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# On the one-sided ergodic Hilbert transform

Idris Assani and Michael Lin

Abstract. Let $T$ be a unitary contraction on a Hilbert space $X$ such that $X = \overline{(I-T)X}$. We answer two questions related to the strongly continuous semi group $\{(I-T)^r : r \geq 0\}$, studied in [**DL**]. We show that the domain of the infinitesimal generator $G$ is precisely the set of functions $f$ for which the one sided ergodic Hilbert transform $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ converges. We also show that the domain of $G$ is not $\bigcup_{0<\alpha<1}(I-T)^{\alpha}X$. The tools used are essentially of a spectral nature.

## 1. Introduction

In 1937, Marcinkiewicz and Zygmund [**MZ**] proved the following.

Theorem A. *Let $\{\xi_n\}$ be an i.i.d. sequence with $\mathbf{E}(|\xi_1|) < \infty$ and $\mathbf{E}(\xi_1) = 0$. If $\xi_1$ is symmetric, or $\mathbf{E}(|\xi_1| \log^+ |\xi_1|) < \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{\xi_n}{n}$ converges a.s.*

This generalizes the corollary to the Khinchine-Kolmogorov, which assumed the existence of the variance. Furthermore, it is also proved in [**MZ**] that if for some $1 < p < 2$ we have $\mathbf{E}(|\xi_1|^p) < \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{\xi_n}{n^{1/p}}$ converges a.s.

Izumi [**Iz**] considered the following problem. Let $\tau$ be an ergodic probability preserving invertible transformation on $(\Omega, \Sigma, \mu)$. Under what conditions does the series $\displaystyle\sum_{n=1}^{\infty} \frac{f \circ \tau^n}{n}$ converge a.e. for every $f \in L_2$ with $\int f\, d\mu = 0$? Halmos [**Ha**] showed that Izumi's conditions are never satisfied, and proved that for atomless spaces there is always some $f \in L_2$ with $\int f\, d\mu = 0$ for which the series does not converge in $L_2$-norm. The existence of $f \in L_\infty$ with $\int f\, d\mu = 0$ such that

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$\sup \Big| \sum_{n=1}^{N} \frac{f\circ \tau^n}{n} \Big| = \infty$ a.e. was established by Dowker and Erdös [**DoE**]. Their result was later improved by del Junco and Rosenblatt [**DeR**] and, independently, by Krzyźewski [**Krz**], Kakutani and Petersen [**KP**].

For $T$ a contraction on a Banach space $X$ and $f \in \overline{(I-T)X}$, we call the series $\sum_{n=1}^\infty \frac{T^n f}{n}$ the *one-sided ergodic Hilbert transform*, and would like to find conditions on $f$ for its convergence. Derriennic and Lin [**DL**] showed that if for some $0 < \alpha \le 1$ $f$ satisfies

$$\Big\| \frac{1}{N} \sum_{k=1}^{N} T^k f \Big\| \le \frac{C}{N^\alpha} \tag{1}$$

then the series $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n^{1-\beta}}$ converges in norm for any $\beta < \alpha$ (and obviously, by (1), $\|\frac{1}{n^{1-\beta}} \sum_{k=1}^n T^k f\| \to 0$). When $f$ satisfies (1) we say that it *has power ergodic rate* $\alpha$. By Kronecker's lemma, convergence of the $\alpha$*-fractional ergodic Hilbert transform* $\displaystyle\sum_{n=1}^{\infty} \frac{T^n f}{n^{1-\alpha}}$ implies that $f$ has power ergodic rate $\alpha$. Convergence of the fractional one-sided ergodic Hilbert transform of course implies convergence of the one-sided ergodic Hilbert transform (see Lemma 2.19 of [**DL**]), and can be considered as a "rate of convergence" for the transform.

Power ergodic rates were first considered by Loève [**Lo**] as a condition for the pointwise ergodic theorem for unitary operators on $L_2$ – see [**Do**], p. 492. Gaposhkin [**G1**] proved that if $T$ is unitary on $L_2(\mu)$ and $f$ has power ergodic rate $\alpha$, then the one-sided ergodic Hilbert transform converges a.e.; in [**G2**] he proved that $\sum_{n=1}^\infty \frac{T^n f}{n^\gamma}$ converges a.e. for $\gamma > \max\{\frac{1}{2}, 1-\alpha\}$. Independently of [**DL**], Assani [**AS1**], [**As2**], [**As3**] defined *power type and logarithmic type Wiener-Wintner functions* for $T$ induced by probability preserving transformations (see [**AsN**] for additional examples), in order to study return times phenomena with a break of duality and convergence and continuity properties of the *rotated ergodic Hilbert transform*. In [**DL**] it was also shown that if $T$ is a mean ergodic positive contraction on $L_1(\mu)$ and $f$ has power ergodic rate, then the one-sided ergodic Hilbert transform converges a.e., but all the fractional transforms may fail to converge a.e. For $1 < p < \infty$ and $T$ power-bounded (not invertible) on $L_p(\mu)$, convergence of the fractional ergodic Hilbert transform for $f$ with ergodic power rate was proved in [**CL1**],[**CL2**],[**W**]; for more precise information when $T$ is Dunford-Schwartz (and $\mu$ is finite) see [**DL**]. Weighted one-sided ergodic Hilbert transforms were considered in [**As4**].

Using for $0 < \alpha < 1$ the power-series expansion $(1-t)^\alpha = 1 - \sum_{j=1}^\infty a_j^{(\alpha)} t^j$ with $a_j^{(\alpha)} > 0$ for every $j$ and $\sum_{j=1}^\infty a_j^{(\alpha)} = 1$, the operator $(I-T)^\alpha = I - \sum_{j=1}^\infty a_j^{(\alpha)} T^j$ was defined in [**DL**] for any contraction $T$ on a Banach space $X$. It was shown that if $(I-T)X$ is not closed, then all the images $(I-T)^\alpha X$ are different, dense in $\overline{(I-T)X}$. When $T$ is mean ergodic, $f \in (I-T)^\alpha X$ if and only if the series $\sum_{n=1}^\infty \frac{T^n f}{n^{1-\alpha}}$ converges. Moreover, if $f$ has power ergodic rate $\beta$, then $f \in (I-T)^\alpha X$ for each $\alpha < \beta$.

For $r > 1$ we define $(I-T)^r = (I-T)^{[r]}(I-T)^{r-[r]}$. It was proved in Theorem 2.22 of [**DL**] that if $T$ is a contraction on a Banach space $X$ with $X = \overline{(I-T)X}$, then the family $\{(I-T)^r : r \geq 0\}$ is a strongly continuous semigroup, and if the one-sided ergodic Hilbert transform converges for $f$, then $f$ is in the domain of definition of the infinitesimal generator $G$, and $Gf = -\sum_{n=1}^{\infty} \frac{T^n f}{n}$. When $(I-T)X$ is not closed, there exists $f \in \overline{(I-T)X}$ such that the transform does not converge, which raises two questions:

(i) Is the domain of $G$ equal to $\bigcup_{0<\alpha<1}(I-T)^\alpha X$?

(ii) Is the domain of $G$ precisely the set of $f$ for which the transform converges?

The first question was raised by the second author at the working session of the Chapel Hill ergodic workshop on February 2004.

Heuristically, the connection between the infinitesimal generator and the one-sided ergodic Hilbert transform can be understood by looking at $(1-t)^r = e^{r\log(1-t)}$ for fixed $t \in [-1,1)$, whose generator is obviously $\log(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$.

Our purpose in this paper is to answer both questions above when $T$ is unitary on a Hilbert space. We will show in section 3 that the answer to the first question is negative, while the second question has a positive answer when $T$ is unitary (or symmetric) on a (complex) Hilbert space (sections 5 and 6). In section 2 we obtain some convergence results, and in section 4 we look at the infinitesimal generator in Banach spaces.

## 2. Convergence of the one-sided ergodic Hilbert transform

Inspired by the condition introduced in [**As1**],[**As2**], we look at $T$ power-bounded on a Banach space $X$ and $f \in X$ which for some $\alpha \geq 0$ satisfies

$$\Big\| \frac{1}{N}\sum_{n=1}^{N} T^n f \Big\| \leq \frac{C}{(\log(N+1))^{1+\alpha}}, \tag{2}$$

and say that $f$ *has logarithmic ergodic rate* $\alpha$. Clearly any $f$ with power ergodic rate has logarithmic ergodic rate. Examples of logarithmic ergodic rate with no power ergodic rate will be given in the next section.

