ERGODIC AND CHAOTIC PROPERTIES OF LIPSCHITZ MAPS ON SMOOTH SURFACES

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Abstract. We construct noninvertible maps of every compact surface and study their chaotic properties from both the measure theoretic and topological points of view. We use some topological techniques employed by others for diffeomorphisms and extend to the noninvertible case.

1. Introduction

In this paper we construct n-to-one dynamical systems on smooth surfaces; some of the maps are smooth and others are continuous but fail to be differentiable on a set of measure zero (usually on finitely many one-dimensional curves). These maps exhibit a variety of chaotic and mixing behavior, both topological and measure theoretic with respect to a smooth measure. There is a rich literature on the subject of dynamical systems on surfaces; we outline it briefly.

The subject of noninvertible continuous and differentiable maps on smooth manifolds goes back to 1969 when Shub extended many dynamical ideas and examples from diffeomorphisms to this setting [25]. Shub's paper contains some fundamental results such as the theorem that any expanding smooth map of $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$, called an expanding endomorphism, is topologically conjugate to an expanding linear map. An expanding endomorphism is (by the definition used in [25]) differentiable at all points; however this notion was weakened in the intervening years to prove that similar dynamical properties hold for expanding and expansive maps on metric spaces (see for example, work of Coven and Reddy [6], and the book by Mañé [16]). All examples that we construct in this paper are made from simple piecewise expanding and piecewise differentiable maps, so we do not need to get into the subtleties of these distinct but overlapping concepts. In fact we end up with expansion and differentiability on a set of full measure but not at every point, so results on expanding and expansive maps to do not directly apply here.

Present in many dynamical settings is a natural measure class; on smooth manifolds it is the class of measures coming from a Riemannian metric. Bernoulli diffeomorphisms of smooth compact surfaces were constructed by Katok in 1979 [13]; these maps preserve a finite smooth measure. Since that

Date: October 15, 2011.

paper many studies have been done on the dynamical properties of invertible maps on smooth surfaces (see eg, [8] and [16] and the references in these).

The purpose of this paper is to use topological methods to construct continuous and smooth noninvertible maps of surfaces that exhibit a variety of measure theoretic behavior with respect to a natural measure on the surface. We assume that every surface is smooth (C^1) and the measure being considered is absolutely continuous with respect to Lebesgue measure in local coordinate charts. We obtain maps that are continuous and Lipschitz, with Lipschitz constant strictly greater than one, but some of them fail to be differentiable at isolated points or on smooth curves. Since many of our constructions involve piecing together maps on surfaces with disks removed, i.e., surfaces with boundary, our maps are generally not expansive [11]. However they exhibit enough expansion in a piecewise way that we obtain chaos and ergodicity on the nonwandering set. Further since expansive maps cannot occur on some surfaces such as the projective plane (see [6, 12, 25]), piecewise expansion is the best one can hope for.

We study topological properties such as transitivity and chaos, and measure theoretic properties such as isomorphism to a one-sided Bernoulli shift, exactness, and ergodicity. One-sided Bernoulli maps and their rigidity were studied in [4]. We extend that study here to include a wider variety of manifolds including some nonorientable surfaces, that admit piecewise smooth maps that are one-sided Bernoulli.

In Section 2, we give definitions for the measurable and topological dynamical properties considered as well as a brief classification of surfaces up to diffeomorphism. In Section 3 we use some basic and familiar one-sided Bernoulli maps of the interval and circle to construct one-sided Bernoulli maps on the Mobius strip, Klein bottle, and real projective plane. These maps are continuous and Lipschitz. We also mention existing examples on the torus and sphere. We then construct smooth noninvertible examples on a two-fold symmetric product. This was defined in [5] and used in [14] to extend topological dynamical properties. We use it to construct ergodic and chaotic many-to-one maps on nonorientable surfaces of any genus ≥ 2 in Section 4. After discussing a few generalizations, we turn to arbitrary orientable surfaces. In Section 5 we introduce maps that are chaotic on their nonwandering set and ergodic with respect to a measure absolutely continuous with respect to Lebesgue measure, but have a fixed point with only one preimage. We use this to extend the technique of blowing up around a fixed point of a diffeomorphism to the noninvertible case in Section 6, where we construction chaotic and ergodic maps on orientable surfaces of any positive genus.

2. Measure theoretic and topological preliminaries

We review some basic definitions in measurable and topological dynamics. While the notions are standard, the same terminology is not always used, so we present our terminology and notation here.

2.1. Measurable properties.

We assume throughout this paper that every space $(X, \mathcal{B}, \mu)$ is a locally compact metric space with metric δ, Borel σ-algebra $\mathcal{B}$ on X, and μ a regular Borel probability measure on $\mathcal{B}$. Moreover, X usually has the structure of a smooth manifold. Infinite measures are always assumed to be σ-finite. We assume that $f : X \to X$ is *nonsingular*; i.e., $f : X \to X$ satisfies: $\mu(A) = 0 \iff \mu(f^{-1}A) = 0$ for every $A \in \mathcal{B}$. We also assume that every point in X has at most finitely many preimages under f. Furthermore in all of our examples we will assume without loss of generality that f is forward nonsingular as well; i.e., that $\mu(A) = 0 \iff \mu(fA) = 0$ for all measurable sets A. For example, any C^1 map of a manifold onto itself whose differential is nonvanishing except at finitely many points is forward and backward nonsingular with respect to the Riemannian volume form (locally equivalent to Lebesgue measure). Let $\mathcal{B}_+ \subset \mathcal{B}$ denote the collection of measurable sets of positive measure. In order to stress the presence of both a topology and a Borel measurable structure, we will refer to $(X, \mathcal{B}, \mu, f)$ as a *nonsingular dynamical system.* If $\mu(f^{-1}A) = \mu(A)$ for all measurable sets A, then we say $(X, \mathcal{B}, \mu, f)$ is a *measure preserving dynamical system*, or more simply, f is *measure preserving.* All of the examples in this paper will be nonsingular with respect to some naturally occurring measure and most of them will be measure-preserving.

Definition 2.1. Let $(X_1, \mathcal{B}_1, \mu_1, f_1)$ and $(X_2, \mathcal{B}_2, \mu_2, f_2)$ be two measure preserving dynamical systems.

- A measurable map $\varphi : X_1 \to X_2$ is a *(measurable) factor map* if there exists a set $Y_1 \in \mathcal{B}_1$ of full measure in X_1 and a set $Y_2 \in \mathcal{B}_2$ of full measure in X_2 such that φ maps Y_1 onto Y_2.
- If the factor map φ is such that $f_1(Y_1) = Y_1$, $f_2(Y_2) = Y_2$, $\varphi \circ f_1 = f_2 \circ \varphi$ on Y_1, and $\mu_2(A) = \mu_1(\varphi^{-1}(A))$ for all $A \in \mathcal{B}_1$, then f_2 is called a *measurable factor* of f_1 *with factor map* φ.
- If the factor map φ is injective on Y_1 we say it is an *isomorphism.* If f_2 is a factor of f_1 and φ is an isomorphism, then we say that the dynamical systems f_1 and f_2 are *isomorphic* (also called measure theoretically isomorphic).
- A nonsingular surjective measurable map $f : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is an *automorphism of* X if there exists $Y \in \mathcal{B}$ of full measure such that the restriction of f to Y is bijective (and $\mu f^{-1} \sim \mu$, but they are not necessarily equal). If f is not an automorphism, then we say f is *noninvertible.*

It was shown in [4] that even in the case of piecewise smooth interval maps, the notion of noninvertibility depends on the measure. We give the definition of an n-to-one map here.

Assume that $(X, \mathcal{B}, \mu, f)$ is a nonsingular dynamical system, not necessarily preserving μ. A partition $\mathcal{P}$ is an ordered countable (possibly finite) disjoint collection of nonempty measurable sets, called atoms, whose union is X ($\mu \bmod 0$). By a result of Rohlin [23] we obtain a partition $\mathcal{P} = \{A_1, A_2, A_3, \dots\}$ of X into at most countably many atoms and satisfying:

(1) $\mu(A_i) > 0$ for each i;
(2) the restriction of f to each A_i, which we will write as f_i, is one-to-one ($\mu \bmod 0$);
(3) each A_i is of maximal measure in $X \setminus \cup_{j<i} A_j$ with respect to property 2;
(4) f_1 is one-to-one and onto X ($\mu \bmod 0$) by numbering the atoms so that
$$\mu(fA_i) \geq \mu(fA_{i+1})$$
for $i \in \mathbb{N}$.

We call any partition $\mathcal{P}$ as defined above a *Rohlin partition* for f. When we say that a nonsingular dynamical system f is *n-to-one*, we mean that every Rohlin partition $\mathcal{P} = \{A_1, A_2, A_3, \dots\}$ satisfying $(1) - (4)$ contains precisely n atoms and that f_i is one-to-one and onto X ($\mu \bmod 0$) for each $i = 1, .., n$.

If f is noninvertible then every Rohlin partition $\mathcal{P}$ contains at least two atoms and generates a non trivial σ-algebra $\mathcal{F}$ such that $f^{-1}\mathcal{F} \subset \mathcal{F}$, namely the σ-algebra generated by

$$\mathcal{F}(\mathcal{P}) \equiv \bigvee_{i \geq 0} f^{-i}(\mathcal{P}). \tag{2.1}$$

The Rohlin partition is a *one-sided generating partition* if $\mathcal{F}(\mathcal{P}) = \mathcal{B}$ up to sets of μ measure 0.

