/p} = v$. Finally, there are also p repelling petals which are defined to be the attracting petals for R^{-1} near a neutral fixed point, and the repelling directions are obtained by rotating the attracting directions by $\dfrac{\pi}{p}$.

Example 1.6. Let $R_a(z) = \frac{z^3-z}{-z^2+az+1}$ be the family in (1); we need to apply Theorem 1.5 to R_a^2, which has fixed points at 0 and ∞ with derivative 1. The derivatives up to $R_a^{(5)}(z)$ are readily computable, so the first five terms of the Taylor series for R_a about 0 can be found; and Proposition 2.1 below gives that $(R_a^2)'(0) = 1$ and $(R_a^2)^{(j)}(0) = 0$ for $j = 2, 3, 4$. From this data the Taylor series expansion of R_a^2 about 0 can be shown to start with: $R_a^2(z) = z - 2a^2z^5 +$ (higher order terms). Using $\alpha = -2a^2$ and $p = 4$, we find that the attracting directions for the 4 petals at the origin are the 4 quartic roots of $1/(8a^2)$, which are rays spaced $\frac{\pi}{2}$ apart. We note that when $a > 0$ the attracting directions are exactly along the coordinate axes in the plane, and when $a = ib$, $b > 0$, the attracting directions form angles of $\frac{\pi}{4}$ with the coordinate axes.

In parallel, using a local change of coordinates at $z_0 = \infty$ we calculate the Taylor series of $1/R_a(1/z)$ at the origin, use the method above and Proposition 2.1 to see that near 0, $1/R_a^2(1/z) = z - 2a^2z^3 +$ (higher order terms). Here, $p = 2$ so the attracting directions for the neutral point at ∞ are the two square roots of $4a^2$, which gives for $\pm 2a$.

It is dynamically significant that when $a > 0$, the fixed points at 0 and ∞ share a common attracting direction along the real axes, while when $a = ib, b > 0$, the attracting directions for the point at ∞ are repelling directions for the point at 0. This causes qualitative differences in the Julia sets and bifurcations in parameter space.

Applying Theorem 1.4, for each map R_a, three critical points are used up by Leau domains of the parabolic points. This property helps to make the analysis of the dynamics accessible.

2. Algebraic properties of $R_a(z) = \dfrac{z^3 - z}{-z^2 + az + 1}$

2.1. **Fixed points and multipliers.** In this section we summarize algebraic properties of the map $R_a(z) = (z^3 - z)/(-z^2 + az + 1)$. We calculate the fixed points of R_a and R_a^2 to show that R_a has no period 2 orbits. We find the multipliers of the fixed points of R_a and determine the number of immediate basin of attraction at each parabolic fixed point of R_a. The next proposition follows from a simple calculation.

Proposition 2.1. For each parameter $a \in \mathbb{C} \setminus \{0\}$, the map R_a has the following properties.

(i) R_a has fixed points 0, ∞, and $(a \pm \sqrt{a^2 + 16})/4$.
(ii) R_a^2 has fixed points 0 with multiplicity 5, ∞ with multiplicity 3, and $(a \pm \sqrt{a^2 + 16})/4$ with multiplicity 1.

Corollary 2.2. ([14], cf. also [9]) For each parameter $a \in \mathbb{C} \setminus \{0\}$, the map R_a has no period 2 cycles.

Proof. The map R_a^2 has 10 fixed points, and by Proposition 2.1 they have all been accounted for by the fixed points of R_a. □

A calculation of the multipliers of the fixed points of R_a and Example 1.6 give the following.

Proposition 2.3. The fixed points of R_a have the following properties.

(i) The multipliers of 0, ∞, and $(a \pm \sqrt{a^2+16})/4$ as fixed points of R_a are -1, -1, and $1 \pm 2\sqrt{a^2+16}/a$, respectively. Hence for any parameter value $a \in \mathbb{C} \setminus \{0\}$, the fixed points 0 and ∞ are parabolic fixed points.

(ii) The number of immediate basin of attraction of 0 and ∞ as parabolic fixed points of R_a are 2 and 1, respectively.

Remarks about some special values of a. By Proposition 3.1 below, each statement we make for a has an analogous one for $-a$.

(i) The fixed points $(a \pm \sqrt{a^2+16})/4$ coincide with each other at $a = 4i$ to form a parabolic fixed point i with multiplier 1. There is one immediate basin of i, consisting of one Leau domain.

(ii) Viewing the map as an analytic map of $\mathbb{C}_\infty$, the fixed point at ∞ has two other preimages, namely $p_1 = \frac{1}{2}(a + \sqrt{a^2+4})$ and $p_2 = \frac{1}{2}(a - \sqrt{a^2+4})$. The preimage is a double pole precisely when $a = 2i$, and the pole is at the critical point i.

(iii) When $a = 8i/\sqrt{3}$, a fixed critical point occurs at $c = \frac{i}{\sqrt{3}}$. This parameter value forms the center of the cardioid shape in Figure 1.

2.2. **Critical points.** We calculate that

$$R_a'(z) = \frac{-(z^4 - 2az^3 - 2z^2 + 1)}{(1 + az - z^2)^2}, \tag{2}$$

and the zeros of this have no "nice" form. However, using the fact that for complex numbers c_j, not necessarily distinct,

$$\prod_{j=1}^{4}(z - c_j) = z^4 - (\sum_{j=1}^{4} c_j)z^3 + (\sum_{j \neq k} c_j c_k)z^2 - (\sum_{j \neq k \neq l} c_j c_k c_l)z + c_1c_2c_3c_4, \tag{3}$$

we can make a few statements about the critical points, using simple algebra and comparing like terms in equations (2) and (3).

Lemma 2.4. Under the assumption that there is no common factor of the numerator and denominator in equation (2):

- The critical points of R_a sum to $2a$ and their product is 1.
- There cannot be a parameter with a multiplicity 4 critical point.
- There cannot be a parameter with a multiplicity 3 critical point.
- There cannot be a parameter with exactly two distinct critical points, each of multiplicity 2.
- If $a > 0$, then there are two real critical points.
- If $a = bi$, $b > 0$, then a variety of possibilities arise, some mentioned below.

Specific values of a and critical points.

(i) One can check by hand using equations (2) and (3) that it can occur that there is a critical point of multiplicity 2, and two other

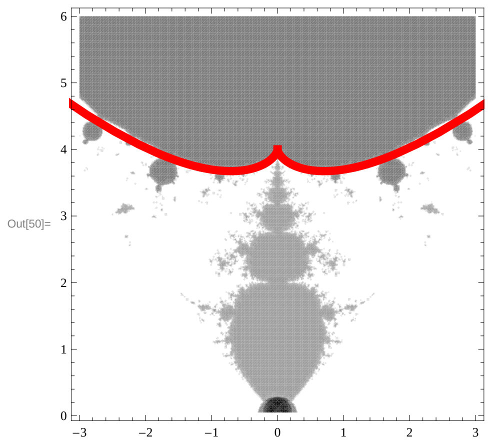

FIGURE 1. A point $a \in \mathbb{C}^*$ is colored according to the corresponding property of R_a^n as $n \to \infty$: dark grey if R_a has an attracting cycle, light grey if 2 critical points iterate to 0 and 2 iterate to ∞, and white if 3 critical points tend to 0 and one to ∞. For a above the red curve, R_a has an attracting fixed point.

distinct critical points. The value $a = 8i/(3\sqrt{3}) \approx 1.5396i$ has a double critical point at $z = \sqrt{3}i$.

(ii) When $a = 2i$, the denominator of $R_a(z)$ is a perfect square, $(z - i)^2$; yielding a double preimage of the neutral fixed point at ∞. Therefore the pole at $z = i$ is a critical point, and there are 3 other distinct critical points.

(iii) For the general case where $a = bi$, $b > 0$, one can easily establish that the coefficient conditions of equations (2) and (3) imply that there are no real critical points. However, purely imaginary critical points and critical points with nonzero real parts occur.