Theorem 2.1. *Let $X$ be a Banach space and $\{f_n\} \subset X$. If for some $\alpha > 0$ we have the estimate*

$$\Big\| \frac{1}{N}\sum_{n=1}^{N} f_n \Big\| \leq \frac{C}{(\log(N+1))^{1+\alpha}}, \tag{2'}$$

*then for every $0 \leq \beta < \alpha$ the series $\sum_{n=1}^{\infty} \frac{(\log n)^\beta f_n}{n}$ converges in norm.*

Proof. Put $S_n = \sum_{k=1}^{n} f_k$ for $n \geq 1$ and $S_0 = 0$. Fix $\beta < \alpha$, and use Abel's summation by parts to obtain

$$\sum_{k=1}^{n} \frac{(\log k)^\beta f_k}{k} = \sum_{k=1}^{n} (S_k - S_{k-1}) \frac{(\log k)^\beta}{k} =$$

$$\frac{(\log n)^\beta S_n}{n} + \sum_{k=1}^{n-1} \Big( \frac{(\log k)^\beta}{k} - \frac{(\log(k+1))^\beta}{k+1} \Big) S_k. \tag{3}$$

Since $\beta < \alpha$, (2') obviously implies $\|\frac{(\log n)^\beta}{n} S_n\| = \|\frac{(\log n)^\beta}{n} \sum_{k=1}^n f_k\| \to 0$. Putting $\phi(t) := (\log t)^\beta / t$ and computing $\frac{d\phi}{dt}$, we see that $\phi$ decreases for $t > e^\beta$. Using the mean value theorem and (2') for estimating individual terms, we obtain

$$\sum_{k=[e^\beta]+1}^{\infty} \Big\| \Big( \frac{(\log k)^\beta}{k} - \frac{(\log (k+1))^\beta}{k+1} \Big) S_k \Big\| \le \sum_{k=2}^{\infty} \frac{(\log k)^\beta C k}{k^2 (\log(k+1))^{1+\alpha}} < \infty. \tag{4}$$

Hence the sum on right hand side of (3) converges. □

Remark. The theorem yields that if (2') holds for every $\alpha < \gamma$, then for every $\beta < \gamma$ the series $\sum_{n=1}^\infty \frac{(\log n)^\beta f_n}{n}$ converges in norm.

Corollary 2.2. *Let $T$ be a mean-bounded operator on a Banach space $X$ (i.e., satisfying $\sup_n \frac{1}{n} \big\| \sum_{k=1}^n T^k \big\| < \infty$), and let $f$ have logarithmic ergodic rate $\alpha > 0$. Then for every $0 \le \beta < \alpha$ the series $\sum_{n=1}^\infty \frac{(\log n)^\beta T^n f}{n}$ converges in norm.*

Remark. Let $\psi_k$ be a positive sequence increasing to $\infty$ with $\sum_{k=1}^\infty \frac{1}{k\psi_k} < \infty$. For $\beta = 0$ and $f_k = T^k f$, (3) becomes

$$\sum_{k=1}^{n} \frac{T^k f}{k} = \frac{S_n}{n} + \sum_{k=1}^{n-1} \Big( \frac{1}{k} - \frac{1}{k+1} \Big) S_k. \tag{5}$$

Norm convergence of the one-sided ergodic Hilbert tranform is similarly proved for any $f \in X$ such that $\big\| \frac{1}{N} \sum_{n=1}^N T^n f \big\| \le \frac{C}{\psi_N}$.

Theorem 2.3. *Let $T$ be a contraction on $L_1(\mu)$ with mean ergodic linear modulus. If $f \in L_1(\mu)$ has positive logarithmic ergodic rate, then the series $\sum_{n=1}^\infty \frac{T^n f}{n}$ converges a.e. and in norm.*

Proof. The $L_1$-norm convergence follows from the previous corollary, with $\beta = 0$. For $\beta = 0$ (4) becomes

$$\sum_{k=1}^{\infty} \Big\| \frac{1}{k(k+1)} S_k f \Big\| < \infty. \tag{6}$$

Applied to the $L_1$-norm, it yields $\sum_{k=1}^\infty \Big| \frac{1}{k(k+1)} S_k f \Big| < \infty$ a.e. by the theorem of Beppo Levi, and together with the pointwise ergodic theorem for $T$ [**ÇoL**], the right hand side of (5) converges a.e. □

Remarks. 1. When $T$ in the theorem is not positive, mean ergodicity of $T$ is not sufficient for the pointwise ergodic theorem [**ÇoL**].

2. The theorem applies when $\mu$ is finite and $T$ is a Dunford-Schwartz operator (i.e., also a contraction of $L_\infty$), in particular when $\tau$ is probability preserving and $Tf = f \circ \tau$.

3. When $\mu$ is $\sigma$-finite infinite and $T$ is Dunford-Schwartz, the previous proof still applies, since we now use the Dunford-Schwartz pointwise ergodic theorem [**DuS**] (see also [**Kr**], p. 65).

When dealing with *one* $L_p(\mu)$ space with $\mu$ $\sigma$-finite, we can take an equivalent probability $\mu'$, and $Vf = (d\mu/d\mu')^{1/p} f$ is an order preserving linear isometry of $L_p(\mu)$ onto $L_p(\mu')$ which preserves pointwise convergence. We can therefore assume in the proofs, with no loss of generality, that $\mu$ is a probability.

THEOREM 2.4. *Let $1 < p < \infty$, and let $T$ be a positive contraction on $L_p(\mu)$. If $f \in L_p(\mu)$ has positive logarithmic ergodic rate, then the series $\sum_{n=1}^\infty \frac{T^n f}{n}$ converges a.e. and in norm. Furthermore, $\sup_{N\ge 1} |\sum_{k=1}^N \frac{T^k f}{k}|$ is in $L_p$.*

PROOF. We assume that $\mu$ is a probability. Then $\|S_n f\|_1 \le \|S_n f\|_p$, so for $f$ with logarithmic ergodic rate $\alpha > 0$, (6) and the argument in the previous proof yield $\sum_{k=1}^\infty \left|\frac{1}{k(k+1)} S_k f\right| < \infty$ a.e., while $\frac{1}{n} S_n f \to 0$ a.e. by Akcoglu's pointwise ergodic theorem [**A**]. This proves a.e. convergence in (5).

By [**A**] we also have $\sup_n |\frac{1}{n} S_n f| \in L_p(\mu)$. Since by (6) $\sum_{k=1}^\infty \left|\frac{1}{k(k+1)} S_k f\right| \in L_p$, (5) yields

$$\sup_{1\le n\le N} \Big|\sum_{k=1}^n \frac{T^k f}{k}\Big| \le \sup_{n\ge 1}\Big|\frac{S_n f}{n}\Big| + \sum_{k=1}^\infty \Big|\frac{1}{k(k+1)} S_k f\Big| \in L_p(\mu).$$

Letting $N \to \infty$ proves the last assertion. □

THEOREM 2.5. *Let $1 < p < \infty$, let $\{f_n\} \subset L_p(\mu)$ satisfy (2') with $\alpha > 0$, and assume $\sup_n \|f_n\|_p = K < \infty$. Then for $0 \le \beta < \min\{\alpha, \frac{\alpha(p-1)+p-2}{p}\}$ the series $\displaystyle\sum_{n=1}^\infty \frac{(\log n)^\beta f_n}{n}$ converges a.e. and in $L_p$-norm, with $\sup_n |\sum_{k=1}^n \frac{(\log k)^\beta f_k}{k}| \in L_p(\mu)$.*

PROOF. Fix $\beta$ satisfying the assumption. We first prove $\frac{(\log n)^\beta}{n}\sum_{k=1}^n f_k \to 0$ a.e., using the method of [**CL1**].