We recall the definition of a Bernoulli shift. Because our emphasis is on noninvertible mappings, we begin with the one-sided Bernoulli shift.

Definition 2.2. Fix an integer $n \geq 2$ and let $\mathcal{A} = \{1, \dots, n\}$ denote a finite state space with discrete topology. Any vector $p = \{p_1, \dots, p_n\}$ such that $p_k > 0$ and $\sum p_k = 1$ determines a measure on $\mathcal{A}$, namely $p(\{k\}) = p_k$. Let $\Omega = \prod_{i=0}^{\infty} \mathcal{A}$ be the product space endowed with the product topology and product measure ρ determined by $\mathcal{A}$ and p. The map σ is the one-sided shift to the left, $(\sigma x)_i = x_{i+1}$. We say σ is a *one-sided Bernoulli shift* and denote it by $(\Omega, \mathcal{D}, \rho; \sigma)$, where $\mathcal{D}$ denotes the Borel σ-algebra generated by the cylinder sets, completed with respect to ρ. The cylinder sets of the form $\mathcal{C}_i = \{x \in \Omega : x_0 = i\}$ form an i.i.d.(independent identically distributed) generating partition for the dynamical system $(\Omega, \mathcal{B}, \rho, \sigma)$ in the sense that for any $k \in \mathbb{N}$,

$$\rho(\mathcal{C}_{i_0} \cap \sigma^{-1}\mathcal{C}_{i_1} \cdots \cap \sigma^{-k}\mathcal{C}_{i_k}) = \rho(\mathcal{C}_{i_0})\rho(\mathcal{C}_{i_1}) \cdots \rho(\mathcal{C}_{i_k}) = p_{i_0}p_{i_1} \cdots p_{i_k}$$

for all $i_j \in \mathcal{A}$, and sets of this form generate $\mathcal{B}$. Any dynamical system isomorphic to a n-to-one Bernoulli shift has a one-sided i.i.d. generating Rohlin partition containing n atoms.

Defining $\Omega_-^+ = \prod_{i=-\infty}^{\infty} \mathcal{A}$, and leaving everything else the same (with ρ the adjusted two-sided product measure), we say that $(\Omega_-^+, \mathcal{D}, \rho; \sigma)$, is an *invertible Bernoulli shift*.

An n-to-one nonsingular dynamical system $(X, \mathcal{B}, \mu, f)$ is said to be *one-sided Bernoulli* if it is isomorphic to some n-state one-sided Bernoulli shift dynamical system $(\Omega, \mathcal{D}, \rho; \sigma)$. One-sided Bernoulli dynamical systems exhibit well-known properties of Bernoulli shifts, such as ergodicity and exactness (defined below).

A sub-σ-algebra $\mathcal{B}_o \subset \mathcal{B}$ is *f-invariant* if $f^{-1}\mathcal{B}_o \subset \mathcal{B}_o$. It is well-known that every factor map gives rise to an f-invariant sub-σ-algebra, $\{\pi^{-1}C\}_{C \in \mathcal{B}_o} \subset \mathcal{B}$, and conversely. We refer the reader to Rohlin [23] for details.

2.1.1. *Ergodicity and exactness.* We adopt the usual convention that for sets $A, B \in \mathcal{B}$, $A \triangle B = (A \setminus B) \cup (B \setminus A)$. The map f is *ergodic* if f has a trivial field of invariant sets, or equivalently, if any measurable set B with the property that $\mu(B \triangle f^{-1}B) = 0$ has either zero or full measure. It follows from the definitions that f is conservative and ergodic if and only if for all sets $A, B \in \mathcal{B}_+$, there is a positive integer n such that $\mu(B \cap f^{-n}A) > 0$.

A map is *exact* if it has a trivial tail field $\cap_{n \geq 0} f^{-n}\mathcal{B} \subset \mathcal{B}$, or equivalently, if any set B with the property $\mu(f^{-n} \circ f^n(B) \triangle B) = 0$ for all n has either zero or full measure. For any set $A \in \mathcal{B}_+$, we define a tail set from it by:

$$\text{Tail}(A) := \cup_{n \in \mathbb{N}} f^{-n} \circ f^n(A).$$

Denoting the tail sets (μ mod 0) by $\mathcal{T} \subset \mathcal{B}$, we have $\cap_{n \geq 0} f^{-n}\mathcal{B} = \mathcal{T}$ (μ mod 0). An equivalent characterization is that f is exact if and only if for every $A \in \mathcal{B}_+$, $\lim_{n \to \infty} \mu(f^n(A)) = 1$ [24].

There is a natural map from $(X, \mathcal{B})$ onto $(X, \mathcal{T})$ which commutes with f, called the *exact decomposition* (of f with respect to μ), and f acts as an automorphism on the factor space. We denote the factor space by $(Y, \mathcal{C}, \nu)$, and the induced automorphism by S. Note that a point in Y is an atom of the measurable partition generated by the relation $x \sim w \iff f^n x = f^n w$ for some $n \in \mathbb{N}$ and ν is the factor measure induced by μ. We call this factor the *maximal automorphic factor*, because if there is a factor map $\varphi : X \to Z$ with induced factor automorphism R, then R is a factor of S. We remark that in general $(Y, \mathcal{C}, S, \nu)$ is a nonsingular surjective map of a Lebesgue space with no specified topology; details of this appear in [23].

2.2. **Some dynamical conventions.** We establish some notation for this paper. For any nonsingular dynamical system $(X, \mathcal{B}, \mu, f)$, by f^k we mean $f \circ f \cdots \circ f$ (k-fold composition). We use the notation $f^{\times k}$ to denote the Cartesian product of k copies of f, with $X^k = X \times \cdots \times X$ (k copies), so $f^{\times k} : X^k \to X^k$ is defined by: $f^{\times k}(x_1, \ldots, x_k) = (f(x_1), \ldots, f(x_k))$, with $x_i \in X$, $i = 1, \ldots, k$.

For any map f, the (positive) *orbit* of $x \in X$ is the set: given by $\mathcal{O}_+(x) = \{f^k(x)\}_{k \in \mathbb{N}}$.

If X is endowed with a Borel structure and a Borel measure μ, then on X^k we use the k-fold product measure, denoted μ^k, using the given measure μ on each copy of X. With respect to the product topology on X^k, μ^k is a Borel measure.

Definition 2.3. Let $(X, \mathcal{B}, \mu, f)$ denote a dynamical system, and consider $x_0 \in X$. We say x_0 is a *periodic point* of f if there exists some $m \in \mathbb{N}$ such that $f^m(x_0) = x_0$. If $m = 1$, then $f(x_0) = x_0$ and we say x_0 is a *fixed point*. The minimum m for which $f^m(x_0) = x_0$ is the *period* of x_0, and $\{x_0, f(x_0), \cdots, f^{m-1}(x_0)\}$ is called a *periodic cycle*.

2.3. **Topological dynamics.** Throughout this section we assume (X, δ) is a compact metric space and $f : X \to X$ is continuous (and hence Borel measurable).

Definition 2.4. We say f is:

(1) *topologically transitive* if for any nonempty open sets $U, V \subseteq X$, there exists an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$; equivalently, f is topologically transitive if there exists a point $x \in X$ such that $\overline{\mathcal{O}_+(x)} = X$.
(2) *topologically weak mixing* if $f^{\times 2}$ is topologically transitive.
(3) *topologically exact* if for every nonempty open set $U \subset X$ there exists $n \in \mathbb{N}$ such that $f^n(U) = X$. This property is also called *locally eventually onto* (l.e.o.) and *finite full*.

It is easy to establish that (3) $\Rightarrow$ (2) $\Rightarrow$ (1), but none of the reverse implications holds. There are many notions of chaotic behavior for a continuous map, but here we work with the following definition, usually known as chaos or Devaney chaos ([7] and cf. [20]).

Definition 2.5. For (X, δ) a compact metric space, and $D \subset X$ a closed infinite set, a continuous map $f : X \to X$ is *chaotic (on D)* if $f(D) \subset D$ and the following hold:

(1) $f|_D$ is topologically transitive;
(2) periodic points are dense in D

Since we assume throughout that X and D are infinite, chaotic maps also exhibit sensitive dependence on initial conditions, which is sometimes included in the definition (see eg., [14] for discussion and references.)

Definition 2.6. Let $(X_1, \mathcal{B}_1, \mu_1, f_1)$ and $(X_2, \mathcal{B}_2, \mu_2, f_2)$ be two dynamical systems. If $\varphi : X_1 \to X_2$ is a continuous surjective map such that $f_2 \circ \varphi(x) = \varphi \circ f_1(x)$ for all $x \in X_1$, we say f_1 and f_2 are *(topologically) semi-conjugate.* If φ is a homeomorphism, then f_1 and f_2 are said to be (topologically) *conjugate*. If X_2 has the quotient topology, φ is called a *quotient map* or a *(topological) factor map.*

Definition 2.7. Assume $f : X \to X$ is a topological dynamical system. A point $x \in X$ is *nonwandering* if for each neighborhood U of x there exists some $n \geq 1$ such that $f_s^n(U) \cap U \neq \emptyset$. The nonwandering set $\Omega(f) \subset X$ is the set of all nonwandering points.