3. DYNAMICAL PROPERTIES OF $R_a(z) = \dfrac{z^3 - z}{-z^2 + az + 1}$

3.1. Reduction of the parameter space of R_a. It is clear that the family of maps $R_a(z)$ varies holomorphically in a and z, for $a \in \mathbb{C}^*$ and $z \in \mathbb{C}_\infty$. By definition, two maps R_a and $R_{a'}$ are *conformally conjugate* to each other, or conjugate, if and only if there exists a $g \in \mathrm{Aut}(\mathbb{C}_\infty)$ such that $g \circ R_{a'} \circ g^{-1} = R_a$ holds. Conjugation respects fixed points and preserves multipliers. We reduce the parameter space of $\{R_a\}$ under conformal conjugacy as follows. Define the equivalence relation $\sim$ on the parameter space of $\{R_a\}$ by $a \sim a'$ if and only if R_a and $R_{a'}$ are conformally conjugate to each other. We form a reduced region $\mathcal{R}$ of the parameter space of $\{R_a\}$ by taking one parameter value from each equivalence class.

Moreover, conjugation preserves the number of the immediate basins of attraction and the number of Leau domains contained in each immediate basin of a parabolic fixed point. Since the parabolic fixed point 0 of R_a has two immediate basins while ∞ has one immediate basin, we may assume that g satisfies $g(0) = 0$ and $g(\infty) = \infty$. With these observations, we see that there are only two cases to consider:

(i) $g : 0 \mapsto 0,\ \infty \mapsto \infty,\ \frac{1}{4}\big(a \pm \sqrt{a^2+16}\big) \mapsto \frac{1}{4}\big(a' \pm \sqrt{a'^2+16}\big)$;

(ii) $g : 0 \mapsto 0,\ \infty \mapsto \infty,\ \frac{1}{4}\big(a \pm \sqrt{a^2+16}\big) \mapsto \frac{1}{4}\big(a' \mp \sqrt{a'^2+16}\big)$.

From this we see that $a = \pm a'$ if and only if R_a is conformally conjugate to $R_{a'}$. Thus we have proved the following.

Proposition 3.1. The reduced parameter space $\mathcal{R}$ of $\{R_a\}$ under conformal conjugacy can be taken as follows.

$$\mathcal{R} = \{a \in \mathbb{C}^* : \mathrm{Re}(a) \geq 0,\ \mathrm{Im}(a) \geq 0\} \cup \{a \in \mathbb{C} : \mathrm{Re}(a) < 0,\ \mathrm{Im}(a) > 0\}.$$

In Figure 1 we show a picture of parameter space; due to the presence of at least two parabolic fixed points, most algorithms produce imperfect views of the space. Here we color a pixel $a = x + iy$ white if three critical points converge to 0 under iteration of R_a, light grey if 2 critical points iterate towards 0 and 2 iterate towards ∞. We color a point dark grey if there is an attracting cycle, which therefore attracts a critical point. (The dark ball around the origin has no significance.) We identify points in $\mathcal{R}$ using symmetry in the following lemma.

Lemma 3.2. For $\phi(z) = -\bar{z}$, we have $\phi \circ R_{-\bar{a}} \circ \phi^{-1} = R_a$.

This implies that although R_a and $R_{-\bar{a}}$ are not conformally conjugate to each other, their dynamics are symmetric about the imaginary axis, and thus the conformally reduced region $\mathcal{R}$ has symmetry about the imaginary axis. Therefore, it suffices to use parameters in

$$\mathcal{R}^{\mathrm{sym}} = \{a \in \mathbb{C}^* : \mathrm{Re}(a) \geq 0,\ \mathrm{Im}(a) \geq 0\}$$

to study how the dynamics of R_a depend on the parameter a.

3.1.1. *Summary of the Dynamics of R_a.* The previous results yield the following. Every map R_a has the property that 0 is a rationally neutral point, which when viewed as a fixed point of R_a^2 with multiplier 1 has multiplicity 5. Therefore $0 \in J(R_a)$, and near the origin we have four attracting petals consisting of two immediate attracting basins under R_a. (The four Leau domains containing the four attracting petals at 0 are fixed by R_a^2.) Around the point at ∞ we have two petals and one immediate attracting basin. This leaves one critical point to track, which we refer to as the *free* critical point. We show below that the free critical point can behave in one of many different ways; however the usual cascade of period 2 dynamics, which includes the entire analog of the period 2 limb from the classical Mandelbrot set are absent, but other types of bifurcations occur.

While 2-cycles are missing, period 4 attracting orbits can occur. One can easily verify that the parameter $a = -4\sqrt{\frac{-4-2i}{5}} \approx -0.869148 + 3.68177i$ has a fixed point with a multiplier which is a quartic root of unity $(-i)$. Moving away from this parameter value (in a specific direction into a period 4 bulb) gives an attracting period four orbit; the parameter $a = -.87 + 3.6i$ can be shown numerically to have an attracting period 4 orbit. In particular the entire Mandelbrot set minus its period two limb occurs, along with additional bifurcations which we discuss in this paper.

3.2. **Dynamics of R_a when $a > 0$.** Throughout this section we assume that $a > 0$ in (1). Recall from Example 1.6 that the neutral point at 0 has its four attracting directions along the coordinate axes, while the point at ∞ has just the real axis, pointing outward (from the origin) in each direction, as its attracting directions. The repelling directions for the point at ∞ are along the imaginary axis. Therefore the real axis intersects Fatou components in the basin of attraction of the origin and others in the basin of attraction of ∞. This accounts for the complicated look of the Julia set around the real axis as shown in Figure 2. Figure 3 shows a cruder global picture of the basins of attraction colored dark for 0 and light for ∞, with the 4 critical points marked as large dots.

Lemma 3.3. $R_a(\bar{z}) = \overline{R_a(z)}$.

This gives us the following dynamical symmetry, described in the next two results.

Corollary 3.4. If $a \in \mathbb{R} \setminus \{0\}$, then $J(R_a)$ and $F(R_a)$ are symmetric with respect to the real axis.

Proposition 3.5. For each $a > 0$, the map R_a has precisely two real critical points; one is in $(0, 1)$, and the other is in $(\frac{a+\sqrt{a^2+4}}{2}, \infty)$. The map R_a also has a pair of complex conjugate critical points.

Proof. Our goal is to locate where (2) vanishes; set

$$R_a'(z) = \frac{-(z^4 - 2az^3 - 2z^2 + 1)}{(1 + az - z^2)^2} = 0.$$

The poles of R_a' are the same as those of R_a, namely $p_1 = \frac{1}{2}(a + \sqrt{a^2+4})$ and $p_2 = \frac{1}{2}(a - \sqrt{a^2+4})$. There is no common factor of the numerator and denominator in equation (2) for any $a > 0$, (and it is not hard to show there is a common factor if and only if $a = \pm 2i$).

Since $p_2 < 0 < 1 < p_1$, R_a' is continuous and real-valued on $(0, 1)$. Since $R_a'(0) = -1$ and $R_a'(1) = \frac{2}{a} > 0$, there is at least one critical point of R_a in $(0, 1)$, proving the first statement.

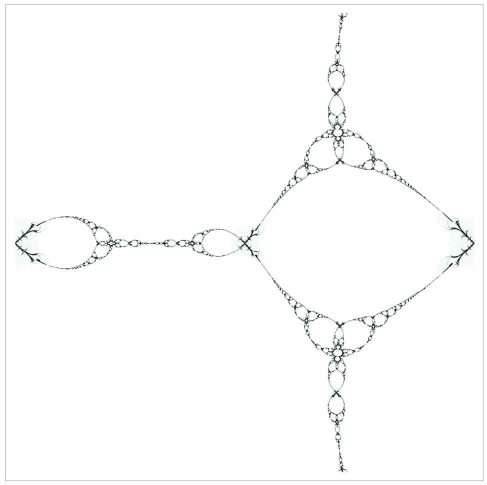

FIGURE 2. The Julia set of R_a with $a = 4$, near the origin. The origin is in the center of the figure where 4 petals meet.