Since $\beta < \frac{\alpha(p-1)+p-2}{p}$ is equivalent to $\frac{p}{p-1}(1+\beta) < p(1+\alpha-\beta)$, we pick $r$ with $\frac{p}{p-1}(1+\beta) < r < p(1+\alpha-\beta)$, and put $n_m = [e^{(m^{1/r})}] + 1$. Then (2') yields

$$\sum_{m=1}^\infty \int \Big|\frac{(\log n_m)^\beta}{n_m}\sum_{k=1}^{n_m} f_k\Big|^p d\mu \le \sum_{m=1}^\infty \Big(\frac{C}{\big(\log(n_m+1)\big)^{1+\alpha-\beta}}\Big)^p \le \sum_{m=1}^\infty \frac{C^p}{m^{(1+\alpha-\beta)p/r}}.$$

By the choice of $r$ the above series converges, so $\sum_{m=1}^\infty \left|\frac{(\log n_m)^\beta}{n_m}\sum_{k=1}^{n_m} f_k\right|^p$ converges a.e., and $\frac{(\log n_m)^\beta}{n_m}\sum_{k=1}^{n_m} f_k \to 0$ a.e.

Fix $n$ and take $m$ with $n_m < n \le n_{m+1}$. Since $(\log t)^\beta/t$ decreases for $t \ge e^\beta$, when $n_m > e^\beta$ we have

$$\Big|\frac{(\log n)^\beta}{n}\Big(\sum_{k=1}^n f_k - \sum_{k=1}^{n_m} f_k\Big)\Big| = \Big|\frac{(\log n)^\beta}{n}\sum_{k=n_m+1}^{n} f_k\Big| \le \frac{(\log n_m)^\beta}{n_m}\sum_{k=n_m+1}^{n_{m+1}} |f_k|.$$

Since $\|f_k\|_p \le K$, we obtain

$$\int \max_{n_m<n\le n_{m+1}} \Big|\frac{(\log n)^\beta}{n}\Big(\sum_{k=1}^n f_k - \sum_{k=1}^{n_m} f_k\Big)\Big|^p d\mu \le K^p (\log n_m)^{\beta p}\Big(\frac{n_{m+1}-n_m}{n_m}\Big)^p.$$

We now use $(m+1)^{1/r} - m^{1/r} \le 1/(rm^{(r-1)/r})$ and $e^t - 1 \le et$ for $0 \le t \le 1$ (noting that $r > 1$) to estimate

$$\frac{n_{m+1}}{n_m} - 1 \le \frac{e^{(m+1)^{1/r}}+1}{e^{(m^{1/r})}} - 1 \le \frac{e}{r\,m^{(r-1)/r}} + \frac{1}{e^{(m^{1/r})}} \le \frac{D}{r\,m^{(r-1)/r}}.$$

This estimate and $r > \frac{p}{p-1}(1+\beta)$ yield

$$\int \sum_{n_m > e^\beta} \max_{n_m < n \le n_{m+1}} \Big| \frac{(\log n)^\beta}{n} \Big( \sum_{k=1}^{n} f_k - \sum_{k=1}^{n_m} f_k \Big) \Big|^p d\mu \le D' \sum_{m=1}^{\infty} \frac{(\log n_m)^{\beta p}}{m^{p(r-1)/r}} < \infty.$$

Hence

$$\max_{n_m < n \le n_{m+1}} \Big| \frac{(\log n)^\beta}{n} \Big( \sum_{k=1}^{n} f_k - \sum_{k=1}^{n_m} f_k \Big) \Big|^p \to 0 \text{ a.e.}$$

Since $m \to \infty$ as $n \to \infty$, the convergence $\frac{(\log n)^\beta}{n} \sum_{k=1}^n f_k \to 0$ a.e. follows from

$$\Big| \frac{(\log n)^\beta}{n} \sum_{k=1}^{n} f_k \Big| \le \max_{n_m < n \le n_{m+1}} \Big| \frac{(\log n)^\beta}{n} \Big( \sum_{k=1}^{n} f_k - \sum_{k=1}^{n_m} f_k \Big) \Big| + \Big| \frac{(\log n_m)^\beta}{n_m} \sum_{k=1}^{n_m} f_k \Big|.$$

Since $\mu$ is assumed a probability and $\beta < \alpha$, the $L_p$-norm convergence of the series in (4) implies its $L_1$-norm convergence, so we have a.e. convergence of the series in (3), and the a.e. convergence of the series is proved. The norm convergence follows from Theorem 2.1.

The beginning of the proof yields $\sup_m \Big| \frac{(\log n_m)^\beta}{n_m} S_{n_m} \Big| \in L_p$, and with the previous computations, the inequality

$$\sup_{n>0} \Big| \frac{(\log n)^\beta}{n} S_n \Big| \le \sup_m \frac{(\log n_m)^\beta}{n_m} \sum_{k=n_m+1}^{n_{m+1}} |f_k| + \sup_m \Big| \frac{(\log n_m)^\beta}{n_m} S_{n_m} \Big|$$

yields $\sup_n \Big| \frac{(\log n)^\beta}{n} S_n \Big| \in L_p$. Since $\beta < \alpha$, the inequality (4) yields that the series in (3) is absolutely convergent to an $L_p$ function, so $\sup_n \Big| \sum_{k=1}^n \frac{(\log k)^\beta f_k}{k} \Big|$ is in $L_p$. □

Remark. When $p = 2$, the theorem yields a.e. convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^\beta f_n}{n}$ for $0 \le \beta < \frac{1}{2}\alpha$. When $1 < p < 2$ we always have $\frac{\alpha(p-1)+p-2}{p} < \alpha$, but for the theorem to be non-void we need $\alpha > (2-p)/(p-1)$ in order to have $\frac{\alpha(p-1)+p-2}{p} > 0$. When $p > 2$, for $\alpha \le p-2$ we can have $\beta$ arbitrarily close to $\alpha$.

Corollary 2.6. *Let $1 < p < \infty$, and let $T$ be power-bounded on $L_p(\mu)$. If $f \in L_p(\mu)$ has logarithmic ergodic rate $\alpha > 0$, then for $0 \le \beta < \min\{\alpha, \frac{\alpha(p-1)+p-2}{p}\}$ the series $\displaystyle\sum_{n=1}^{\infty} \frac{(\log n)^\beta T^n f}{n}$ converges a.e. and in norm, with $\displaystyle\sup_n \Big| \sum_{k=1}^{n} \frac{(\log k)^\beta T^k f}{k} \Big|$ in $L_p$.*

## 3. Convergence of the transform without power ergodic rate

In this section we show that when $T$ is unitary on a Hilbert space $H$ and $(I-T)H$ is not closed, there may be elements with positive logarithmic ergodic rate, so the transform converges, which have no power ergodic rate. In this case $\bigcup_{\beta>0}(I-T)^\beta H$ is a strict (dense) subspace of the domain of $G$ (the infinitesimal generator of the semigroup $\{(I-T)^r{}_{|\overline{(I-T)H}}\}$).

Throughout this section, $T$ is unitary on a Hilbert space $H$. Recall (e.g., [**Q**]; see also [**Kr**], p. 95) that for each $f \in H$ there is a (unique) positive measure $\sigma_f$ on the Borel subsets of the unit circle with Fourier coefficients $\hat{\sigma}_f(n) = \langle T^n f, f\rangle$, called the spectral measure of $f$. We identify the unit circle with the periodic interval $[-\frac{1}{2}, \frac{1}{2})$, so $\hat{\sigma}_f(n) = \int_{-1/2}^{1/2} e^{2\pi i n t} d\sigma_f(t)$. It is well-known that we then have

$$\Big\|\sum_{k=1}^{n} T^k f\Big\|^2 = \int_{-1/2}^{1/2} \Big|\sum_{k=1}^{n} e^{2\pi i k t}\Big|^2 d\sigma_f(t) = \int_{-1/2}^{1/2} \frac{\sin^2(\pi n t)}{\sin^2(\pi t)} d\sigma_f(t). \tag{7}$$

THEOREM 3.1. *Let $T$ be a unitary operator on $H$ and $0 \neq f \in H$.*

*(i) For $0 < \alpha < 1$, $f$ has power ergodic rate $\alpha > 0$ if and only if there exists $C > 0$ such that $\sigma_f([-\delta, \delta]) \le C\delta^{2\alpha}$ for $\delta > 0$.*

*(ii) $f$ has logarithmic ergodic rate $\alpha \ge 0$ if and only if there exists $C > 0$ such that*

$$\sigma_f([-\delta, \delta]) \le C/|\log \delta|^{2+2\alpha} \quad \textit{for} \ \ \delta > 0. \tag{8}$$

PROOF. If $f$ has logarithmic or power ergodic rate, then $f \in \overline{(I-T)H}$, which is equivalent to $\sigma_f(\{0\}) = 0$. Also the conditions on $\sigma_f$ in (i) or (ii) imply $\sigma_f(\{0\}) = 0$, so actually we deal with $f \in \overline{(I-T)H}$.