One can see that $\Omega(f)$ is closed and $f^{-1}(\Omega(f)) = \Omega(f)$.

2.4. **The classification of surfaces.** Throughout this paper a surface refers to a Hausdorff topological space X, such that each point $x \in X$ has a neighborhood U homeomorphic to an open disk in $\mathbb{R}^2$. We also assume X is endowed with a C^r differential structure, for some $r \geq 1$. We refer to X as a smooth surface. We recall here the well-known classification theorem and assume the reader has some familiarity with the terms used (see eg. [12] for details).

Theorem 2.8. Every compact connected smooth surface is diffeomorphic to a sphere, a connected sum of n tori, or a connected sum of n projective planes.

The smooth structure on a surface X defines a measure class which is invariant under differentiable maps and maps that are differentiable on sets of full measure. This follows from the fact that there is a collection of Borel sets $\mathfrak{N} \subset \mathcal{B}$ such that for any smooth local chart (U, φ) on X, the set $B \in \mathfrak{N}$ satisfies: $\varphi(U \cap B) \subset \mathbb{R}^2$ has Lebesgue measure 0. The sets in $\mathfrak{N}$ are the null sets. If $(X, \mathcal{B})$ is a surface the σ-algebra of Borel sets, a measure μ is *absolutely continuous* if in any smooth local chart μ is given by integrating a non-negative density function. When the density function is strictly positive we sometimes say μ is equivalent to Lebesgue measure by a slight abuse of notation.

We review some common conventions used in the next sections. Letting $\mathbb{S}^1$ denote the 1-dimensional circle, we set $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, which we also frequently denote additively as $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The term *disk* always refers to a set diffeomorphic to the open unit disk in $\mathbb{R}^2$.

The genus of $\mathbb{T}^2$ is one, (so $\mathbb{T}^2 = 1\mathbb{T}$), and by $n\mathbb{T}$ we denote the compact orientable surface of genus n; i.e., the connected sum of n tori. Let $N = \mathbb{T}^2 - \{\text{disk}\}$. We can decompose $n\mathbb{T}$ into n copies of N, labeled $N_0, N_1, \ldots N_{n-1}$ as shown in Figure 1, glued together so that $\cap_{i=0}^{n-1} N_i = \{A, B\}$ (exactly two points). We can also view $n\mathbb{T}$ as a branched covering of the torus $\mathbb{T}^2$ with branch points A and B (see [22] for its use to construct surface homeomorphisms), but we take a different approach here. For each $i =$

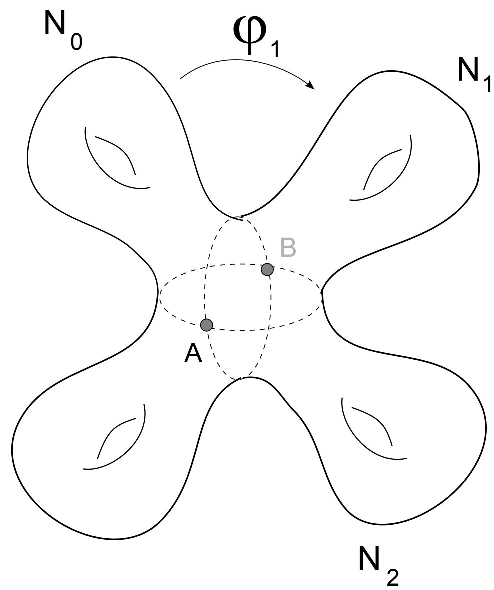


FIGURE 1. Construction of N_i in $4T$

$0,\dots,n-1$, N_i is diffeomorphic to N on its interior. Moreover, the boundary of N_i, ∂N_i is homeomorphic to $\mathbb{S}^1$ via a map which is a diffeomorphism except at exactly two points (see Figure 1).

After embedding the N_i's in $\mathbb{R}^3$ we define some homeomorphisms $\varphi_i : N_0 \to N_i$, $i = 0,\dots,n$, with $\varphi_0 = \varphi_n$, to be rotations about the line in $\mathbb{R}^3$ through the points A and B as shown in Figure 1. Clearly each φ_i is a diffeomorphism except at A and B. Later on it becomes useful to define φ_k for every $k \in \mathbb{N}$ by writing $k \equiv i (\bmod n)$, and defining $\varphi_k : N_0 \to N_i$ by just setting $\varphi_k = \varphi_i$.

A similar construction is used for nonorientable surfaces. We begin by letting $\mathbb{P}$ denote the two-dimensional real projective plane. We let $\mathbb{M}$ denote the Mobius band, the compact nonorientable surface diffeomorphic to $\mathbb{P} \setminus \{\text{disk}\}$, with $\partial\mathbb{M}$ diffeomorphic to $\mathbb{S}^1$. If we write $n\mathbb{P}$ for the connected sum of n projective planes, $n \geq 2$, we can decompose $n\mathbb{P}$ into n homeomorphic copies of $\mathbb{M}$, labeled $M_0, M_1, \dots M_{n-1}$, with boundaries identified so that $\cap_{i=0}^{n-1} M_i = \{A, B\}$.

While the nonorientable surface $n\mathbb{P}$ does not embed in $\mathbb{R}^3$, we can adapt the construction above to obtain analogs of the φ_i maps. Let U be a small neighborhood of $\cup_{i=0}^{n-1} \partial M_i \subset n\mathbb{P}$. The neighborhood U can be imbedded in $\mathbb{R}^3$ as illustrated in Figure 2, and we again define some maps $\varphi_i : \mathbb{M}_0 \to \mathbb{M}_i$, $i = 0,\dots,n-1$, and set $\varphi_n = \varphi_0$. The φ_i's restricted to U can be defined by rotations about a line through A and B just as in the orientable case. Each φ_i can then be extended in an obvious way to all of M_0, and is a diffeomorphism except at the points A and B.

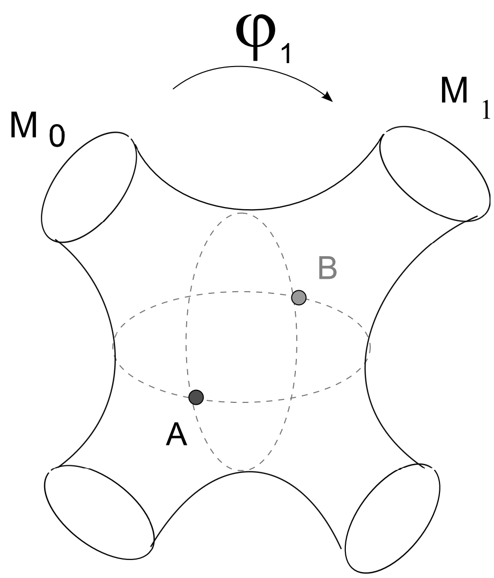


FIGURE 2. The local picture of of $\mathbb{M}_i$ in $4\mathbb{P}$

Finally we note that the sphere $\mathbb{S}^2$ is frequently viewed as the Riemann sphere, denoted $\mathbb{C}_\infty$, giving $\mathbb{S}^2$ an analytic structure as well as a smooth one.

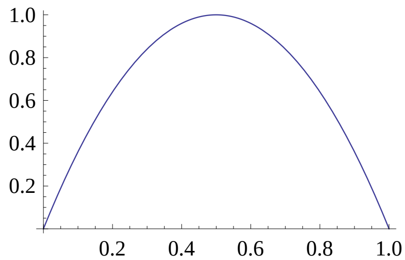
We endow every surface X with a Borel structure by letting open sets generate the σ-algebra of measurable sets, and we use m to denote a measure which is equivalent to Lebesgue measure in every coordinate chart. Since the surface is at least C^1 it has a Riemannian metric and the measure m has a locally differentiable description.

3. The Basic Examples of One-sided Bernoulli Maps

We begin with a list of examples of classical one-sided Bernoulli maps of one and two-dimensional manifolds. In each case, the measure used is equivalent to Lebesgue measure on X. These provide some of the basic building blocks for constructing maps on surfaces later in this paper.

Example 3.1. On each manifold we use a smooth measure determined by the dimension of the manifold. Let $d > 1$ be an integer.

(1) $f_1(x) = dx \bmod 1$ on $X = [0, 1)$ or $X = [0, 1]/0 \sim 1 \cong \mathbb{R}/\mathbb{Z}$. f_1 is d-to-1.
(2) $f_2(z) = z^d$ on $X = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. f_2 is d-to-1.
(3) $f_3(x) = f_2^{\times 2}$ on $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. f_3 is d^2-to-one.
(4) Viewing the torus $\mathbb{T}^2$ as a 2-fold branched covering of $\mathbb{C}_\infty$, for each f_3 above we obtain a rational map $f_4 : \mathbb{C}_\infty \to \mathbb{C}_\infty$, which is d^2-to-one.
(5) For any $d \in \mathbb{N}$, $d \geq 2$, there exist d^2-to-one and $2d^2$-to-one (one-sided) Bernoulli rational maps of $\mathbb{C}_\infty \cong \mathbb{S}^2$ [2].