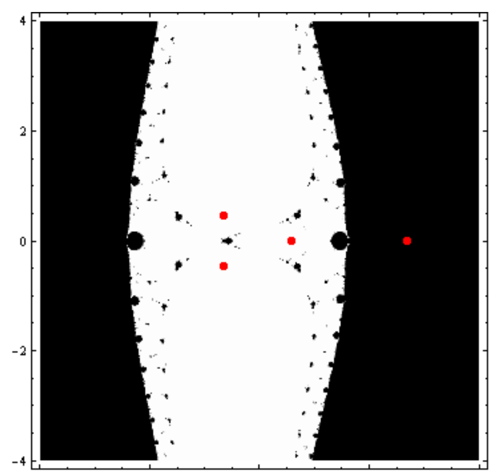

FIGURE 3. The Julia set of R_a with $a = 1$ with the attracting basin of 0 shown (white) and the attracting basin of ∞ (black), and the critical points marked (in red).

We also have the following limits: for $z \in \mathbb{R}$,

$$\lim_{z\to\pm\infty} R_a'(z) = -1, \quad \lim_{z\to p_1} R_a'(z) = +\infty, \quad \lim_{z\to p_2} R_a'(z) = -\infty. \tag{4}$$

From (4) we see that there is another zero of R_a' in (p_1, ∞), proving the second statement.

Finally we claim there are no other real critical points of R_a. Since all coefficients in R_a' are real, there must be a pair of complex conjugate (non-real) critical points of R_a.

We now prove the claim. A straightforward calculation gives

$$R_a''(z) = \frac{2a(az^3 + 3z^2 + 1)}{(1 + az - z^2)^3}.$$

To determine the sign of R_a'', we set $f(z) = az^3 + 3z^2 + 1$. Then $f'(z) = 3az^2 + 6z$ and $f'(z) = 0$ if and only if $z = -\frac{2}{a}$ or 0. Note that f is continuous on $\mathbb{R}$, and $f(-\frac{2}{a}) = \frac{4}{a^2} + 1 > 0$ and $f(0) = 1 > 0$. Hence f has just one $\mathbb{R}$-intercept, say x_f, and it satisfies $x_f < -\frac{2}{a}$. Thus $f < 0$ on $(-\infty, x_f)$ and $f > 0$ on (x_f, ∞). Note that at $z = x_f$, the numerator of R_a' is $-z^4 + 2(az^3 + 3z^2 + 1) - 4z^2 - 3 = -z^4 - 4z^2 - 3 < 0$. Therefore $x_f < -\frac{2}{a} < p_2 < p_1$. Therefore on $\mathbb{R}$ the behavior of R_a' is as follows:

- On $(-\infty, x_f]$, the map R_a' is continuous, monotone increasing, and takes only negative values.
- On (x_f, p_2), the map R_a' is continuous, monotone decreasing, and takes only negative values.
- On (p_2, p_1), the map R_a' is continuous and monotone increasing.
- On (p_1, ∞), the map R_a' is continuous and monotone decreasing.

Therefore there cannot be any real zeros of R_a' other than the two given above. □

Proposition 3.6. The forward orbit of the critical point in $(0, 1)$ under R_a converges to the fixed point at 0. More generally,

$$\lim_{n\to\infty} R_a^n(z) = 0 \quad \forall z \in (0, 1).$$

To prove this proposition we need the following lemma.

Lemma 3.7. For every $z \in (0, 1)$ and $a > 0$, $0 < R_a^2(z) < z$.

Proof. We have that

$$R_a^2(z) = -\frac{z(z^2 - 1)(z^6 - 3z^4 + 2az^3 + 3z^2 - a^2z^2 - 2az - 1)}{(1 + az - z^2)(z^6 + az^5 - 3z^4 - a^2z^4 + 3z^2 - az - 1)}. \tag{5}$$

and

$$z - R_a^2(z) = \frac{a^2z^5(2z^2 - az - 2)}{(1 + az - z^2)(z^6 + az^5 - 3z^4 - a^2z^4 + 3z^2 - az - 1)}. \tag{6}$$

We determine the sign of each factor in equations (5) and (6) on $(0, 1)$ to conclude that $0 < R_a^2(z) < z$ holds on $(0, 1)$. □

Proof of Proposition 3.6. Lemma 3.7 implies that for any $z \in (0,1)$ the sequence $\{R_a^{2n}(z)\}$ is monotone decreasing and bounded below by 0. Hence for each $z \in (0,1)$ it must be that $\lim_{n\to\infty} R_a^{2n}(z) = \alpha_z$ for some $\alpha_z \in [0,1)$. Since R_a^2 is continuous on $[0,1)$ we have

$$\lim_{n\to\infty} R_a^{2n+2}(z) = R_a^2(\lim_{n\to\infty} R_a^{2n}(z)) = R_a^2(\alpha_z) = \alpha_z,$$

that is, α_z is fixed by R_a^2. The fixed points of R_a^2 are 0, ∞, and $\frac{a\pm\sqrt{a^2+16}}{4}$, of which $\frac{a-\sqrt{a^2+16}}{4} < 0$ and $\frac{a+\sqrt{a^2+16}}{4} > 1$. Hence $\alpha_z = 0$. We showed in Example 1.6 that the attracting directions at 0 are along the coordinate axes in the plane. This implies that for each $z \in (0,1)$, there exists an $N_z > 0$ such that $R_a^{2n}(z)$ lie in a petal at 0 for all $n \geq N_z$. Since this petal is mapped to the other petal in the same immediate basin at 0, we see that $R_a^{2N_z+1}(z)$ lies in this other petal, and hence $\lim_{n\to\infty} R_a^{2n+1}(z) = 0$. Thus $\lim_{n\to\infty} R_a^n(z) = 0$ for every $z \in (0,1)$. □

Proposition 3.8. The forward orbit of the critical point in $(\frac{a+\sqrt{a^2+4}}{2}, \infty)$ under R_a converges to the fixed point at ∞. More generally,

$$\lim_{n\to\infty} R_a^n(z) = \infty \quad \forall z \in (p_1, \infty).$$

Proof. The proof this proposition is similar to that of Proposition 3.6, using the following lemma. □

Lemma 3.9. The inequality $R_a^2(z) > z$ holds for all $z \in (\frac{a+\sqrt{a^2+4}}{2}, \infty)$.

Proof. We check the sign of each factor in the numerator and denominator of the right side of equation (6) on $(\frac{a+\sqrt{a^2+4}}{2}, \infty)$. □

Corollary 3.10. At least one complex critical point is in an immediate basin of attraction of the parabolic fixed point 0 of R_a.

Proof. The map R_a has four distinct critical points, two real ones and a pair of complex conjugate ones. The forward orbit of the real critical point in $(\frac{a+\sqrt{a^2+4}}{2}, \infty)$ diverges to ∞, and hence this critical point does not lie in the immediate basins of attraction of 0. Since there are two distinct immediate basins at 0, at least two critical points must be in the immediate basins of 0. This gives the result. □

The main result in this section is that for every nonzero real parameter a, three critical points of R_a lie in the immediate basins of attraction of the parabolic fixed point at 0, and the fourth lies in the immediate attracting basin of the fixed point at ∞. Recall that the parabolic fixed point 0 of R_a has two immediate basins, each consisting of two disjoint Leau domains. Denote by B_r the immediate basin of 0 whose attracting directions lie on the real axis; denote by B_i the immediate basin of 0 whose attracting directions lie on the imaginary axis. By Lemma 3.3 neither of the two Leau domains in B_i intersects the real axis.

Corollary 3.10, combined with Lemma 3.3, implies that one of the following two cases must occur:

Case 1: Both complex critical points lie in the same Leau domain of B_r.
Case 2: Both complex critical points lie in B_i; moreover, each of the two Leau domains in B_i contains just one complex critical point.