Part (i) is Theorem 3 of Kachurovskii [**Ka**]. Ideas of the proof will be used for proving (ii).

Assume that (8) holds. The trigonometric identity used in (7) shows that the integrand on the right hand side of (7) is bounded by $n^2$. Graphing $\sin(\pi t)$ we see that $\sin(\pi t) \ge 2t$ for $0 \le t \le \frac{1}{2}$. Hence (7) yields

$$\Big\|\sum_{k=1}^{n} T^k f\Big\|^2 \le n^2 \sigma_f([-1/n, 1/n]) + \frac{1}{4}\int_{\{\frac{1}{n} \le |t| \le 1/2\}} \frac{1}{t^2} d\sigma_f(t). \tag{9}$$

By (8), the first term on the right hand side is bounded by $C(n/|\log n|^{1+\alpha})^2$. It remains to dominate last integral. Following [**Ka**] put $s_k = \sigma_f(\{0 < |t| < 1/k\})$, so $\sigma_f(\{1/(k+1) \le |t| < 1/k\}) = s_k - s_{k+1}$. Using (8) and the fact that $t/(\log t)^{2+2\alpha}$ is increasing for $t > e^{2+2\alpha}$, we obtain

$$\int_{\{\frac{1}{n} \le |t| \le 1/2\}} \frac{1}{t^2} d\sigma_f(t) \le \sum_{k=1}^{n-1} (k+1)^2 (s_k - s_{k+1}) \le s_1 + \sum_{k=1}^{n-1} (2k+1) s_k \le$$

$$4\sum_{k=1}^{n-1} k s_k \le 4C \sum_{k=1}^{n-1} \frac{k}{(\log k)^{2+2\alpha}} \le 4C\big(C' + (n - e^{2+2\alpha})\frac{n}{(\log n)^{2+2\alpha}}\big) \le \frac{C" n^2}{(\log n)^{2+2\alpha}}.$$

Now (9) yields that $f$ has logarithmic ergodic rate $\alpha$.

Now we assume (2). For $|t| \le 1/2n$ we have $|\sin(\pi n t)| \ge |\sin(\pi t)|$, so (7) yields

$$\sigma_f([-1/2n, 1/2n]) \le \int_{-1/2n}^{1/2n} \frac{\sin^2(\pi n t)}{\sin^2(\pi t)} d\sigma_f(t) \le \Big\|\sum_{k=1}^{n} T^k f\Big\|^2 \le$$

$$\frac{C^2}{(\log(n+1))^{2+2\alpha}} \le \frac{2^{2+2\alpha} C^2}{(\log(2n))^{2+2\alpha}}.$$

This proves (8) when $\delta = 1/(2n)$. For general $\delta > 0$ use $n = [1/2\delta]$. □

In the remainder of this section we deal only with separable Hilbert spaces. In this case, it is well-known (e.g., [**Q**], p. 27) that for a unitary operator $T$ there exists a finite measure $\sigma_m$, unique up to equivalence, with the following properties:

(i) For every $f \in H$ we have $\sigma_f << \sigma_m$.

(ii) If $0 \leq \sigma << \sigma_m$, there exists $f \in H$ with $\sigma_f = \sigma$.

In particular, there exists $g \in H$ with $\sigma_g = \sigma_m$. The measure $\sigma_m$ is called *the maximal spectral type of* $T$.

The property $H = \overline{(I-T)H}$, which is equivalent to $\sigma_f(\{0\}) = 0$ for every $f \in H$, is now equivalent to $\sigma_m(\{0\}) = 0$. In general, $\sigma'_m := \sigma_m - \sigma_m(\{0\})\delta_0$ (where $\delta_0$ is the Dirac measure at 0) is the maximal spectral type of the restriction of $T$ to $\overline{(I-T)H}$.

Let $E(.)$ be the resolution of the identity of $T$, i.e., the projection valued measure such that $Tf = \int_{-1/2}^{1/2} e^{2\pi it} E(dt)f$. Then $\sigma_f(.) = \langle E(.)f, f\rangle = \|E(.)f\|^2$, so $\sigma_m(A) = 0$ if and only if $E(A) = 0$. From Proposition 4.5.10 in [**Pe**] we obtain the following.

THEOREM 3.2. *Let $T$ be unitary on a separable Hilbert space $H$. Then the topological support of $\sigma_m$ is the spectrum, i.e., if $A$ is an open subset of $[-1/2, 1/2]$ with $\sigma_m(A) = 0$, then for every $t \in A$ we have $e^{2\pi it} - T$ invertible.*

THEOREM 3.3. *Let $T$ be a unitary operator on a separable Hilbert space $H$, with maximal spectral type $\sigma_m$. If $\sigma_m(\{1/(k+1) < |t| \leq 1/k\}) > 0$ for every positive integer $k$, then for every $\alpha > 0$ there exists $f \in H$ with logarithmic ergodic rate $\alpha$, which has no logarithmic rate $\gamma > \alpha$ (hence has no power ergodic rate).*

PROOF. As observed above, the spectral type of $T'$, the restriction of $T$ to $H' := \overline{(I-T)H}$, is $\sigma'_m$. Hence $\sigma'_m(\{1/(k+1) < |t| \leq 1/k\}) > 0$ for every $k$, and since the spectrum is closed, the previous result yields that $1 = e^{2\pi i0}$ is in the spectrum of $T'$, so $(I-T')H' = (I-T)H$ is not closed. We thus may assume that $H = \overline{(I-T)H}$ and $(I-T)H$ not closed. Let $g \in H$ satisfy $\sigma_g = \sigma_m$.

Put $A_k = \{1/(k+1) < |t| \leq 1/k\}$. By assumption $\|E(A_k)g\|^2 = \sigma_m(A_k) > 0$, so we can define the elements $g_k := \frac{E(A_k)g}{\|E(A_k)g\|}$. Fix $\alpha > 0$, and put $a_k = \left(\frac{1}{(\log k)^{2+2\alpha}} - \frac{1}{(\log(k+1))^{2+2\alpha}}\right)^{1/2}$. Since $\{A_k\}$ are disjoint, $\{g_k\}$ are orthogonal, and since $\sum_{k=2}^{\infty} |a_k|^2 = \frac{1}{(\log 2)^{2+2\alpha}}$ , the series $f = \sum_{k=2}^{\infty} a_k g_k$ is defined in $H$. Note that $\sigma_f(0) = 0$, and with the property $E(A \cap B) = E(A)E(B)$ we have

$$\sigma_f([-\frac{1}{n}, \frac{1}{n}]) = \|E(\{|t| \leq \frac{1}{n}\})f\|^2 = \sum_{k=n}^{\infty} |a_k|^2 = \frac{1}{(\log n)^{2+2\alpha}}. \tag{10}$$

By Theorem 3.1(ii) $f$ has logarithmic ergodic rate $\alpha$. Since for any fixed $\gamma > \alpha$ there is no $C$ with $1/(\log n)^{2+2\alpha} \leq 1/(\log n)^{2+2\gamma} \leq C/n^{2\beta}$, the equality in (10) and Theorem 3.1(ii) show that $f$ has no logarithmic ergodic rate $\gamma$ (so cannot have a power ergodic rate). □

COROLLARY 3.4. *Let $(\Omega, \Sigma, \mu)$ be a separable space $\sigma$-finite measure space, and let $T$ be the unitary operator induced on $L_2(\mu)$ by an invertible aperiodic measure preserving transformation. Then for every $\alpha > 0$ there exists a function $f \in L_2(\mu)$ with logarithmic ergodic rate $\alpha$ and no power ergodic rate, $\sum_{k=1}^{\infty} \frac{T^k f}{k}$ converges a.e. and in $L_2$, with $\sup_n |\sum_{k=1}^{n} \frac{T^k f}{k}| \in L_2(\mu)$.*