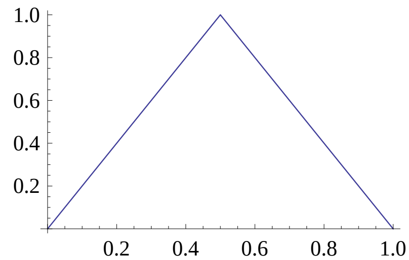


FIGURE 3. Two one-sided Bernoulli maps with extra symmetry

(6) On $X = [0,1]$, we consider the map from the logistic family given by: $f_6(x) = 4x(1-x)$. This gives a map with the property that for all $x \in X$ $f_6(x) = f_6(1-x)$; the full tent map f_7, with slopes ± 2 also has the same symmetry. f_6 and f_7 are 2-to-one and are shown in Figure 3.

All of the maps f_i above, $i = 1, \dots, 7$, share the following properties.

Proposition 3.2. If m denotes smooth measure on X, with respect to some invariant probability measure $\mu \sim m$, each $(X, \mathcal{B}, \mu, f_i)$ satisfies:

(a) f_i is isomorphic to a one-sided (noninvertible) Bernoulli shift;
(b) $f_i^{\times k}$ is ergodic and exact on X^k for each $k \in \mathbb{N}$;
(c) $f_i^{\times k}$ is one-sided Bernoulli on X^k.

Proof. Properties (a)-(c) of the examples given in 3.1 are well known, but we give brief explanations here. For the map f_1, using the base d expansion of a number $x \in [0,1)$, and dividing the interval into intervals $A_j = [j/d, (j+1)/d)$, for $j = 1, 2, \dots, d$ gives a d-to-1 Rohlin partition for f_1. The map $\varphi(x) = \varphi(.x_0x_1x_2\dots) = \{x_0, x_1, x_2, \dots\} \in \Omega$ implements the isomorphism to the one-sided $(1/d, 1/d, \cdots, 1/d)$ Bernoulli shift using Lebesgue measure. The map f_2 is clearly conjugate to f_1 via the map $\exp : [0,1] \to \mathbb{S}^1$, $\exp(t) = e^{2\pi it}$ since for $d \in \mathbb{N}, d \geq 2$,

$$\exp(f_1(t)) = e^{2\pi idt} = (e^{2\pi it})^d = f_2(\exp(t)).$$

The map f_3 is implemented by the linear transformation $A(x,y) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$, which is well-known to be isomorphic to the $(1/d^2, 1/d^2, \dots, 1/d^2)$ one- sided Bernoulli map. The maps in (5) come from classical Lattès examples, and explicit isomorphisms to one-sided Bernoulli shifts are constructed in [2]. Finally the maps f_6 and f_7 are well-known to be isomorphic to f_1 (see eg., [4, 7]).

Proof of (b): We first show that $f_i^{\times k}$ is ergodic. Let $\mathcal{B}^k$ denote the σ-algebra of Borel sets on X^k. We fix an i and k, and set $F = f_i^{\times k}$. f_i is weak mixing if and only if $f_i \times f_i$ is ergodic and weak mixing, and f_i Bernoulli implies it is mixing, which implies $f_i \times g$ is ergodic for every measure preserving transformation g of a measure space $(Y, \mathcal{F}, \nu)$ (cf. also [9]). Using induction on k, the ergodicity of F follows. The exactness follows from the fact that f_i is Bernoulli, so we turn to the proof of (c).

Assume $f_i = \sigma$ is an d-to-one Bernoulli shift, $(\Omega, \mathcal{D}, \rho; \sigma)$, and set $k = 2$. We consider the alphabet $\mathcal{A}_2$ on d^2 symbols labeled by pairs with $\mathcal{A}_2 = \{(1,1), (1,2), \cdots, (1,d), (2,1), (2,2), \cdots, (d,d)\}$. Given the generating partition for $\mathcal{D}$ given by: $C_j = \{\omega \in \Omega : \omega_0 = j\}$, we now consider the sets $C^{ij} = C_i \times C_j \in \Omega^2$, $i, j = 1, \ldots, d$. We have $\mu^2(C^{ij}) = \mu(C_i)\mu(C_j) = p_i p_j$. In this way we construct a generating i.i.d. partition for the Bernoulli measure with probability distribution: $q = \{q_{ij}\}$, with $q_{ij} = p_i p_j$, $i, j = 1, \ldots, d$. The shift map $\sigma \times \sigma(\omega, \zeta)$ is the obvious shift map defined by:

$$[(\sigma \times \sigma)(\omega, \zeta)]_i = (\omega_{i+1}, \zeta_{i+1}).$$

This makes the 2-fold product into a one-sided Bernoulli shift. To show the result on the k-fold product, we use induction on k. For the inductive step we take the 2-fold product of a d^{k-1} state one-sided Bernoulli shift with a d-to-one Bernoulli shift and proceed as above. □

The result above leads to a more general result, whose short proof we give here.

Proposition 3.3. Let $(X, \mathcal{B}, \mu, f)$ and $(Y, \mathcal{F}, \nu, g)$ be finite measure preserving exact dynamical systems. Then $(X \times Y, \mathcal{B} \times \mathcal{F}, \mu \times \nu, f \times g)$ is measure theoretically exact as well, and hence ergodic.

Proof. If we consider any set of the form $A \times B$, with $\mu(A) > 0$ and $\nu(B) > 0$, then it follows immediately that $(\mu \times \nu)(\mathrm{Tail}(A \times B)) = 1$, because each of f and g are exact. Let $\mathcal{C} = \{C \in \mathcal{B} \times \mathcal{F} : (\mu \times \nu(\mathrm{Tail}(C)) = 1\}$. It is easy to see that $\mathcal{C}$ is a monotone class, and contains finite unions of rectangles, which generate $\mathcal{B} \times \mathcal{F}$, so it follows from standard measure theoretic techniques (see eg., [26]) that $\mathcal{C} = \mathcal{B} \times \mathcal{F}$. □

We also have a topological version of Proposition 3.2 whose proof is standard (see eg., [14]).

Proposition 3.4. Each $(X, \mathcal{B}, \mu, f_i)$ satisfies:

(a) f_i is topologically exact and chaotic.
(b) $f_i^{\times k}$ is topologically exact and chaotic.
(c) If f_i is j-to-one, then the topological entropy, $h_{top}(f_i)$, is $\log j$.

3.1. **One-sided Bernoulli maps of some nonorientable surfaces.** We begin by constructing one-sided Bernoulli maps for a few basic nonorientable surfaces.

3.1.1. *The Mobius band* $\mathbb{M}$*, Klein bottle* $\mathbb{K}$*, and real projective plane* $\mathbb{P}$. We begin with interval maps; on $I = [0,1]$, we consider the map from the logistic family given by: $g(x) := f_6(x) = 4x(1-x)$; we could just as well use the tent map f_7 in what follows. As is clear from the graph in Figure 3 (or the equation), for all $x \in I$,

$$g(x) = g(1-x). \tag{3.1}$$

On $I \times I$ we define $G(x,y) = (g(x), g(y))$, and we show it extends to a well-defined map of each of $\mathbb{M}$, $\mathbb{K}$ and $\mathbb{P}$, using the identifications given below in Figures 4 and 5. Using f_6, differentiability fails at the point $0 = 1$ on $\mathbb{S}^1$, and f_7 fails to be differentiable at $x = \frac{1}{2}$ and $x = 0 = 1$. In each case we have finitely many intersecting smooth curves on $\mathbb{M}$, $\mathbb{K}$ and $\mathbb{P}$ where G fails to be differentiable.

Using (3.1), we have for each $x, y \in I$

$$G(0,y) = G(0,1-y) = (0, g(y)) \ = \ G(1,y) = G(1,1-y) \tag{3.2}$$

$$G(x,0) = G(1-x,0) = (g(x),0) \ = \ G(x,1) = G(1-x,1) \tag{3.3}$$

$$G(0,y) = G(1,y) = (0,g(y)) \ \text{ and } \ G(x,0) = G(x,1) = (g(x),0) \tag{3.4}$$

Now we identify points on the boundary of $I \times I$ in the classical way described below to obtain these basic nonorientable surfaces:

(1) $\mathbb{M}$: $(0,y) \sim (1,1-y)$;
(2) $\mathbb{P}$: $(0,y) \sim (1,1-y)$ and $(x,0) \sim (1-x,1)$
(3) $\mathbb{K}$: $(0,y) \sim (1,1-y)$, and $(x,0) \sim (x,1)$

We then have the following result.

Theorem 3.5. There exist Lipschitz one-sided Bernoulli maps of $\mathbb{M}$, $\mathbb{P}$, and $\mathbb{K}$, which are smooth except on one smooth curve on $\mathbb{M}$, and on $\mathbb{P}$ and $\mathbb{K}$, on two smooth curves intersecting in one point.

Proof. We use Equation (3.2) to see that G is well-defined on the quotient space $\mathbb{M}$, and Equations (3.2) and (3.3) to see that G is well-defined on $\mathbb{P}$; we use Equations (3.2) and (3.4) to see that G is well-defined on $\mathbb{K}$. Since we have only made identifications on a set of measure 0, we have not changed properties of the maps, and Proposition 3.2 holds. The maps are smooth on the interior of $I \times I$, and one-sided limits exist for $g'(x)$ as x approach the boundaries of I. However the one-sided limits do not agree, so differentiability fails at the identified sides.