We prove that Case 1 cannot occur; suppose Case 1 does hold. Proposition 3.6 says that the forward orbit of the critical point in $(0, 1)$ eventually lands in the attracting petals of B_r. Then B_i contains no critical points, which is a contradiction. We summarize the location of the critical points in the following.

Theorem 3.11. For $a > 0$, the critical points of R_a satisfy the following:

- The critical point in $(0, 1)$ is in an immediate basin of the parabolic fixed point 0.
- The complex conjugate critical points lie in the other immediate basin of 0; moreover, each Leau domain of this immediate basin contains just one complex critical point.
- The critical point in $(\frac{a+\sqrt{a^2+4}}{2}, \infty)$ is in the immediate basin of the parabolic fixed point ∞.

Using Lemma 3.2 we have a similar result for negative parameters.

Theorem 3.12. For $a < 0$, the critical points of R_a, (c_1, c_2, c_3, c_4), satisfy the following: $c_1 \in (-1, 0)$ and lies in an immediate basin of 0; $c_2 = \overline{c_3}$ and they lie in the other immediate basin of 0. Moreover, each Leau domain of this immediate basin contains just one complex critical point. Finally $c_4 \in (-\infty, \frac{a-\sqrt{a^2+4}}{2})$, and is in the immediate basin of the fixed point at ∞.

Example 3.13. The dynamics of R_4. We use $a = 4$ and $a = 1$ as typical examples. Figure 2 shows the Julia set of R_4 (in black); the range shown is $[-1, 1] \times [-i, i]$. Note that the dynamics of R_4 are symmetric about the real axis.

The map R_4 has four distinct critical points. The critical point approximately at 8.2409 (out of figure) is in the immediate basin of ∞. The critical point approximately at -0.434807 is in the immediate basin of 0 with nonempty overlap with the real axis. The critical points approximately at $-0.337856 \pm 0.406119i$ are in the other immediate basin of 0 with nonempty overlap with the imaginary axis. The complex conjugate critical points lie in different Leau domains.

Figure 3 shows these features, using $a = 1$.

The dynamics of R_a along the real line in $\mathcal{R}$ are stable (see [15] for definitions); that is, moving the parameter a off the real axis a little does not change the qualitative behavior of the critical points. Figure 4 uses the same algorithm as that of Figure 3, and we see the same dynamics with regard to critical points and the basins of attraction of the two parabolic fixed points.

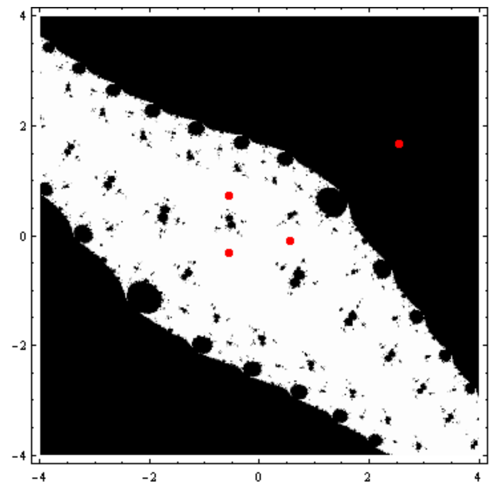

FIGURE 4. The Julia set of R_a with $a = 1 + i$ with the attracting basin of 0 shown (white) and the attracting basin of ∞ (black), and the critical points marked (in red).

However in the next section we see that the critical behavior is quite different for parameters along the imaginary axis so there are bifurcations in a-space.

3.3. **Dynamics of R_a when a lies along the imaginary axis.** We now turn to more complicated dynamics that occur in the family R_a. Throughout this section we suppose that $a = bi$ with $b > 0$. The critical behavior for purely imaginary parameters, and therefore the dynamics of R_a^n, are quite different from those for real parameters. For example we show that attracting fixed points arise when $b > 4$; we also exhibit bifurcations in the parameter plane even when there are no attracting cycles. In that case, there are subdivisions of the parameter plane determined by how many critical points go to which neutral fixed point, 2 and 2 critical points converging to 0 and ∞ respectively, or 3 and 1.

Define $I = \{iy : y \in \mathbb{R}\}$ to be the imaginary axis. We have the following easily proved result, which leads to symmetry in the Julia sets.

Lemma 3.14. If $a = bi$, then $R_a(-\bar{z}) = -\overline{R_a(z)}$ and R_a maps I into I.

Corollary 3.15. For any $a = bi$, with $b \neq 0$, $J(R_a)$ and $F(R_a)$ are symmetric about the imaginary axis.

3.3.1. *Dynamics of R_a if $a = bi$ for $b > 4$.* Under this assumption an attracting fixed point occurs so the dynamics persist for parameters off the imaginary axis as well.

Proposition 3.16. For $a = bi$, where $b > 4$, the fixed point of R_a, $(a - \sqrt{a^2 + 16})/4$, is an attracting fixed point.

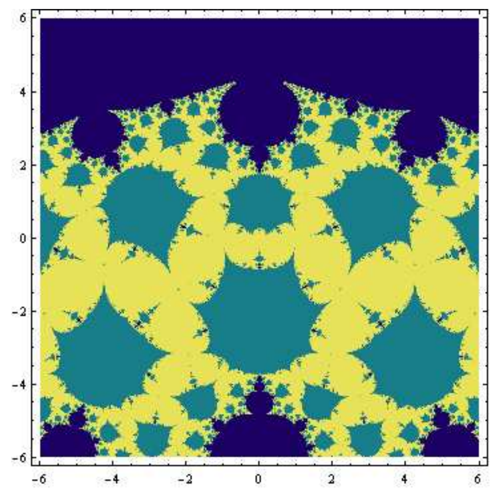

FIGURE 5. The Julia set of R_a with $a = 8i/\sqrt{3}$ with the attracting basins of: 0 (yellow) and ∞ (dark blue), and the attracing basin of a superattracting fixed point at $\frac{i}{\sqrt{3}}$ (light blue).

Proof. If $b \in [4, \infty)$, the multiplier of the fixed point $(a - \sqrt{a^2+16})/4$, $1 - \frac{2\sqrt{b^2-16}}{b}$, is monotone decreasing from 1 to -1. □

In Figure 5 we show the attracting basins of R_a when $a = \frac{8i}{\sqrt{3}}$. There is a superattracting fixed point at $z_0 = \frac{i}{\sqrt{3}}$. As a moves away from this value, the existence of an attracting fixed point persists.

3.3.2. *Along the imaginary axis,* $a = bi$ *for* $0 < b \leq 4$. In this section we show that the dynamics of R_a are quite diverse. In Figure 6 we show the Julia set by coloring the attracting basins of the fixed points at ∞ (light) and 0 (dark). We also show the critical orbits to see that two orbits lie in the same component of the immediate basin of attraction of ∞.

To exploit consequences of Lemmas 3.14 and Corollary 3.15, we conjugate R_a so that the positive imaginary axis is mapped to the positive real axis and reverse 0 and ∞. Set $\varphi(z) = i/z$; write $a = bi$, where $b > 0$. Then

$$S_b(z) = (\varphi \circ R_a \circ \varphi^{-1})(z) = \frac{-z(z^2 - bz + 1)}{z^2+1}.$$

Proposition 3.17. The critical points of S_b satisfy the following:

(i) If $0 < b < \frac{8}{3\sqrt{3}}$, then S_b has two pairs of complex conjugate critical points with nonzero imaginary parts.

(ii) If $b = \frac{8}{3\sqrt{3}}$, then S_b has a double real critical point and a pair of complex conjugate critical points with nonzero imaginary parts.