PROOF. It was proved by A. Ionescu Tulcea [**IT**] that under the aperiodicity assumption the spectrum of $T$ is the whole unit circle, so the previous theorem can be applied. The additional properties of $f$ follow from Corollary 2.6. $\square$

## 4. Weak forms of the one-sided ergodic Hilbert transform

As indicated in the introduction, it was proved in [**DL**] that if $T$ is a contraction on a Banach space $X$ with $X = \overline{(I-T)X}$, then $\{(I-T)^r : r \geq 0\}$ is a strongly continuous semigroup on $X$; we denote its infinitesimal generator by $G$. If the one-sided ergodic Hilbert transform converges for $f$, then $f$ is in $\mathbf{D}(G)$, the domain of definition of $G$, and $Gf = -\sum_{n=1}^{\infty} \frac{T^n f}{n}$. Recall that $\mathbf{D}(G)$ is defined as the set of $f$ for which the limit of $\frac{(I-T)^\alpha f - f}{\alpha}$ as $\alpha \to 0^+$ exists, and this limit defines $Gf$.

PROPOSITION 4.1. *Let $T$ be a power-bounded operator in a Banach space $X$ with $X = \overline{(I-T)X}$. If $\sum_{j=1}^{n} \frac{T^j f}{j}$ converges weakly, say to $g \in X$, as $n \to \infty$, then $f \in \mathbf{D}(G)$, and $Gf = -g$.*

PROOF. An appropriate modification of the proof of Proposition 2.21 of [**DL**] yields that $\frac{1}{\alpha}[(I-T)^\alpha - I]f + g$ converges weakly to 0 as $\alpha \to 0^+$. By a result of Yosida (see Theorem 10.5.4 in [**HP**], pp. 318-319) we have that necessarily $\lim_{\alpha \to 0^+} \|\frac{1}{\alpha}[(I-T)^\alpha - I]f + g\| \to 0$. $\square$

PROPOSITION 4.2. *Let $T$ be power-bounded on a reflexive Banach space $X$ and $f \in X$. Then $\sum_{j=1}^{n} \frac{T^j f}{j}$ converges weakly if and only if*

$$\sup_n \left\| \sum_{j=1}^{n} \frac{T^j f}{j} \right\| < \infty. \tag{11}$$

PROOF. By the Banach-Steinhaus theorem, weak convergence implies boundedness. Assume now (11). Then for every $T^*$-invariant functional $f^*$ we have $\langle f, f^* \rangle = 0$, so $f \in \overline{(I-T)X}$. By reflexivity $X^* = \{f^* : T^* f^* = f^*\} \oplus \overline{(I-T^*)X^*}$. For a functional of the form $g^* = (I-T^*)h^*$ we have

$$\langle \sum_{j=1}^{n} \frac{T^j f}{j}, g^* \rangle = \langle Tf, h^* \rangle - \langle \frac{T^{n+1} f}{n}, h^* \rangle + \sum_{j=2}^{n-1} \langle (\frac{1}{j} - \frac{1}{j-1}) T^j f, h^* \rangle.$$

By power-boundedness $\left\| \frac{1}{n} T^{n+1} f \right\| \to 0$, so for every $g^* = (I-T^*)h^*$ the series $\sum_{j=1}^{\infty} \langle \frac{T^j f}{j}, g^* \rangle$ converges. Convergence for $g^*$ in a dense subspace of $X^*$ and the boundedness condition (11) yield convergence of $\sum_{j=1}^{\infty} \langle \frac{T^j f}{j}, g^* \rangle$ for every functional $g^* \in X^*$. $\square$

LEMMA 4.3. *Let $T$ be power-bounded on a Banach space $X$ and $\{b_j\}$ a non-increasing sequence of non-negative numbers. Then for $f \in \overline{(I-T)X}$ we have*

$$\lim_{\alpha\to 0^+} \left\| \sum_{j=2}^{\infty} \frac{\alpha b_j T^j f}{j(j-1)^\alpha} \right\| = 0.$$

PROOF. For $\alpha > 0$ the operator $R_\alpha := \sum_{j=2}^{\infty} \frac{\alpha b_j T^j}{j(j-1)^\alpha}$ is well-defined in the operator norm topology, and it is easy to check that $\sup_{0<\alpha\le 1} \|R_\alpha\| < \infty$. Hence the set of $f$ for which $\lim_{\alpha\to 0^+} \|R_\alpha f\| = 0$ is closed, and the lemma will be proved if for $f = (I-T)g$ we have $\lim_{\alpha\to 0^+} \|R_\alpha f\| = 0$. But for such $f$

$$R_\alpha f = \frac{1}{2}\alpha b_2 T^2 g + \alpha \sum_{j=3}^{\infty} \left[ \frac{b_j}{j(j-1)^\alpha} - \frac{b_{j-1}}{(j-1)(j-2)^\alpha} \right] T^j g$$

and the monotonicity yields $\|R_\alpha f\| \le \alpha b_2 \|g\| \sup_n \|T^n\| \to 0$. □

We want to characterize the domain of the infinitesimal generator of the semi-group $\{(I-T)^r : r \ge 0\}$.

For $0 < \alpha < 1$ denote by $A_j^{(\alpha)}$ the ratio $\frac{a_j^{(\alpha)}}{\alpha}$ (where $(1-t)^\alpha = 1 - \sum_{j=1}^\infty a_j^{(\alpha)} t^j$). Simple computations shows that $A_j^{(\alpha)} = \frac{1}{j}(1-\alpha)(1-\frac{\alpha}{2}) \cdots (1-\frac{\alpha}{j-1})$ for $j \ge 2$.

LEMMA 4.4. *Let $T$ be power-bounded on a Banach space $X$ and $f \in \overline{(I-T)X}$. Then*

$$\lim_{\alpha\to 0^+} \sum_{j=2}^{\infty} \frac{a_j^{(\alpha)}}{\alpha} T^j f = \lim_{\alpha\to 0^+} \sum_{j=2}^{\infty} \frac{T^j f}{j(j-1)^\alpha}$$

*– if one limit exists so does the other, and the limits are equal.*

PROOF. For fixed $0 < \alpha < 1$ put $L_j^{(\alpha)} := \log\left[(1-\alpha)(1-\frac{\alpha}{2}) \cdots (1-\frac{\alpha}{j-1})\right]$. For $j \ge 2$ we have

$$L_j^{(\alpha)} = -\sum_{l=1}^{j-1} \sum_{n=1}^{\infty} \frac{1}{n} (\frac{\alpha}{l})^n = -\sum_{l=1}^{j-1} (\frac{\alpha}{l}) - B_j^{(\alpha)}$$

where

$$(*) \qquad B_j^{(\alpha)} = \sum_{n=2}^{\infty} \sum_{l=1}^{j-1} \frac{1}{n} (\frac{\alpha}{l})^n = \sum_{n=2}^{\infty} \frac{\alpha^n}{n} \sum_{l=1}^{j-1} (\frac{1}{l})^n > 0.$$

We can rewrite $L_j^{(\alpha)}$ as $L_j^{(\alpha)} = -\left[\alpha \log(j-1) + \alpha[\sum_{l=1}^{j-1}(1/l) - \log(j-1)] + B_j^{(\alpha)}\right]$, and obtain

$$A_j^{(\alpha)} = \frac{1}{j} \cdot \frac{1}{(j-1)^\alpha} \cdot \exp\left( -\alpha[\sum_{l=1}^{j-1}(1/l) - \log(j-1)]\right) \exp\left( -B_j^{(\alpha)}\right).$$