$\square$

3.2. **Maps of symmetric products.** Our goal is to extend maps of simple surfaces to other surfaces retaining as much of the chaotic behavior as possible, both topological and measure theoretic. To move to arbitrary nonorientable surfaces, we need to construct one-sided Bernoulli maps of $\mathbb{M}$ with some additional symmetry needed for our topological construction.

We extend an idea to use symmetric products from [14], applied to the measurable setting. We note from the outset that there are two distinct but related topological constructions called *symmetric products*. In dimension 2 the definitions agree so we provide only the definition used in [14],which was

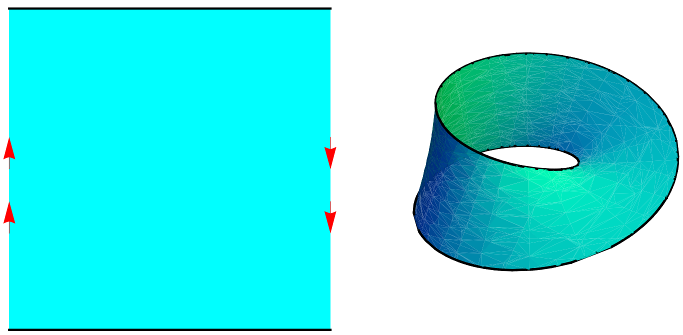


FIGURE 4. The Mobius band

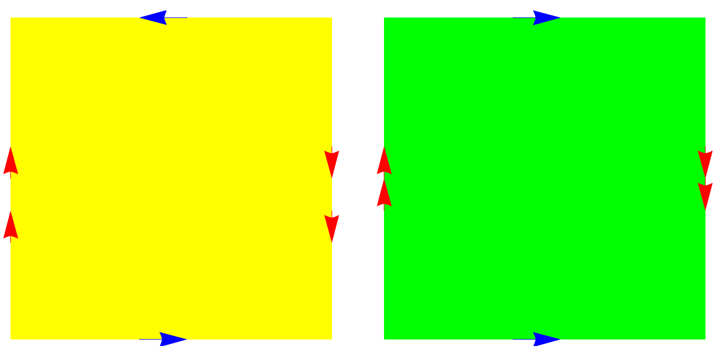


FIGURE 5. The real projective plane and Klein bottle

introduced by Borsuk and Ulam in 1931 [5]; the other definition appears in [21].

Assume (X, δ) is a bounded and connected metric space, and define the *hyperspace of* X, denoted 2^X, to be the collection of all nonempty compact subsets of X. The space 2^X inherits a metric from X as follows. Given $A \in 2^X$, and $\varepsilon > 0$, we define the ε-neighborhood of A, denoted $N_\varepsilon(A)$ by:

$$N_\varepsilon(A) = \{x \in X \,:\, \inf\{\delta(x, y) \,:\, y \in A\} < \varepsilon\},$$

which gives rise to the Hausdorff metric:

$$\delta_H(A, B) = \inf\{\varepsilon \geq 0 \,:\, A \subset N_\varepsilon(B) \,\text{and}\, B \subset N_\varepsilon(A)\}$$

for all $A, B \subset 2^X$. If X is compact and connected, this makes $(2^X, \delta_H)$ into a compact connected metric space.

Definition 3.6. The *k-fold symmetric product*, denoted X^{*k} is the subset of 2^X consisting of all nonempty subsets of X containing at most k points.

Clearly $X \subset X^{*1} \subset X^{*k}$ for $k \geq 2$ since each $x \in X$ forms a one-point subset. A point in X^{*2} consists of either an unordered pair $\{x, y\}$ with $x, y \in X$, $x \neq y$, or a single point $x \in X$. For any continuous map

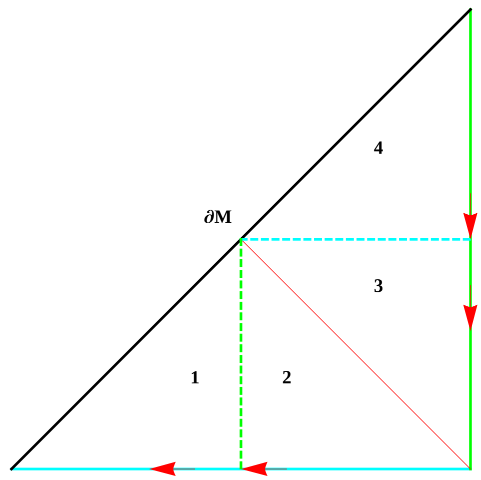


FIGURE 6. A Bernoulli partition for f^{*2} on $\mathbb{M}$ if $f(z) = z^2$. Dotted lines map onto the solid lines of corresponding color.

$f : X \to X$ we can define a map f^{*k} in a natural way. Namely for $A \subset X^{*k}$, define $f^{*k}(A) = f(A)$ (this is just the map f applied to a set in X). Clearly f^{*k} is a topological factor (quotient map) of the map $f^{\times k}$, and each fiber in the factor map π contains finitely many points in $X^{\times k}$. In particular we can define a continuous map $\pi : X^k \to X^{*k}$ such that the diagram commutes by $\pi(x_1, x_2, \ldots, x_k) = \{x_1, x_2, \ldots, x_k\}$.

$$\begin{array}{ccc} X^k & \xrightarrow{f^{\times k}} & X^k \\ \downarrow \pi & & \downarrow \pi \\ X^{*k} & \xrightarrow{f^{*k}} & X^{*k} \end{array} \tag{3.5}$$

Moreover if we put a Borel measure μ^k on $X^{\times k}$ then we have an induced measure structure on X^{*k} and the measure μ^{*k} is preserved by f^{*k} if f preserves μ on X; i.e., f^{*k} is a measurable factor.

Given any integer $d \geq 1$, we consider the map $f(z) = z^d$ on $\mathbb{S}^1$, which induces a one-sided Bernoulli map on f^{*2} on $(\mathbb{S}^1)^{*2}$. The smooth structure on $(\mathbb{S}^1)^{\times 2}$ induces one on $(\mathbb{S}^1)^{*2}$, which makes it diffeomorphic to $\mathbb{M}$ (see [14]). which is d^2-to-one with d^2 distinct preimages for m^{*2} a.e. $x \in \mathbb{M}$. In Figure 6 we show the Mobius band realized as $(\mathbb{S}^1)^{*2}$ (we actually show I^{*2} so the identification of the sides as shown is needed to get $\mathbb{M}$). Take $f(z) = z^2$, so that f^{*2} is 4-to-one; in Figure 6 we show four fundamental regions for the map f^{*2}; the interior of each region maps injectively onto the interior of $\mathbb{M}$. These are atoms of a generating i.i.d. Rohlin partition. Moreover, the map f^{*2} is smooth on $\mathbb{M}$ and preserves the factor measure m^{*2} induced by $m \times m$.

The following proposition summarizes the properties of f^{*2}.

Proposition 3.7. If $f(z) = z^d$ for some integer $d \geq 2$, then the dynamical system $(X^{*2}, \mathcal{B}^{*2}, m^{*2}, f^{*2})$ is:

(1) smooth
(2) ergodic
(3) chaotic
(4) topologically exact
(5) exact with respect to m^{*2}
(6) one-sided Bernoulli on d^2 states
(7) $h_{top}(f^{*2}) = 2 \log d$

We now use these maps to construct Lebesgue ergodic and chaotic continuous maps of arbitrary nonorientable surfaces. If we define $g^* = f^{*2} : \mathbb{M} \to \mathbb{M}$, then we have some symmetries worth noting. The one point sets (the diagonal in Figure 6) behave as follows: using additive notation on I (i.e., identifying $z = e^{2\pi i\theta}$ on $\mathbb{S}^1$ with $\theta \in I$) to match what is shown in Figure 6,

$$\begin{aligned} g^*(\{\theta\}) &= g^*(\theta, \theta) = (f(\theta), f(\theta)) \\ &= (2\theta, 2\theta) \mod 1 = -(f(-\theta), f(-\theta)) = -g^*(\{1-\theta\}) \end{aligned} \tag{3.6}$$

4. Extending the examples to nonorientable surfaces

We want to construct ergodic and chaotic maps on nonorientable compact surfaces of genus > 1. As discussed in Section 2.4, it is equivalent to consider connected sums of $\mathbb{P}$.