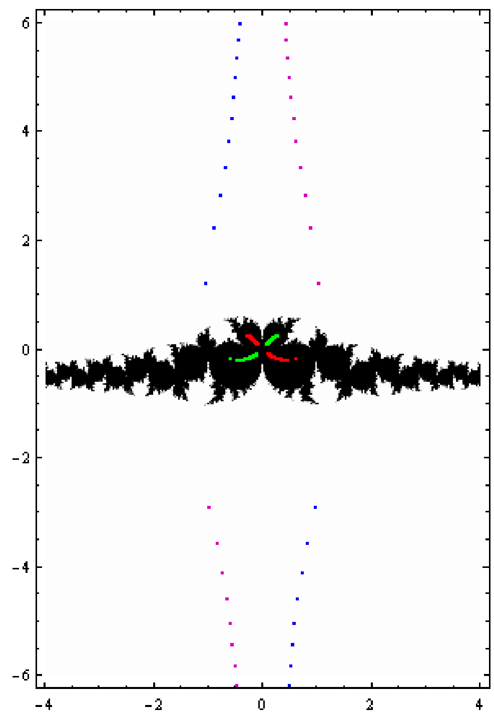

FIGURE 6. The Julia set of R_a with $a = i$. The basin of attraction of 0 is black, the basin of attraction of ∞ is white. The four critical orbits are shown in color.

(iii) If $\frac{8}{3\sqrt{3}} < b < 4$, then S_b has two distinct real critical points; one is in $(0, \frac{1}{\sqrt{3}})$ and the other is in $(\frac{1}{\sqrt{3}}, \frac{b}{2}]$. S_b also has a pair of complex conjugate critical points with nonzero imaginary parts.

Proof. It is easy to calculate that

$$S_b'(z) = \frac{-(z^4 + 2z^2 - 2bz + 1)}{(z^2+1)^2} \quad \text{and} \quad S_b''(z) = \frac{-b(6z^2-2)}{(z^2+1)^3}.$$

Solving $6z^2 - 2 = 0$, we obtain $z = \pm\frac{1}{\sqrt{3}}$. Note that $S_b'(-\frac{1}{\sqrt{3}}) = -1 - \frac{3\sqrt{3}}{8}b < 0$ and $S_b'(\frac{1}{\sqrt{3}}) = -1 + \frac{3\sqrt{3}}{8}b$. Also, note that S_b' is continuous on $\mathbb{R}$, and $\lim_{z\to\pm\infty} S_b'(z) = -1$, when $z \in \mathbb{R}$, holds. We summarize the behavior of S_b' on $\mathbb{R}$ as follows:

- On $(-\infty, -\frac{1}{\sqrt{3}}]$, the map S_b' is monotone decreasing and takes only negative values.
- On $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, the map S_b' is monotone increasing.
- On $[\frac{1}{\sqrt{3}}, \infty)$, the map S_b' is monotone decreasing.

Case (i): If $0 < b < \frac{8}{3\sqrt{3}}$, then $S_b'(\frac{1}{\sqrt{3}}) < 0$. From this we see that $S_b' < 0$ on $\mathbb{R}$ and S_b' has no zeros on the real axis. The rest of the claim follows from the symmetry about the real axis of the dynamics of S_b.
Case (ii): If $b = \frac{8}{3\sqrt{3}}$, then the map S_b has a double critical point $z = \frac{1}{\sqrt{3}}$, and one solves for the other two critical points $\frac{1}{\sqrt{3}}(-1 \pm 2\sqrt{2}i)$.
Case (iii): If $\frac{8}{3\sqrt{3}} < b < 4$, then $S_b'(\frac{1}{\sqrt{3}}) > 0$. By the shape of the graph of S_b' on $\mathbb{R}$, we see there are just two real critical points of S_b, one in $(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and the other in $(\frac{1}{\sqrt{3}}, \infty)$. In fact, since $S_b'(0) = -1 < 0$ and $S_b'(\frac{b}{2}) = \frac{-(b^2-4)^2}{(b^2+4)^2} \leq 0$ (the equality holds when $b = 2$), we see that the two real critical points of S_b are in $(0, \frac{1}{\sqrt{3}})$ and in $(\frac{1}{\sqrt{3}}, \frac{b}{2}]$, respectively. □

The next proposition and its corollaries are the main results of this section. For each parameter b, we know that three critical points of S_b lie in 3 distinct immediate basins of parabolic fixed points 0 and ∞.

Proposition 3.18. The orbit of the free critical point of S_b either lies in the attracting basin of 0 or lands on $0 \in J(S_b)$ after a finite number of iterations.

Corollary 3.19. For every $0 < b < 4$, there is a dichotomy among the components of $F(S_b)$: each Fatou component is eventually mapped to the immediate basin of either 0 or ∞. Also, the orbit of each point in $\mathbb{R} \cap F(S_b)$ converges to 0.

Interpreting the results for the original family R_a yields the following.

Corollary 3.20. For $a = bi$, $b \in (0, 4)$, the free critical point c_f of R_a is in the attracting basin of ∞ or is a prepole in the sense that $R_a^k(c_f) = \infty$ for some $k \in \mathbb{N}$. In the first case $c_f \in F(R_a)$ and in the second case $c_f \in J(R_a)$.

Proof of Corollary 3.19. Sullivan's No Wandering Domains Theorem (see, e.g., [3], [6], [17]) implies that each Fatou component eventually becomes periodic and is of one of the four types listed in Section 3.4. Since all critical orbits converge to parabolic fixed points, the topological closure of the critical orbits cannot contain the boundary of a Herman ring or a Siegel disk. By the same token there are no attracting periodic points. Hence each Fatou component is eventually mapped to an immediate basin of a parabolic periodic point. Since all four critical orbits converge to 0 or ∞, we have proved the first statement.

Recall that S_b maps $\mathbb{R}$ to $\mathbb{R}$. Since two repelling directions of ∞ lie on the real axis, any Fatou component with non-empty intersection with $\mathbb{R}$ must be associated to 0. □

We now prove Proposition 3.18. The proof is split up into the cases given in Proposition 3.17 and proved separately; the proof of Case (iii) is further divided into 3 cases.

Proof of Case (i): Assume $0 < b < \frac{8}{3\sqrt{3}}$. By Proposition 3.17, a pair of complex conjugate critical points is associated to the parabolic fixed point 0 of S_b, while the other pair to ∞. Since both attracting directions of 0 are on the real axis and there is symmetry about the real axis, the two critical points associated to 0 are in the same Leau domain. Hence the free critical point is attracted to 0.
Proof of Case (ii): We suppose $b = \frac{8}{3\sqrt{3}}$. Two complex conjugate critical points associated to 0 coincide with each other to become a double critical point $\frac{1}{\sqrt{3}}$ of S_b. The other two critical points then must be in distinct immediate basins of attraction of ∞.
Case (iii): We first assume that $\frac{8}{3\sqrt{3}} < b < 2$. We prove the following.

Proposition 3.21. If $\frac{8}{3\sqrt{3}} < b < 2$, then the forward orbits under S_b of both of the real critical points converge to the parabolic fixed point 0.

To prove this proposition we need the following lemma.

Lemma 3.22. For every $z \in (0, \frac{b}{2})$, $0 < S_b^2(z) < z$.

Proof. Straightforward calculations show that

$$S_b^2(z) = \frac{z(z^2 - bz + 1)(z^6 - bz^5 + 3z^4 + (3 - b^2)z^2 + bz + 1)}{(z^5 + 1)(z^6 - 2bz^5 + (3 + b^2)z^4 - 2bz^3 + 3z^2 + 1)}. \tag{7}$$

and

$$z - S_b^2(z) = \frac{b^2 z^3(2z^2 - bz + 2)}{(z^2 + 1)(z^6 - 2bz^5 + (3 + b^2)z^4 - 2bz^3 + 3z^2 + 1)}. \tag{8}$$

An argument similar to that used in the proof of Lemma 3.7, checking the sign of each factor on the right side of equations (7) and (8), gives the result. □

Proof of Proposition 3.21. The claim that the forward orbit of any point in $(0, \frac{b}{2})$ under S_b is attracted to the parabolic fixed point 0 follows from an argument similar to that of Proposition 3.6 and the fact that the attracting directions at 0 are on the real axis. Since both of the two real critical points are in $(0, \frac{b}{2})$, we are done. □

If $b = 2$, the map S_b has a real critical point 1, which is mapped to $0 \in J(S_b)$ and the proposition is proved.