We denote below $f_j = T^j f$ and $b_j := \sum_{l=1}^{j-1}(1/l) - \log(j-1)$. Since $\log j < \sum_{l=1}^{j-1}(1/l) < 1 + \log(j-1)$, we have $0 < b_j < 1$. The boundedness of $\{\|f_j\|\}$ together with $B_j^{(\alpha)} > 0$ yield that the two series $\sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \exp(-\alpha b_j)(1 - \exp\big(-B_j^{(\alpha)}\big))$ and $\sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \exp(-\alpha b_j)$ are absolutely convergent, and we can write

$$\sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \exp(-\alpha b_j) - \sum_{j=2}^{\infty} A_j^{(\alpha)} f_j =$$

$$(**) \qquad \sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \exp(-\alpha b_j)\big(1 - \exp(-B_j^{(\alpha)})\big).$$

Now we show that the last series converges absolutely to zero as $\alpha \to 0^+$. First we observe that (*) yields

$$(1 - \exp\big(-B_j^{(\alpha)}\big)) \leq B_j^{(\alpha)} \leq \sum_{n=2}^{\infty} \frac{\alpha^n}{n} \sum_{l=1}^{\infty} (\frac{1}{l})^n \leq \sum_{n=2}^{\infty} \frac{\alpha^n}{n}\big(1 + \frac{1}{n-1}\big) \leq$$

$$|\log(1-\alpha)| - \alpha + \alpha^2 \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = |\log(1-\alpha)| - \alpha + K\alpha^2$$

Put $C = \sup_j \|f_j\|$. Since $\exp(-\alpha b_j) \leq 1$ and $\sum_{j=2}^{\infty} \frac{1}{j(j-1)^\alpha} \leq \frac{\alpha+1}{\alpha}$, the sum of the norms of the terms in the series (**) is estimated by

$$C \sum_{j=2}^{\infty} \frac{1}{j(j-1)^\alpha} B_j^{(\alpha)} \leq C\big(|\log(1-\alpha)| - \alpha + K\alpha^2\big)\frac{\alpha+1}{\alpha} =$$

$$(\alpha+1)C\bigg(\frac{|\log(1-\alpha)|}{\alpha} - 1 + K\alpha\bigg),$$

which converges to 0 as $\alpha \to 0^+$. Going back to the formula of the series (**), we have shown that

$$\lim_{\alpha \to 0^+} \sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \exp(-\alpha b_j) = \lim_{\alpha \to 0^+} \sum_{j=2}^{\infty} A_j^{(\alpha)} f_j$$

– one limit exists if and only if the other exists, and they are equal.

So far we have only used the boundedness of $\{f_j\}$. It remains to show that

$$(***) \qquad \lim_{\alpha \to 0^+} \sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \exp(-\alpha b_j) = \lim_{\alpha \to 0^+} \sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha}$$

– one limit exists if and only if the other exists, and they are equal. For fixed $\alpha$ the difference of the two series is

$$(+) \qquad \sum_{j=2}^{\infty} \frac{\alpha b_j f_j}{j(j-1)^\alpha} + \sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \big[1 - \exp(-\alpha b_j) - \alpha b_j\big].$$

The sequence $\{b_j\}$ decreases to a limit $b \geq 0$, since

$$b_{j+1} - b_j = \frac{1}{j} - \log j + \log(j-1) \leq \int_{j-1}^{j} \frac{dx}{x} - \log j + \log(j-1) = 0.$$

The first series in (+) tends to 0 as $\alpha \to 0^+$ by the previous lemma. Since $b_j < 1$ and $0 \le t + e^{-t} - 1 \le t^2$ for $t > 0$, we estimate the last series in (+) by

$$\Big\| \sum_{j=2}^{\infty} \frac{f_j}{j(j-1)^\alpha} \cdot \big[1 - \exp(-\alpha b_j) - \alpha b_j\big] \Big\| \le \alpha^2 \sum_{j=2} \frac{C}{j(j-1)^\alpha} \le \alpha(\alpha+1)C,$$

which tends to 0 as $\alpha \to 0^+$. This yields (***) and proves the lemma. $\square$

Corollary 4.5. *Let $T$ be a contraction on a Banach space $X$ with $X = \overline{(I-T)X}$. Then $f \in \mathbf{D}(G)$ if and only if $\sum_{j=1}^{\infty} \frac{T^j f}{j^{1+\alpha}}$ converges as $\alpha \to 0^+$, and then*

$$Gf = -\lim_{\alpha \to 0^+} \sum_{j=1}^{\infty} \frac{T^j f}{j^{1+\alpha}}$$

Proof. By definition $a_1^{(\alpha)} = \alpha$ for $0 < \alpha < 1$, so

$$\frac{(I-T)^\alpha f - f}{\alpha} = -Tf - \sum_{j=2}^{\infty} \frac{a_j^{(\alpha)}}{\alpha} T^j f.$$

Combined with the previous lemma this yields $Gf = -Tf - \lim_{\alpha \to 0^+} \sum_{j=2}^{\infty} \frac{T^j f}{j(j-1)^\alpha}$, with equal domains of convergence.

To complete the proof, it is enough to show that

$$\lim_{\alpha \to 0^+} \Big\| \sum_{j=2}^{\infty} \frac{T^j f}{j^{1+\alpha}} - \sum_{j=2}^{\infty} \frac{T^j f}{j(j-1)^\alpha} \Big\| = 0.$$

Fix $\epsilon > 0$ and take a positive integer $N$ with $\log \frac{N}{N-1} < \epsilon$. Then $\lim_{\alpha \to 0} \frac{(\frac{N}{N-1})^\alpha - 1}{\alpha} = \log \frac{N}{N-1} < \epsilon$. We now estimate

$$\Big\| \sum_{j=2}^{\infty} \frac{T^j f}{j^{1+\alpha}} - \sum_{j=2}^{\infty} \frac{T^j f}{j(j-1)^\alpha} \Big\| \le$$

$$\Big\| \sum_{j=2}^{N} \frac{T^j f}{j^{1+\alpha}} \Big(1 - (\frac{j}{j-1})^\alpha\Big) \Big\| + \Big\| \sum_{j=N+1}^{\infty} \frac{T^j f}{j^{1+\alpha}} \Big(1 - (\frac{j}{j-1})^\alpha\Big) \Big\|$$

Since $N$ is fixed (indpendently of $\alpha$), the first sum tends to 0 as $\alpha \to 0^+$. The second sum (the infinite series) is bounded by

$$\|f\| \sum_{j=N+1}^{\infty} \frac{1}{j^{1+\alpha}} \Big((\frac{j}{j-1})^\alpha - 1\Big) \le \|f\| \Big((\frac{N}{N-1})^\alpha - 1\Big) \sum_{j=N+1}^{\infty} \frac{1}{j^{1+\alpha}} \le$$

$$\|f\| \Big((\frac{N}{N-1})^\alpha - 1\Big) \frac{1}{\alpha N^\alpha}.$$

Hence $\limsup_{\alpha \to 0^+} \Big\| \sum_{j=2}^{\infty} \frac{T^j f}{j^{1+\alpha}} - \sum_{j=2}^{\infty} \frac{T^j f}{j(j-1)^\alpha} \Big\| \le \|f\|\epsilon$, which completes the proof. $\square$

## 5. The one-sided ergodic Hilbert transform of $T$ unitary defines the infinitesimal generator of $\{(I-T)^r : r \geq 0\}$

In this section we show that when $T$ is a unitary operator on a Hilbert space $H$, the domain of the infinitesimal generator $G$ of the semigroup $\{(I-T)^r : r \geq 0\}$ restricted to $\overline{(I-T)H}$ is the same as the domain of the one-sided ergodic Hilbert transform.

We start by a spectral characterization of weak convergence of the one-sided ergodic Hilbert transform, using Proposition 4.2.