We fix an integer $d > 1$ and consider $g^* : M_0 \to M_0$ to be the map constructed in Section 3.2, coming from the map $z \mapsto z^d$ on $\mathbb{S}^1$. The map g^* is d^2-to-one on M_0. Then the map defined for $x \in M_i$ by:

$$G(x) = \varphi_{i+1} \circ g^* \circ \varphi_i^{-1}(x), \quad \text{for } i = 0, \ldots, n-1 \tag{4.1}$$

is clearly well-defined away from ∂M_i. It remains to check that if $x \in n\mathbb{P}$ satisfies $x \in \partial M_i$ and $x \in \partial M_{i-1}$, then $G(x)$ is uniquely defined. Equivalently, we need to verify that for all $x \in \partial M_{i-1} \cap \partial M_i$,

$$\varphi_{i+1} \circ g^* \circ \varphi_i^{-1}(x) = \varphi_i \circ g^* \circ \varphi_{i-1}^{-1}(x) \tag{4.2}$$

In the decomposition illustrated in Figure 2, we can choose (and label) points $A = \{0\} = (0,0)$ and $B = \{1/2\} = (1/2, 1/2)$ from Figure 6. A point $x \in \partial M_i$ is of the form $\theta_x = \varphi_i^{-1}(x) \in \partial M_0$. This corresponds to a point making an angle of θ_x with A; similarly $\varphi_{i-1}^{-1}(x) \in \partial M_0$ corresponds to a point making an angle $-\theta_x$ with A. Then Eqn (3.6) shows that $g^*(\varphi_i^{-1}(x))$ and $g^*(\varphi_{i-1}^{-1}(x))$ also have opposite angles since $g^*(-\theta_x) = -g^*(\theta_x) \mod 1$. Therefore $G(x) = \varphi_{i+1} \circ g^* \circ \varphi_i^{-1}(x) = \varphi_i \circ g^* \circ \varphi_{i-1}^{-1}(x)$ is well-defined for every point $x \in X$. This is illustrated in Figure 7. We note that the point A is fixed for G, and $G(B) = A$.

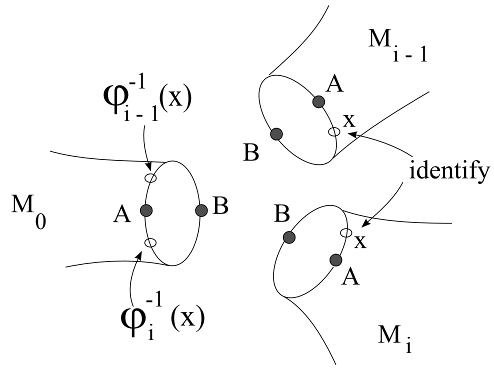


FIGURE 7. Using the symmetry of g^* so G is well-defined

An easy inductive argument shows that to iterate G, if $x \in M_i$, then for any $k \in \mathbb{N}$ we can write:

$$G^k(x) = \varphi_{i+k} \circ (g^*)^k \circ \varphi_i^{-1}(x), \quad \text{for } i = 0, \dots, n-1 \tag{4.3}$$

using the convention given earlier that $\varphi_{i+k} = \varphi_{i+k(\bmod n)}$.

Our construction leads to the following result.

Theorem 4.1. Given any nonorientable compact surface X of genus ≥ 2, and $d \in \mathbb{N}$, $d \geq 2$, there exists a map $G : X \to X$ which is locally Lipschitz on X (Lipschitz in each coordinate chart), continuous, and smooth except at two points, and satisfying:

(i) G preserves a smooth probability measure m_n on X.
(ii) G is ergodic with respect to m_n, but is not exact.
(iii) G is isomorphic to an n-point extension of a one-sided Bernoulli shift
(iv) G is transitive and chaotic, but not topologically exact.
(v) $h_{top}(G) = 2 \log d$.

Proof. Without loss of generality we set $X = \cup_{i=0}^{n-1} M_i = n\mathbb{P}$, noting that the union is not disjoint as shown in Figure 1. Since $g^* : M_0 \to M_0$ preserves m^{*2}, for any measurable $C \subset M_i$, we set

$$m_i^{*2}(C) = m^{*2}(\varphi_i^{-1} C); \tag{4.4}$$

m_i^{*2} is a probability measure supported on M_i. We now define a probability measure $m_n = \frac{1}{n} \sum_{i=0}^{n-1} m_i^{*2}$ on X. Given an arbitrary Borel set $B \subset X$, write $B = \cup_{i=0}^{n-1} C_i$, where the C_i's are disjoint and each $C_i \subset M_i$. Since $m_n(M_i \cap M_j) = 0$ if $i \neq j$, the decomposition of B is unique only up to sets of measure 0 (because each C_i may contain points in $\partial M_i \cap \partial M_j$, for $j = i-1$ or $i+1$, which could just as well belong to C_j). For any $j = 0, \dots, n-1$, given $C_j \in \mathcal{B} \cap M_j$, $m_n(C_j) = \frac{1}{n} m^{*2}(\varphi_j^{-1} C_j)$. We define the map G as in equation 4.1, and therefore by definition we have that $G^{-1}(C_{j+1}) \subset M_j$.

Then by equation 4.4,

$$m_n(G^{-1}C_{j+1}) = \frac{1}{n}m^{*2}(\varphi_j^{-1}[\varphi_j\circ(g^{*2})^{-1}\circ\varphi_{j+1}(C_{j+1})]) = \frac{1}{n}m^{*2}((g^{*2})^{-1}(\varphi_{j+1}C_{j+1})),$$

and since g^{*2} preserves m^{*2},

$$m_n(G^{-1}C_{j+1}) = \frac{1}{n}m^{*2}(\varphi_{j+1}C_{j+1}) = m_n(C_{j+1}). \tag{4.5}$$

Note that the modification for $C_0 \subset M_0$ is obvious since $G^{-1}C_0 \subset M_{n-1}$. Since equation (4.5) holds for each C_{j+1}, $m_n(G^{-1}B) = m_n(B)$ for all $B \in \mathcal{B}$.

This proves (i).

Assume $G^{-1}B = B$, and $m_n(B) > 0$; then for all $k \in \mathbb{Z}$, $G^k(B) = B$, so $B \cap M_i$ has positive measure for all i. Let $B_j = B \cap M_j$; since $G^{-n}(B_j) = B_j$ and $(g^{*2})^n$ is ergodic on M_0, we have that $m_n(B_j) = \frac{1}{n}$; i.e., B_j fills M_j up to a set of measure 0. Since $G(B_j) \subset M_{j+1}$, it follows that $m_n(B) = 1$ and G is ergodic. Since for each j, $m_n(M_j) = 1/n$ and satisfies

$$M_j = \cup_{k\in\mathbb{N}} G^{-k}(G^k(M_j)),$$

we see that there are nontrivial tail sets, proving (ii).

To show (iii), consider the one-sided 4-state Bernoulli shift defined by g^{*2} on $\mathbb{M}$. We set $Y = \mathbb{M} \times \{0, 1, \ldots, n-1\}$, and give Y the product measure ν, using m^{*2} and uniformly distributed point mass measure on $\{0, 1, \ldots, n-1\}$. consider the map: $S(z, j) = (g^{*2}(z), j + 1(\mathrm{mod} n))$. Clearly $S : Y \to Y$, and S is an n-point extension over the Bernoulli map g^{*2}. We now define $\eta : X \to Y$ by $\eta(x) = (z, j)$ if $x \in M_j$ and $\varphi_j^{-1}(x) = z \in M_0$. Then η is a measure theoretic isomorphism, and $\eta \circ G(x) = S \circ \eta(x)$ for m_n almost every $x \in X$. This proves (iii).

The proofs of (iv) and (v) are similar to some given in [14], but we give a few details here in our setting. To show topological transitivity, it is enough to consider open sets $U \subset M_i$ and $V \subset M_j$. If we first project the sets onto $\mathbb{M}$, the topological exactness of g^{*2} on $\mathbb{M}$ gives a $k_0 \in \mathbb{N}$ such that $(g^{*2})^{k_0}(U) = \mathbb{M}$. Now use any $k \geq k_0$ for G to take U onto M_j. To show periodic points are dense, we use the corresponding property of g^{*2} on $\mathbb{M}$; if for example an arbitrary open set $U \subset M_0$ has a periodic point x of period p under g^{*2}, then for $\varphi_i(x) \in \varphi_i U \cap M_i$, satisfies $G^{np}(\varphi_i(x)) = \varphi(x)$ as well. G fails to be topologically exact for the same reason it fails to be exact.

The map G has entropy at least as great as that of g^{*2} since g^{*2} is a (topological) factor. But G is still only d^2-to-one so we have not increased the entropy.

Finally G is clearly continuous everywhere, and since g^{*2} is smooth on $\mathbb{M}$, with constant derivative mapping (viewed in local additive coordinates as $\begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$), we have only lost differentiability at the points A and B, so G is Lipschitz and piecewise expanding.

□

4.1. **Generalizations of the construction.** The construction of $g*$ and G on X is actually quite general and we mention a few extensions. First, we note that a similar construction would work for maps of $\mathbb{T}^2 \setminus \{\text{disk}\}$, with the same symmetry required on the boundary. Since constructing an ergodic or chaotic d-to-one map of $\mathbb{T}^2 \setminus \{\text{disk}\}$ with the appropriate boundary symmetry is difficult, we take a different approach for the orientable case in Section 5.

Moreover the technique used leads to the following proposition.

Theorem 4.2. Suppose $(\mathbb{S}^1, \mathcal{B}, m, f)$ is any nonsingular d-to-one dynamical system satisfying the following conditions:

(1) f is continuous on $\mathbb{S}^1$ and differentiable except at finitely many points;
(2) f is topologically exact;
(3) f is weak mixing;
(4) In additive coordinates, $f(1-x) = -f(x)$ for all $x \in [0,1]$.

Then for any nonorientable compact surface X of genus > 1, f defines a d^2-to-one nonsingular map G on X with respect to a smooth measure μ, is ergodic and chaotic, and G is continuous and differentiable μ-a.e.