We finish Case (iii) by assuming $2 < b < 4$. The zeros of S_b, 0 and $\frac{b \pm \sqrt{b^2-4}}{2}$, are all real, and they satisfy the inequality

$$0 < \frac{b - \sqrt{b^2 - 4}}{2} < \frac{b + \sqrt{b^2 - 4}}{2} < b.$$

Since $S_b'(\frac{b-\sqrt{b^2-4}}{2}) = \frac{\sqrt{b^2-4}}{b} > 0$ and $S_b'(\frac{b+\sqrt{b^2-4}}{2}) = -\frac{\sqrt{b^2-4}}{b} < 0$, the map S_b has one critical point in $I_1 = (0, \frac{b-\sqrt{b^2-4}}{2})$ and the other one in $I_2 = (\frac{b-\sqrt{b^2-4}}{2}, \frac{b+\sqrt{b^2-4}}{2})$.

Proposition 3.23. For $a = bi$, where $2 < b < 4$, the forward orbits of both of the real critical points converge to 0. More generally,

$$\lim_{n\to\infty} S_b^n(z) = 0 \quad \forall z \in (0, \frac{b+\sqrt{b^2-4}}{2}).$$

To prove this proposition we need two lemmas.

Lemma 3.24. For $a = bi$, where $2 < b < 4$, the inequalities $0 < S_b(z) < z$ hold on $I_2 = (\frac{b-\sqrt{b^2-4}}{2}, \frac{b+\sqrt{b^2-4}}{2})$.

Proof. To show that $0 < S_b$ on I_2, recall that the continuous function S_b has zeros at 0 and $\frac{b\pm\sqrt{b^2-4}}{2}$, none of which is in I_2. Hence S_b is either always positive or always negative on I_2. Since S_b takes the positive value $\frac{b(b^2-4)}{2(b^2+4)}$ at $\frac{b}{2} \in I_2$, we have that $S_b > 0$ on I_2. To show that $S_b(z) < z$ on I_2, recall that S_b has fixed points at 0 and $\frac{b\pm\sqrt{b^2-16}}{4}$, none of which is in I_2. Hence $z - S_b$ is either always positive or always negative on I_2. Since $z - S_b(z)$ takes the positive value $\frac{4b}{b^2+4}$ at $\frac{b}{2} \in I_2$, we have that $z - S_b(z) > 0$ on I_2. □

Lemma 3.25. For $a = bi$, where $2 < b < 4$, the inequalities $0 < S_b^2(z) < z$ hold on $I_1 = (0, \frac{b-\sqrt{b^2-4}}{2})$.

Proof. By checking the sign of each factor of equations (7) and (8) on I_1, the result follows.

□

Proof of Proposition 3.23. Lemma 3.24 implies that for any $z \in I_2$, with $I_2 = (\frac{b-\sqrt{b^2-4}}{2}, \frac{b+\sqrt{b^2-4}}{2})$, the sequence $\{S_b^n(z)\}$ is monotone decreasing, as long as $S_b^n(z) \in I_2$, and bounded below by 0. Hence for each $z \in I_2$ it must be either that $\lim_{n\to\infty} S_b^n(z) = \beta_z$ for some $\beta_z \in [\frac{b-\sqrt{b^2-4}}{2}, \frac{b+\sqrt{b^2-4}}{2})$ or that there exists a smallest $N = N_z > 0$ such that $S_b^N(z) \leq \frac{b-\sqrt{b^2-4}}{2}$. If the first case were to happen, then since S_b is continuous on $\mathbb{R}$, we have

$$\lim_{n\to\infty} S_b^{n+1}(z) = S_b(\lim_{n\to\infty} S_b^n(z)) = S_b(\beta_z) = \beta_z,$$

that is, β_z is fixed by S_b. The fixed points of S_b are 0, ∞, and $\frac{b\pm\sqrt{b^2-16}}{4}$, none of which is in $[\frac{b-\sqrt{b^2-4}}{2}, \frac{b+\sqrt{b^2-4}}{2})$, a contradiction. Hence there exists a smallest $N = N_z$ for which $S_b^N(z) \in (0, \frac{b-\sqrt{b^2-4}}{2}]$. If $S_b^N(z) = \frac{b-\sqrt{b^2-4}}{2}$, then $S_b^{N+1}(z) = S_b(\frac{b-\sqrt{b^2-4}}{2}) = 0$, and we are done. Otherwise, with Lemma 3.25 and an argument similar to that of Proposition 3.6, we obtain that $\lim_{n\to\infty} S_b^{2n}(z) = 0$. Since both attracting directions at 0 of S_b are on the real axis, we have that $\lim_{n\to\infty} S_b^n(z) = 0$. □

Finally, if $a = 4i$, then the map R_a has three parabolic fixed points: 0, ∞, and i. The fixed point at i has multiplicity 2 and hence multiplier 1. There is one immediate basin of i consisting of one Leau domain. Thus the critical points of R_a lie in four distinct immediate basins of attraction of parabolic points.

3.4. **Connectivity results for $J(R_a)$.** In this section we show that under the assumption that each Fatou component contains at most one critical value (possibly a double critical value), the Julia set of R_a is connected. This is largely due to the fact that 3 of the 4 critical points must lie in disjoint Fatou components that never intersect under forward iteration. That leaves only one free critical point, and the hypotheses guarantee this does not disconnect the Julia set. The only way that $J(R_a)$ can fail to be connected is if one of the immediate basins of attraction of either 0 or ∞ has infinite connectivity. First, we establish that no Herman rings can occur for R_a. This follows from the following result of Shishikura (cf. [18]).

Theorem 3.26. ([18], Theorem 4.3.1) For a rational map R of degree $k \geq 2$, let $N(c)$ be the number of critical points in $F(R)$; let $N(ir)$ be the number of irrationally neutral periodic points for R; let $N(HR)$ be the number of Herman rings. Then

$$N(c) + N(ir) + 2N(HR) \leq 2k - 2.$$

Corollary 3.27. For every $a \in \mathbb{C}^*$, R_a has no Herman rings.

Proof. Let a be arbitrary; $k = \deg(R_a) = 3$ and $2k - 2 = 4$. We showed in Proposition 2.3 that of the 4 critical points available, there are at least two critical points in the immediate attracting basins of 0 and at least one critical point in the immediate attracting basin of ∞. If there were a Herman ring, then $N(c) + 2N(HR) = 3 + 2 = 5 > 4$, which is a contradiction. □

We next make a few observations about the valency of points for R_a; most can be found for example in [3]. From the discussion above, there are at least 3, and at most 4, distinct critical points of R_a. Let $v_a(z_0)$ denote the valency of R_a at z_0; by this we mean the order of R_a at z_0, the number of solutions of $R_a(z) = R_a(z_0)$ at z_0. Let $c_1, \ldots, c_k$ denote the k distinct critical points of R_a; $k = 3$ or 4. Since the degree of R_a is 3, the Riemann–Hurwitz relation gives that $\sum(v_a(c_j) - 1) = 4 = 2\deg(R_a) - 2$, where $j = 1, \ldots, k$. So we have either 4 simple critical points c_j, with $v_a(c_j) = 2$, or exactly 1 double critical point, label it c_1, with $v_a(c_1) = 3$; and then $v_a(c_2) = v_a(c_3) = 2$. We make heavy use of the following version of the Riemann–Hurwitz relation in what follows.