Proposition 5.1. *Let $T$ be a unitary operator on $H$ and $0 \neq f \in \overline{(I-T)H}$ with spectral measure $\sigma_f$. Then*

$$\sup_N \Big\| \sum_{j=1}^N \frac{T^j f}{j} \Big\| < \infty \quad \Longleftrightarrow \quad \int_{0<|t|<1/2} (\log|t|)^2 d\sigma_f(t) < \infty.$$

Proof. Since $f \in \overline{(I-T)H}$, we have $\sigma_f(\{0\}) = 0$. Hence all integrals below with respect to $\sigma_f$ are in fact over $[-\frac{1}{2}, \frac{1}{2}) - \{0\}$. The spectral theorem gives us the equality

$$\Big\| \sum_{j=1}^N \frac{T^j f}{j} \Big\|^2 = \int \Big[ \big( \sum_{n=1}^N \frac{\cos 2\pi n t}{n} \big)^2 + \big( \sum_{n=1}^N \frac{\sin 2\pi n t}{n} \big)^2 \Big] d\sigma_f(t).$$

We recall a few properties of some trigonometric series:

(i) By [**Z**] ( (2.28) on p. 191), There exists a finite constant $C$ such that for all $0 < |t| < 1/2$,

$$(++) \qquad \sup_N \Big| \sum_{n=1}^N \frac{\cos 2\pi n t}{n} \Big| \leq \log(\frac{1}{2\pi|t|}) + C.$$

(ii) By [**Z**] ( (2.8) on p. 5), for each $0 < |t| \leq 1/2$,

$$(+++) \qquad \lim_N \sum_{n=1}^N \frac{\cos 2\pi n t}{n} = \log \frac{1}{2 \sin \pi |t|}$$

(iii) The partial sums $\sum_{n=1}^N \frac{\sin 2\pi n t}{n}$ are uniformly bounded on $[-1/2, 1/2)$ ([**Z**], p. 61).

Thus, if $\sup_N \| \sum_{j=1}^N \frac{T^j f}{j} \| < \infty$, then by (+++) and Fatou's lemma we have

$$\int (\log \frac{1}{|2 \sin \pi t|})^2 d\sigma_f(t) = \int \liminf_{N\to\infty} \big( \sum_{n=1}^N \frac{\cos 2\pi n t}{n} \big)^2 d\sigma_f(t) \leq \sup_N \Big\| \sum_{j=1}^N \frac{T^j f}{j} \Big\| < \infty.$$

This proves one implication. For the reverse implication, as the partial sums $\sum_{n=1}^N \frac{\sin 2\pi n t}{n}$ are uniformly bounded on $[-1/2, 1/2)$, we have

$$\sup_N \Big\| \sum_{j=1}^N \frac{T^j f}{j} \Big\|^2 < \infty \quad \Longleftrightarrow \quad \sup_N \int \big( \sum_{n=1}^N \frac{\cos 2\pi n t}{n} \big)^2 d\sigma_f(t) < \infty.$$

Because of (++) this last condition is satisfied as soon as

$$\int_{0<|t|<1/2} (\log|t|)^2 d\sigma_f(t) < \infty,$$

since $\sigma_f(\{0\}) = 0$. This proves that $\sup_N \big\| \sum_{j=1}^N \frac{T^j f}{j} \big\| < \infty$. □

Proposition 5.1 has the following immediate corollary, which justifies the interest in [**As1**] and [**As3**] and in this paper for logarithmic ergodic rates.

Corollary 5.2. *Let $T$ be unitary on $H$. If the series $\displaystyle\sum_{j=1}^n \frac{T^j f}{j}$ converges weakly, then $f$ has logarithmic ergodic rate 0.*

Proof. By Proposition 5.1 the weak convergence of the series implies the finiteness of the integral $\int_{0<|t|<1/2} [\log|t|]^2 d\sigma_f(t)$. From this we conclude without difficulty that

$$\sigma_f([-\delta, \delta]) \le C/|\log\delta|^2 \quad \text{for} \quad \delta > 0.$$

Hence $f$ has logarithmic ergodic rate 0, by Theorem 3.1. □

Lemma 5.3. *For $0 < |t| \le 1/2$ we have*

$$\lim_{\alpha\to 0+} \Big| \sum_{n=1}^{\infty} \frac{\cos 2\pi nt}{n^{1+\alpha}} - \sum_{n=1}^{\infty} \frac{\cos 2\pi nt}{n} \Big| = 0,$$

*and*

$$\lim_{\alpha\to 0+} \Big| \sum_{n=1}^{\infty} \frac{sin 2\pi nt}{n^{1+\alpha}} - \sum_{n=1}^{\infty} \frac{sin 2\pi nt}{n} \Big| = 0.$$

Proof. We give a proof of the first equality. The second follows by a similar reasoning.

We fix $t$ with $0 < |t| \le 1/2$. By (++), there exists $M_t$ such that

$$\Big| \sum_{k=N}^{N+\ell} \cos 2\pi kt \Big| \le M_t \quad \forall N, \ell > 0. \tag{12}$$

By Abel's summation (Theorem 1.2.2 of [**Z**]), for $\alpha \ge 0$ (12) yields

$$\Big| \sum_{k=N}^{N+\ell} \frac{\cos 2\pi kt}{k^{1+\alpha}} \Big| \le \frac{M_t}{N^{1+\alpha}} \le \frac{M_t}{N} \quad \forall N, \ell > 0. \tag{13}$$

Inequality (13) yields that (for fixed $t$) the series $\displaystyle\sum_{k=1}^{\infty} \frac{\cos 2\pi kt}{k^{1+\alpha}}$ converges uniformly in $\alpha \ge 0$, and $\displaystyle\Big| \sum_{k=N}^{\infty} \frac{\cos 2\pi kt}{k^{1+\alpha}} \Big| \le \frac{M_t}{N}$ for every $\alpha \ge 0$. These imply

$$\Big| \sum_{k=1}^{\infty} \frac{\cos 2\pi kt}{k^{1+\alpha}} - \sum_{k=1}^{\infty} \frac{\cos 2\pi kt}{k} \Big| \le \Big| \sum_{k=1}^{N} \frac{\cos 2\pi kt}{k^{1+\alpha}} - \sum_{k=1}^{N} \frac{\cos 2\pi kt}{k} \Big| + \frac{2M_t}{N}. \tag{14}$$

Keeping $N$ fixed, (14) yields

$$\limsup_{\alpha\to 0^+}\Big|\sum_{k=1}^{\infty}\frac{\cos 2\pi kt}{k^{1+\alpha}}-\sum_{k=1}^{\infty}\frac{\cos 2\pi kt}{k}\Big|\le\frac{2M_t}{N},$$

which yields the first equality of the lemma. The second equality is obtained by a similar argument. □

LEMMA 5.4. *Let $T$ be a unitary operator on $H$ and $0\neq f\in\overline{(I-T)H}$ with spectral measure $\sigma_f$. Then*

$$\sup_{0<\alpha<1/2}\Big\|\sum_{j=1}^{\infty}\frac{T^j f}{j^{\alpha+1}}\Big\|<\infty\quad\Longleftrightarrow\quad\int_{-1/2}^{1/2}[\log|t|]^2 d\sigma_f(t)<\infty.$$

PROOF. We have $\sigma_f(\{0\})=0$ since $f\in\overline{(I-T)H}$. If we assume that

$$\sup_{0<\alpha<1/2}\Big\|\sum_{j=1}^{\infty}\frac{T^j f}{j^{\alpha+1}}\Big\|\le M<\infty,$$

then by the spectral theorem we have

$$\sup_{0<\alpha<1/2}\int\Big|\sum_{k=1}^{\infty}\frac{\cos 2\pi kt}{k^{1+\alpha}}\Big|^2 d\sigma_f(t)\le M^2<\infty.$$

Lemma 5.3 and Fatou's lemma yield

$$\int\Big|\sum_{n=1}^{\infty}\frac{\cos 2\pi nt}{n}\Big|^2 d\sigma_f(t)=\int\liminf_{\alpha\to 0^+}\Big|\sum_{k=1}^{\infty}\frac{\cos 2\pi kt}{k^{1+\alpha}}\Big|^2 d\sigma_f(t)\le M^2<\infty,$$

and by (+++) we have $\int\big(\log\frac{1}{2\sin\pi|t|}\big)^2 d\sigma_f(t)<\infty$.