Proof. We use the symmetric product f^{*2} to define an ergodic and chaotic map on $\mathbb{M}$. We then use the decomposition given in Section 2.4 to extend f^{*2} to G on X. □

We can also reduce the measure theoretic entropy of the maps constructed. Given any $p \in (0,1)$, $p \neq \frac{1}{2}$, set $q = 1 - p$. Then we consider the following piecewise affine map, defined in [4] that satisfies the hypotheses of Theorem 4.2. Define $T_p : \mathbb{S}^1 \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ as follows.

$$T_p(x) = \begin{cases} \frac{1}{p}\, x & \text{if } x \in [0, \frac{p}{2}) \\ \frac{1}{1-p}(x - \frac{p}{2}) + \frac{1}{2} & \text{if } x \in [\frac{p}{2}, \frac{1}{2}) \\ \frac{1}{1-p}(x - \frac{1}{2}) & \text{if } x \in [\frac{1}{2}, 1 - \frac{p}{2}) \\ \frac{1}{p}(x - (1 - \frac{P}{2})) + \frac{1}{2} & \text{if } x \in [1 - \frac{p}{2}, 1) \end{cases}$$

See Figure 8 for a graph of T_p.

Then T_p preserves m and is mixing and chaotic (but not one-sided Bernoulli [4]), and the measure theoretic entropy $h_m(T_p) = -p \log p - (1-p)\log(1-p)) < \log 2$. Varying the choice of p gives maps T_p, hence T_p^{*2} and then the corresponding G_p of arbitrarily small measure theoretic entropy.

5. Ergodic and chaotic dynamical systems on orientable surfaces

In this section our goal is to use a technique called blowing up a fixed point of a diffeomorphism to construct explicit examples of expanding ergodic and chaotic maps on any compact orientable surface. The technique we use is defined on diffeomorphisms, and we extend the ideas here to noninvertible

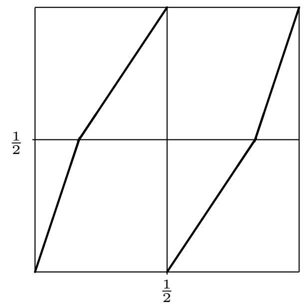


FIGURE 8. The graph of T_p with $p \approx \frac{1}{3}$

continuous maps with some differentiability. The technique forces us to restrict the chaotic behavior to a set of measure $1-\varepsilon$, where ε can be made as small as we want. However the resulting map is differentiable on a set of full measure. In particular we construct explicit examples of continuous maps on orientable surfaces of genus n, denoted $n\mathbb{T}$.

5.1. **Fixed points of many-to-one maps.** The blowup construction described below depends on the existence of a fixed point only having itself as its preimage, which can cause difficulties in the many-to-one setting. If a fixed point exists with no other preimages, this would imply that there exists a point x whose grand orbit, $\mathcal{O}^{\pm}(x) = \cup_{i=0}^{\infty} \cup_{j=0}^{\infty} F^{-j}(F^i x)$, is simply $\{x\}$. For a map exhibiting chaotic or ergodic behavior, this is rare, but not impossible. For example, it is classical (see eg. [3]) that any rational map of the sphere with a finite grand orbit is conjugate to $R(z) = z^d$, $d \geq 2$, and the point is either 0 or ∞, and in either case is (super)attracting. In the case of the torus $\mathbb{T}^2 = \mathbb{C}/\Lambda$ for some lattice Λ, holomorphic maps of degree $d \geq 2$ always have fixed points with d distinct preimages [18].

5.2. **An expanding piecewise smooth circle map.** We define the parametrized family of maps for each $\beta \in (\frac{1}{4}, \frac{1}{2})$. We set $s = 1 + 4\beta$, and $\alpha = \beta/s$; note that with the given interval chosen for β, we have $\alpha \in (\frac{1}{8}, \frac{1}{6})$, and $s \in (2, 3)$. Then we set $(\mathbb{S}^1, \mathcal{B}, m; f_s)$, where

$$f_s(x) = \begin{cases} s\,x & \text{for } x \in [0, \alpha) \\ -s(x-1/2) - 1/2 & \text{for } x \in [\alpha, 2\alpha) \\ -s(x-1/2) + 1/2 & \text{for } x \in [2\alpha, 1-2\alpha) \\ -s(x-1/2) + 3/2 & \text{for } x \in [1-2\alpha, 1-\alpha) \\ s(x-1) + 1 & \text{for } x \in [1-\alpha, 1] \end{cases}$$

with $\mathcal{B}$ the σ-algebra of Lebesgue measurable sets. Each map has constant slope s and defines a circle map as shown in Figure 9

Let $F_s : \mathbb{R} \to \mathbb{R}$ denote the lift of f_s. Since $F_s(0) = 1$, and $F_s(1) = 0$, we have that $\deg(f_s) = -1$ as shown in Figure 10. It also has periodic orbits of period 3; therefore by ([1], Thm 4.4.20) $h_{top}(f_s) > 0$. Since f_s is expanding with $|f_s'(x)| = s$, it follows that $h_{top}(f_s) = \log s$ [19].

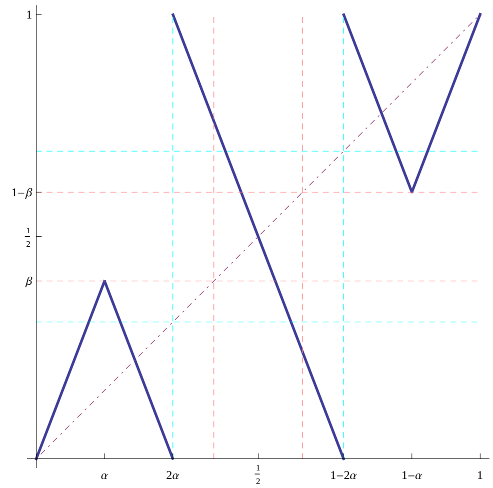


FIGURE 9. An expanding circle map with slope $\pm(1+4\beta)$

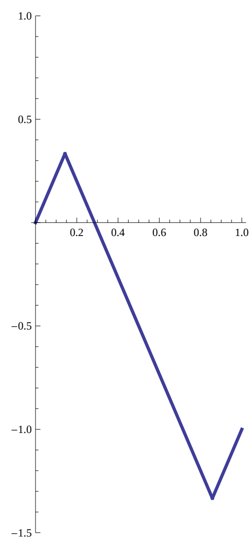


FIGURE 10. The lift to $\mathbb{R}$ of a degree -1 map with slope $s = \pm(1+4\beta)$

We have the following properties of the map. Many of these properties are classical properties of piecewise monotone interval maps (see eg. [17]).

Theorem 5.1. For each $s \in (2,3)$ or equivalently for each $\beta \in (\frac{1}{4}, \frac{1}{2})$, the following hold.

(1) $f_s(1-x) = 1 - f_s(x)$ for all $x \in [0, 1/2]$. In particular, $f_s(1/2) = 1/2$, so $p = \frac{1}{2}$ is a repelling fixed point.
(2) The nonwandering set $\Omega(f_s) = [0,1] \setminus (\beta, 1-\beta)$,

(3) on $\Omega(f_s)$, f_s is exactly 3-to-one except at the turning points (α, β) and $(1-\alpha, 1-\beta)$,
(4) there exists an absolutely continuous invariant probability measure $\mu \ll m$, supported on $\Omega(f_s)$.
(5) f_s is ergodic with respect to m and μ, and weakly Bernoulli, hence exact w.r.t. μ.
(6) Writing $\tilde{f}_s = f_s|_{\Omega(f_s)}$, $\tilde{f}_s$ is transitive, topologically exact, and chaotic.
(7) $\tilde{f}_s$ is weakly Bernoulli but is not one-sided Bernoulli.

Proof. (1) is easy to verify. Each open interval in $[0, \alpha]$ and $[\alpha, 2\alpha]$ is mapped onto $[0, \beta]$ after finitely many iterations of f_s. Points in $[2\alpha, \beta]$ are mapped by f_s diffeomorphically into the interval $[1-2\alpha, 1]$. By the symmetry in (1), any interval in $[1-2\alpha, 1]$ is mapped onto $[1-\beta, 1]$. The subinterval $[1-\beta, 1-\alpha]$ is mapped diffeomorphically onto an interval in $[0, \alpha]$. Therefore any open interval containing a point in $\Omega(f_s) = [0, 2\alpha] \cup [2\alpha, \beta] \cup [1-\beta, 1-2\alpha] \cup [1-2\alpha, 1]$ is mapped eventually onto $\Omega(f_s)$. (3) and (4) follow from the Folklore Theorem for expanding maps (see V. Thm 2.1 of [17]). The topological exactness follows from the proof of (1); transitivity follows from topological exactness, and Devaney chaos follows from the transtivity [15], and also from the positive entropy [19]. (7) Since $h_{top}(\tilde{f}_s) < \log 3$, it is not isomorphic to the $\{1/3, 1/3, 1/3\}$ one-sided Bernoulli shift. Then it would be a $\{p_1, p_2, p_3\}$ shift, and the fact that the automorphism $\varphi(x) = 1 - x$ commutes with $\tilde{f}_s$ makes this impossible ([4], Cor. 2.23). □

Remarks 5.2. We can do a similar construction with a degree 1 circle map by reflecting the graph of f_s across the line $x = \frac{1}{2}$,i.e., by using $g_s(x) = f_s(1-x)$ instead.