Proposition 3.28. Let $U \neq \mathbb{C}$ be a simply connected domain bounded by a Jordan curve, and let W be a connected component of $R_a^{-1}(U)$. Suppose there are no critical values of R_a on ∂U. Then there exists an $m = 1, 2$, or

3 such that R_a is an m-fold map of W onto U and

$$\chi(W) = m - \delta_a(W), \tag{9}$$

where $\delta_a(W) = \sum_{z \in W}(v_a(z) - 1) = \sum_{c_j \in W}(v_a(c_j) - 1)$ and $\chi(W)$ is the Euler characteristic of W.

Corollary 3.29. Under the hypotheses above,

(i) if there are no critical values in U, then $R_a^{-1}(U)$ consists of 3 disjoint homeomorphic copies of U;
(ii) if there is one critical value u_1 in U, coming from a simple critical point, then $R_a^{-1}(U)$ consists of two simply connected regions, W_0 a homeomorphic image of U bounded by a Jordan curve and W_1, a simply connected domain that maps by R_a onto U by a 2-fold ramified covering (and contains a simple critical point c_1);
(iii) if there is one critical value $u_2 \in U$, coming from a double critical point c, then $W = R_a^{-1}(U)$ consists of a simply connected region and R_a gives a 3-fold branched cover of U by W.

Proof. Under the assumptions in (i), $\delta_a(W) = 0$ and $m = 1$ for each component of $R_a^{-1}(U)$. To show (ii) holds, there must be a component W_1 of $R_a^{-1}(U)$ with $\delta_a(W_1) = 1$ and $m = 2$. If W_0 is a component containing no critical point, $\delta_a(W_0) = 0$ and W_0 is homeomorphic to U. Since $\deg(R_a) = 3$, by (9) there are no other possibilities. In case (iii), in any neighborhood of c the map R_a looks locally like $z \mapsto z^3$ so the degree of the branched covering map R_a from W to U is 3 and hence $\chi(W) = 1$. □

The last case is to assume that one Leau domain L_0 contains two distinct critical points, which means its paired domain, L_1 contains two critical values. Recall that a Leau domain is always either simply connected or infinitely connected [17].

Proposition 3.30. Consider two Leau domains L_0 and L_1 whose union forms an immediate basin of attraction for one of the parabolic fixed points, 0 or ∞ for R_a. Then R_a maps L_0 into L_1, and for some integer $m = 1, 2$, or 3, R_a is an m-fold map of L_0 onto L_1, and if L_0 contains two critical points counting multiplicity, then

$$\chi(L_0) + 2 = 3\chi(L_1). \tag{10}$$

In particular, L_0 is simply connected if and only if L_1 is simply connected.

We cannot eliminate the case where both might be infinitely connected; however we prove a result that shows connectivity of the Julia set of R_a for most of the parameters discussed above. In Figure 6 we show a Julia set where two critical values lie in the same Leau domain associated to the point at ∞.

The approach to the proof of the next result is similar to a result for rational maps with two critical points given by Milnor in [16] (see also [12]).

Theorem 3.31. If the critical values of R_a which are in the Fatou set belong to disjoint Fatou components, then $J(R_a)$ is connected.

Proof. Because of the rationally neutral fixed point at 0 we have a nonempty Fatou set. For any parameter a, the map R_a has either 3 or 4 critical values. By Proposition 2.3 three critical points of R_a, and hence also critical values, lie in distinct immediate basins of attraction of the parabolic fixed points 0 and ∞. If there is a critical point of R_a that is not simple, then there are exactly 3 critical values and they each lie in one of these distinct basins.

We now consider a loop γ in any Fatou component W of R_a. It is enough to show that γ shrinks to a point in W since $J(R_a)$ is connected if and only if each component of $F(R_a)$ is simply connected.

We established in Corollary 3.27 that there are no Herman rings so some forward image $R_a^k(\gamma)$ lies in a simply connected region $U \subset F(R_a)$ which is one of these:

(i) a linearizing neighborhood of an attracting periodic point;
(ii) a Böttcher neighborhood of a superattracting periodic point;
(iii) an attracting Leau petal for a periodic parabolic point;
(iv) a cycle of Siegel disks.

Furthermore since U is simply connected, we can choose it so that its boundary is a simple closed curve not passing through a critical value, and by hypothesis U contains either 0 or one critical value. Then, using induction on k, we show that $R_a^{-k}(U)$ consists of only simply connected components. Therefore γ is shrinkable to a point.

For the inductive argument, we first show that $R_a^{-1}(U)$ has only simply connected components. We apply Corollary 3.29 to each of these three cases.
Case 1: If U contains no critical value, then $R^{-1}(U)$ consists of three simply connected regions bounded by three disjoint curves.
Case 2: If U contains one critical value w_0 associated to a simple critical point c_0, then the valency of c_0 is 2 and w_0 is contained in the connected component of $F(R_a)$ containing U. Then $R_a^{-1}(U)$ consists of two simply connected regions, one a homeomorphic image of U bounded by a simple closed curve and the other is simply connected and maps by R_a onto U by a 2-fold ramified covering (and contains c_0).
Case 3: If U contains a critical value of a double critical point, then the valency is 3 and U is contained in a simply connected Leau petal. The other 2 critical points must lie in the remaining two immediate basins of attraction. Therefore there cannot be any critical relations $R^m(c_i) = R^n(c_j)$, $j \neq i$, since they are contained in disjoint forward periodic Leau domains. Then $R_a^{-1}(U)$ consists of a simply connected region bounded by a curve and which maps onto U by a ramified 3-fold covering.

The inductive step on k is the same. Each simply connected component of $R_a^{-1}(U)$ contains no critical value in its boundary and contains either 0 or 1 critical value in it, so we repeat the argument to obtain that each component of $R^{-k}(U)$ is simply connected; therefore the curve γ shrinks to a point. □

This result leads immediately to the following application to our setting.

Theorem 3.32. The Julia set of R_a is connected if R_a satisfies any one (or possibly more) of the following conditions:

(i) There exists an attracting periodic orbit.
(ii) There exists a cycle of Siegel disks.
(iii) There is a critical point in $J(R_a)$.
(iv) R_a has a multiple critical point.
(v) $a \in \mathbb{R} \setminus \{0\}$.
(vi) $a = ib$, $b \in [4, \infty)$.
(vii) There exists a parabolic periodic point in addition to 0 and ∞.

Proof. All of these follow from Theorem 3.31. Theorem 3.11 shows that (v) implies the hypotheses of Theorem 3.31 are satisfied. Proposition 3.16 shows that (vi) implies (i); and the proof that (i) implies connectivity of $J(R_a)$ is clear because an attracting orbit requires a critical point in its immediate basin of attraction. □

4. Measure theoretic properties and Hausdorff dimension of $J(R_a)$

In this section we apply some results of others to our parametrized family of maps. Since the literature on the topics of Haudsdorff dimension and conformal measures for rational maps is quite deep, we do not attempt a full history of the results, but stick to a few key references such as [1, 7], and refer to the extensive bibliographies in these. Suppose R is a rational map with critical set $C = \{c_1, \ldots, c_k\}$. The *postcritical set* of R is: $P(R) = \overline{\cup_{n\geq 1} R^n(C)}$, where $\overline{A}$ denotes the topological closure of a set A.

We call a rational map R *parabolic* if there are no critical points in $J(R)$, but $P(R) \cap J(R) \neq \emptyset$. Clearly many of the maps in the family R_a are parabolic; all maps with an attracting cycle or with all critical points attracted to 0 or ∞ (but not landing on either of them). This seems to be the generic case. A continuous map $f : X \to X$ of a compact metric space (X, ρ) is *(positively) expansive* if and only if there exists a constant $\beta > 0$ such that if $\rho(f^n(x), f^n(y)) \leq \beta$ for some $x, y \in X$ and every $n = 0, 1, 2, ...$ then $x = y$. It was shown in ([7], Theorem 4) that every parabolic rational map is expansive on its Julia set; more precisely a rational map $R|_{J(R)}$ is expansive if and only if $J(R) \cap C = \emptyset$.