5.3. **Moving the maps to surfaces.** We now consider the two-dimensional torus as $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \cong \mathbb{R}^2/\mathbb{Z}^2$, with $\mathbb{S}^1 = [0, 1]/0 \sim 1$. For each $s \in (2, 3)$, we consider the map $g = f_s^{\times 2}$, so $g : \mathbb{T}^2 \to \mathbb{T}^2$ is given by $g(x, y) = (f_s(x), f_s(y))$. The following hold.

Theorem 5.3. If m_2 denotes normalized 2-dimensional Lebesgue measure on $\mathbb{T}^2$, then g is differentiable m_2 almost everywhere on $\mathbb{T}^2$ and satisfies the following.

(1) Given any $\varepsilon > 0$, there exists an $s \in (2, 3)$ and a probability measure $\nu \ll m_2$ with $m_2(\text{Supp}) > 1 - \varepsilon$, and ν is preserved under g.
(2) The support of ν is the nonwandering set $\Omega(f_s)$.
(3) On $\Omega(f_s)$, g is:
    (a) topologically exact
    (b) chaotic
    (c) exact w.r.t. ν
    (d) weakly Bernoulli
(4) g is 9-to-one for all $(x, y) \in \Omega(f_s)$ except at the turning points: (α, β), $(1-\alpha, \beta)$, $(\alpha, 1-\beta)$, and $(1-\alpha, 1-\beta)$.
(5) g has a fixed point $P = (\frac{1}{2}, \frac{1}{2})$ with only one preimage (itself).

(6) $h_{top}(g) = 2\log s$

Proof. These properties follow from the construction of g. Any nonempty open set of the Cartesian product $I \times I$ contains a basic open set of the form $U \times V$, where U and V are open and nonempty in I. Hence there exist nonnegative integers m and n such that $f^m(U) = \mathbb{S}^1$ and $f^n(V) = \mathbb{S}^1$. Let $N = \max\{m, n\}$. Then $(f^{\times 2})^N = f^N(U) \times f^N(V) = \mathbb{S}^1 \times \mathbb{S}^1$. □

6. Higher genus constructions

We turn to a classical procedure of blowing up a map around a fixed point of a diffeomorphism; there are many sources for this construction (for example, [8]).

6.1. **The blowup construction.** Letting $\mathbf{0} \in \mathbb{R}^2$ denote the origin, assume that h is a homeomorphism of $\mathbb{R}^2$ with $h(\mathbf{0}) = \mathbf{0}$, and h is differentiable near $\mathbf{0}$. Let $Dh_{\mathbf{0}}$ denote the usual derivative mapping of h defined on $T_{\mathbf{0}}\mathbb{R}^2$, the tangent space of $\mathbb{R}^2$ at $\mathbf{0}$.

Define $Y = [0, \infty) \times \mathbb{S}^1$ with polar coordinates on Y given by (r, θ) with $r \geq 0$ and $\theta \in [0, 2\pi)$. Let $q : Y \to \mathbb{R}^2$ be the quotient map taking (r, θ) to $x = r\cos\theta$ and $y = r\sin\theta$. The boundary circle $\Sigma = \{x \in Y : r = 0\} \subset Y$ satisfies $q(\Sigma) = \mathbf{0} \in \mathbb{R}^2$.

Let $S^1_{\mathbf{0}}\mathbb{R}^2 = \{u \in T_{\mathbf{0}}\mathbb{R}^2 \text{ s.t. } ||u|| = 1\}$. Since in polar coordinates $u = (1, \theta)$, $\theta \in [0, 2\pi)$, clearly $S^1_{\mathbf{0}}\mathbb{R}^2 \cong \mathbb{S}^1 \cong \Sigma$ via the map

$$u = (1, \theta) \mapsto \theta. \tag{6.1}$$

We define a map: $\widehat{Dh_{\mathbf{0}}} : \Sigma \to \Sigma$ by $\widehat{Dh_{\mathbf{0}}}(\theta) = \rho$, if $Dh_{\mathbf{0}}(u) = (t, \rho)$ in polar coordinates. The map $\widehat{Dh_{\mathbf{0}}}$ gives the angular part of $Dh_{\mathbf{0}}$ applied to a unit vector.

Definition 6.1. The *blowup* $\hat{h}$ *of* h *about* $\mathbf{0}$ is defined by $\hat{h} : Y \to Y$,

$$\hat{h}(r, \theta) = h(r, \theta) \text{ for } r > 0,$$

and

$$\hat{h}(0, \theta) = \widehat{Dh_{\mathbf{0}}}(\theta) \text{ when } r = 0,$$

Remarks 6.2. We give an equivalent version of blowing up and some remarks.

(1) Letting $\mathbb{S}^1$ represent the unit vectors of $\mathbb{R}^2$ (as in equation 6.1), we see that $(0, \infty) \times \mathbb{S}^1$ is homeomorphic to $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ via the correspondence $(t, u) \mapsto tu$. Then on $[0, \infty) \times \mathbb{S}^1$ we define the dynamical system:

$$\hat{h}(t, u) = \begin{cases} \left(||h(tu)||, \frac{h(tu)}{||h(tu)||}\right) & \text{if } t > 0 \\ \left(0, \frac{Dh_{\mathbf{0}}(u)}{||Dh_{\mathbf{0}}(u)||}\right) & \text{otherwise} \end{cases}$$

Since in our examples, h is affine near P, on the boundary circle Σ (corresponding to $\{0\} \times \mathbb{S}^1$), either $\hat{h}(0, \theta) = (0, \theta)$ or $\hat{h}(0, \theta) = (0, \theta + \pi)$.

(2) The map $\hat{h} : Y \to Y$ is continuous.

(3) The boundary circle Σ is invariant under the action of $\hat{h}$.
(4) The blowup map is local; in particular h does not need to be a global homeomorphism since the construction only uses the fact that it is a homeomorphism in a neighborhood of a fixed point. We could have $h : D \to D$ for some open disk $D \subset \mathbb{R}^2$ of radius ρ, h a homeomorphism onto its image, and for some point $P \in D$, $h(P) = P$. We also assume that Dh_P exists. Then $\hat{h}$ is defined as above by giving D local coordinates with the origin at p.

6.2. **The map H_s on $n\mathbb{T}$.** From Theorem 5.3 we have a map g on $\mathbb{T}^2$ with a fixed point $P = (\frac{1}{2}, \frac{1}{2})$ which has only itself as a preimage. We blow up g at P to produce a map $\hat{g}$ defined on $N = \mathbb{T}^2 \setminus \{\text{disk}\}$. We note that using the notation from above, $\widehat{Dg_P}(\theta) = \theta + \pi$, so $\hat{g}|_{\partial N}$ is rotation by π (or equivalently -π). Hence the condition required to produce a well-defined map on $n\mathbb{T}$, equivalent to equation (4.2), is that

$$\widehat{Dg_P}(-\theta) = -\theta + \pi = -(\theta - \pi) = -\widehat{Dg_P}(\theta)(\text{mod} 2\pi), \tag{6.2}$$

and this is clearly satisfied.

Therefore $\hat{g}$ extends to $G_s : n\mathbb{T} \to n\mathbb{T}$ as follows.

$$G_s(x) = \varphi_{i+1} \circ \hat{h} \circ \varphi_i^{-1}, \quad \text{for } i = 0, \ldots, n-1 \tag{6.3}$$

In this construction, using angular coordinates for the circle, we label points A and B as shown in Figure 2, corresponding to the angles 0 and π respectively (on Σ_i, $i = 0, \ldots n-1$).

Theorem 6.3. For each $s \in (2, 3)$, the map G_s satisfies the following properties.

(1) The points A and B form a repelling period two orbit of G_s.
(2) Using normalized Lebesgue measure m on $n\mathbb{T}$, for any $\varepsilon > 0$, there is some s such that G_s has a nonwandering set Ω_s of measure $> 1 - \varepsilon$.
(3) on Ω_s, G_s is nine-to-one m_s a.e.
(4) there exists an absolutely continuous invariant probability measure $\mu_s \ll m$, supported on Ω_s.
(5) G_s is ergodic with respect to μ. G_s is not exact; there is an automorphic factor isomorphic to rotation on n points.
(6) $G_s|_{\Omega_s}$ is transitive and chaotic.
(7) G_s is isomorphic to an n-point extension over a Bernoulli shift.

Proof. To prove (2), given ε we find f_s so that the wandering set of f_s has measure less than ε/n. To prove ergodicity, first suppose that C is a Borel measurable invariant subset of $n\mathbb{T}$ of positive measure. Then $m(C \cap Y_i) > 0$ for some i, and since $G_s^{-j}C = C$ for all $j \in \mathbb{N}$, we also have that $m(C \cap Y_j) > 0$ for all $j = 0, \ldots, n-1$. Let $C_j = C \cap Y_j$; then $G_s^{-n}(C_j) = C_j$, so $m(Y_j \setminus C_j) = 0$ and G_s is ergodic.

The rest of the proof is almost identical to that of Theorem 4.1.

□

Finally we note that this method allows for the construction of maps on surfaces preserving the properties in Theorem 6.3, orientable or not, as long as one one can blow up a fixed point to produce a map with the symmetry of the boundary circle needed to satisfy equation 4.2.

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