A large body of results that hold for hyperbolic rational maps (where $P(R) \cap J(R) = \emptyset$) have been extended to the parabolic setting (see [1, 7]). We mention only a few here as they apply to our family of maps R_a. Since $J(R_a) \neq \mathbb{C}_\infty$, without loss of generality we assume that we conjugate R_a by a Mobius transformation in order to assume that $\infty \notin J(R_a)$ so we can use the Euclidean metric in what follows.

Definition 4.1. Consider a set $B \subset \mathbb{C}$. For any $\epsilon > 0$, let $\mathcal{O}_\epsilon(B)$ be the collection of countable coverings $(U_j)_{j\in\mathbb{N}}$ of B by balls of diameter $\leq \epsilon$.

Given a fixed $\epsilon > 0$ and $t > 0$, define

$$\mathcal{H}^t_\epsilon(B) := \inf\Big\{ \sum_j (\mathrm{diam}(U_j))^t : (U_j)_{j\in\mathbb{N}} \in \mathcal{O}_\epsilon(B)\Big\},$$

and

$$\mathcal{H}^t(B) = \sup_{\epsilon>0} \mathcal{H}^t_\epsilon(B) = \lim_{\epsilon\to 0} \mathcal{H}^t_\epsilon(B).$$

If $t < s$, then $\mathcal{H}^s_\epsilon(B) \leq \epsilon^{s-t}\mathcal{H}^t_\epsilon(B)$, so $\mathcal{H}^t(B) = \infty$ if $\mathcal{H}^s(B) > 0$ and $\mathcal{H}^s(B) = 0$ if $\mathcal{H}^t(B) < \infty$. The *Hausdorff dimension* of B, denoted $HD(B)$, is the unique value t such that

$$\mathcal{H}^t(B) = \begin{cases} \infty & \text{if } t < HD(B); \\ 0 & \text{if } t > HD(B). \end{cases} \tag{11}$$

Let μ denote a Borel probability measure on $\mathbb{C}_\infty$ supported on $J(R_a)$.

Definition 4.2. The *Hausdorff dimension* of μ is given by:

$$HD(\mu) := \inf\{HD(A) : A \subseteq \mathbb{C} \text{ is Borel measurable and } \mu(A) = 1.\}.$$

Denker and Urbanski [7] proved the following for an expansive map.

Theorem 4.3. ([7], Thm 15) Assume R is a rational map with $\deg(R) \geq 2$, such that $R|_{J(R)}$ is expansive. Let $M_+(R)$ denote the set of invariant probability measures for R with positive entropy. Then:

$$h := HD(J(R)) = \sup\{HD(\mu), \mu \in M_+(R)\}. \tag{12}$$

We now apply some estimates from [1] to our setting. We set

$$h_a = HD(J(R_a)).$$

Let $\mathcal{N}$ denote the collection of parabolic periodic points for the parabolic map R_a; then $0, \infty \in \mathcal{N}$, but an additional point might be included as well, for example if $a = 4i$ (there are many other parameters where R_a has 3 parabolic periodic points).

Looking at a higher iterate of R_a if necessary, write $T \equiv R_a^q$, so that $T'(\zeta) = 1$ for all $\zeta \in \mathcal{N}$. Then T^{-1} is defined in a neighborhood of ζ such that $T^{-1}\zeta = \zeta$ and we write the Taylor series expansion of T^{-1} about ζ as: $T^{-1}(z) = z + \alpha(z-\zeta)^{s+1} + O(z^{s+2})$, where as before $s + 1 \geq 2$ is the multiplicity of the fixed point ζ.

Let $\gamma(\zeta) = \dfrac{s+1}{s}$; and define $\gamma_0 = \min\{\gamma(\zeta) : \zeta \in \mathcal{N}\}$.

Theorem 4.4. ([1]) For each a such that R_a is parabolic, $1/\gamma_0 < h_a < 2$.

Corollary 4.5. For each a such that R_a is parabolic, $4/5 < h_a < 2$ and $m(J(R_a)) = 0$, where m denotes normalized Lebesgue measure on $\mathbb{C}_\infty$.

Proof. For each $a \in \mathcal{R}$, we have that 0 is a fixed point of R_a^2 of multiplicity 5, hence $\gamma_0 \leq \frac{5}{4}$. Then $4/5 \leq 1/\gamma_0 < h_a < 2$ and the result follows. ☐

This improves estimates given in [1].

4.1. **Some measure theoretic properties of R_a.** Properties of measures supported on Julia sets of rational maps with parabolic fixed points were studied in detail by Denker and Urbanski in many papers (eg, [7], [1]). We begin with the definition of a conformal measure [20].

Definition 4.6. Given $h \in \mathbb{R}$, a Borel probability measure ν is called an *h-conformal measure* if it is supported on $J(R_a)$ and satisfies:

$$\nu(R_a(B)) = \int_B |R_a'(z)|^h d\nu$$

for every Borel set B such that $R_a|_B$ is injective.

For the next result we combine several results of Aaronson, Denker, and Urbanski rephrased for our setting ([1], Theorems 8.7, 9.8, and 9.9).

Theorem 4.7. Suppose R_a satisfies one of the following:

(i) There is no critical point in $J(R_a)$;
(ii) If the free critical point of R_a is in $J(R_a)$ then it has a finite forward orbit;

then there exists a unique h-conformal measure ν, with $h = h_a$, and ν is nonatomic. There also exists an ergodic R_a-invariant measure $\mu_a \sim \nu$. The measure μ_a is either σ-finite or finite depending on $HD(J(R_a))$. In particular, if $h_a > 2/\gamma_0 \geq \frac{8}{5}$, then $\mu_a(J(R_a)) < \infty$ and if $h_a \leq 1$, then $\mu_a(J(R_a)) = \infty$.

There are more precise statements regarding the finiteness of the measure μ_a given in [1] but we do not develop them here so we omit them. In any case, in order to determine whether or not the invariant conformal measure ν is finite, it is sufficient to know h_a. Whenever $J(R_a)$ is connected, $HD(J(R_a)) \geq 1$ (but this does not help). Using (12) one can obtain a better lower bound for h_a, at least numerically, by estimating $HD(\rho_a)$, where ρ_a is the Mañé–Lyubich measure of maximal entropy ($\log 3$) (cf. eg. [13]). We do this using the equality (cf. [7]):

$$HD(\rho_a) = \frac{\log 3}{\chi_{R_a}},$$

where χ_{R_a} is the Lyapunov exponent, reasonably well-approximated by the following algorithm. Choose a "random" number with respect to the maximal entropy measure ρ_a by choosing a Lebesgue (computer-selected) random number and then taking a randomly chosen backward path for a few hundred iterations; this gives a point that is likely a generic point z_0 for ρ_a; it is justified by the result of [13]. Calculate the backward average $\frac{1}{n}\sum_{j=0}^{n-1} \log |R_a'(z_j)|$ for a randomly chosen backward path of z_0 (so $R_a^j(z_j) = z_0$). This gives a good approximation to χ_{R_a}, hence to $HD(\rho_a)$. While we obtain lower bounds that are greater than one, we cannot conclude from this information whether or not the measure μ_a, equivalent to the h_a-conformal measure is finite. However, it gives interesting information about the Hausdorff dimension of the measure of maximal entropy and a strict lower bound for h_a so we include it here.

Example 4.8. Some sample estimates of $HD(\rho_a)$, where ρ_a is the Mañé–Lyubich measure. We only estimate values where we know the Julia set is connected (so $h_a \geq 1$). We give the approximation to the nearest .01.

(i) For $a = \frac{8}{\sqrt{3}}i$, with a superattracting fixed point, $HD(\rho_a) \approx 1.32$.
(ii) For $a = 4$, $HD(\rho_a) \approx 1.17$.
(iii) For $a = -1.75 + 3.75i$, the approximate location of a period three attractor, $HD(\rho_a) \approx 1.34$.
(iv) For $a = 4i$ (3 parabolic points), $HD(\rho_a) \approx 1.39$.

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