Conversely, assume that $\int_{-1/2}^{1/2}[\log|t|]^2 d\sigma_f(t)<\infty$. Abel's summation and (++) yield

$$\Big|\sum_{k=1}^{N}\frac{\cos 2\pi kt}{k^{1+\alpha}}\Big|\le\max_{1\le n\le N}\Big|\sum_{k=1}^{n}\frac{\cos 2\pi kt}{k}\Big|\le K|\log|t||$$

for every $\alpha>0$, and by property (iii) in the proof of Proposition 5.1 we have

$$\Big|\sum_{k=1}^{N}\frac{\sin 2\pi kt}{k^{1+\alpha}}\Big|\le\max_{1\le n\le N}\Big|\sum_{k=1}^{n}\frac{\sin 2\pi kt}{k}\Big|\le C.$$

Using the spectral theorem, we obtain

$$\sup_{0<\alpha<1/2}\sup_{N\ge 1}\Big\|\sum_{k=1}^{N}\frac{T^k f}{k^{1+\alpha}}\Big\|^2\le C^2+K^2\int_{-1/2}^{1/2}[\log|t|]^2 d\sigma_f(t)<\infty,$$

which yields the claim of the lemma. □

Based on these lemmas one can prove the main result of this section, which answers the second question mentioned in the introduction: for $T$ unitary, the one-sided ergodic Hilbert transform is the infinitesimal generator of the semigroup $\{(I-T)^r : r\ge 0\}$.

THEOREM 5.5. *Let $T$ be a unitary operator on a Hilbert space $H$ with $H = \overline{(I-T)H}$, and let $G$ be the infinitesimal generator of the strongly continuous semigroup $\{(I-T)^r : r \geq 0\}$. Then the following are equivalent for $f \in H$:*

*(i) The one-sided ergodic Hilbert transform $\sum_{n=1}^{\infty} \frac{T^n f}{n}$ converges in norm.*

*(ii) The one-sided ergodic Hilbert transform converges weakly.*

*(iii) $f \in \mathbf{D}(G)$.*

*If either condition holds, then*

$$\lim_{N\to\infty} \sum_{n=1}^{N} \frac{T^n f}{n} = -Gf.$$

PROOF. Obviously $(i) \Rightarrow (ii)$, and by Proposition 4.1 weak convergence of $\sum_{n=1}^{N} \frac{T^n f}{n}$ implies that $f \in \mathbf{D}(G)$ and the (weak) limit is $-Gf$.

Assume now that $f$ is in the domain of $G$. By Corollary 4.5 $\sum_{j=1}^{\infty} \frac{T^j f}{j^{1+\alpha}}$ converges strongly (to $-Gf$) as $\alpha \to 0^+$. Hence

$$\sup_{0<\alpha<1/2} \left\| \sum_{j=1}^{\infty} \frac{T^j f}{j^{1+\alpha}} \right\| < \infty.$$

By lemma 5.4 we have $\int_{0<|t|<1/2} (\log|t|)^2 d\sigma_f(t) < \infty$, which yields, by Proposition 5.1, that $\sup_N \| \sum_{j=1}^{N} \frac{T^j f}{j} \| < \infty$. From Proposition 4.2 we conclude that the sequence $\sum_{j=1}^{n} \frac{T^j f}{j}$ converges weakly, necessarily to $-Gf$.

Recall that if $\{f_n\} \subset H$ converges weakly to $f$ and $\|f_n\| \to \|f\|$, then

$$\|f_n - f\|^2 = \|f_n\|^2 - 2\mathcal{R}e\langle f_n, f\rangle + \|f\|^2 \longrightarrow 2\|f\|^2 - 2\mathcal{R}e\langle f, f\rangle = 0.$$

Since the weak convergence yields $\|Gf\| \leq \liminf_n \left\| \sum_{j=1}^{n} \frac{T^j f}{j} \right\|$, to obtain strong convergence it is therefore sufficient to show that $\limsup_n \left\| \sum_{j=1}^{n} \frac{T^j f}{j} \right\| \leq \|Gf\|$. The convergence for every $t \in (-1/2, 1/2)$ of $\{\sum_{n=1}^{N} \frac{\sin 2\pi nt}{n}\}$ [**Z**] (p. 5) together with the uniform boundedness of the sums yield, by Lebesgue's theorem, that

$$(15) \qquad \lim_{N\to\infty} \int \left( \sum_{n=1}^{N} \frac{\sin 2\pi nt}{n} \right)^2 d\sigma_f(t) = \int \left( \sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{n} \right)^2 d\sigma_f(t).$$

We have shown already that $\int_{0<|t|<1/2} (\log|t|)^2 d\sigma_f(t) < \infty$, which yields, by (++), that $\sup_N \left( \sum_{n=1}^{N} \frac{\cos 2\pi nt}{n} \right)^2$ is $\sigma_f$-integrable. Hence (+++) yields, by Lebesgue's theorem, that

$$(16) \qquad \lim_{N\to\infty} \int \left( \sum_{n=1}^{N} \frac{\cos 2\pi nt}{n} \right)^2 d\sigma_f(t) = \int \left( \sum_{n=1}^{\infty} \frac{\cos 2\pi nt}{n} \right)^2 d\sigma_f(t).$$

Using (15), (16), Lemma 5.3 and Fatou's lemma, we conclude that

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} \frac{T^n f}{n} \right\|^2 = \int \left[ \left( \sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{n} \right)^2 + \left( \sum_{n=1}^{\infty} \frac{\cos 2\pi nt}{n} \right)^2 \right] d\sigma_f(t) \leq$$

$$\lim_{\alpha\to 0^+} \int \left[ \left( \sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{n^{1+\alpha}} \right)^2 + \left( \sum_{n=1}^{\infty} \frac{\cos 2\pi nt}{n^{1+\alpha}} \right)^2 \right] d\sigma_f(t) = \lim_{\alpha\to 0^+} \left\| \sum_{n=1}^{\infty} \frac{T^n f}{n^{1+\alpha}} \right\|^2.$$

By Corollary 4.5, the final term is $\| -Gf\|^2$, so $\lim_{N\to\infty} \left\| \sum_{n=1}^N \frac{T^n f}{n} \right\| \leq \|Gf\|$, which completes the proof. □

## 6. The one-sided ergodic Hilbert transform of $T$ symmetric defines the infinitesimal generator of $\{(I-T)^r : r \geq 0\}$

In this section we show that when $T$ is a symmetric contraction (a power-bounded symmetric operator must be a contraction!) on a Hilbert space $H$, the domain of the infinitesimal generator $G$ of the semigroup $\{(I-T)^r : r \geq 0\}$ restricted to $\overline{(I-T)H}$ is the same as the domain of strong convergence of the one-sided ergodic Hilbert transform.

Theorem 6.1. *Let $T$ be a symmetric contraction on a Hilbert space $H$ with $H = \overline{(I-T)H}$, and let $G$ be the infinitesimal generator of the strongly continuous semigroup $\{(I-T)^r : r \geq 0\}$. Then the following are equivalent for $f \in H$:*

*(i) The one-sided ergodic Hilbert transform $\sum_{n=1}^\infty \frac{T^n f}{n}$ converges in norm.*

*(ii) The one-sided ergodic Hilbert transform converges weakly.*

*(iii) $f \in \mathbf{D}(G)$.*

*If either condition holds, then*

$$\lim_{N\to\infty} \sum_{n=1}^N \frac{T^n f}{n} = -Gf.$$

proof. As in the proof of Theorem 5.5, we have to show only that if $f \in \mathbf{D}(G)$, then $\sum_{n=1}^N \frac{T^n f}{n}$ converges in norm.

Since $T$ is symmetric, its spectrum is a closed subset of the interval [-1,1]. For any $f \in H$ the spectral measure $\sigma_f$, supported on the spectrum, has no atom at 1 because $T$ has no fixed points.

For each $|t| < 1$ we have, by Lebesgue's bounded convergence theorem (applied to the counting measure),

$$\lim_{\alpha\to 0^+} \sum_{k=1}^\infty \frac{t^k}{k^{1+\alpha}} = \sum_{k=1}^\infty \frac{t^k}{k}. \tag{17}$$

Equation (17) holds also for $t=-1$, since $\left| \sum_{k=j}^n \frac{(-1)^k}{k^{1+\alpha}} \right| \leq \frac{1}{j^{1+\alpha}} \leq \frac{1}{j}$ for any $\alpha \geq 0$, by Abel's summation by parts (see Theorem I.2.2 in [**Z**]). For any $\alpha \geq 0$, Abel's summation yields also

$$\left| \sum_{k=1}^n \frac{t^k}{k^{1+\alpha}} \right| \leq |t| \quad \text{for} \quad -1 \leq t \leq 0. \tag{18}$$

By the spectral theorem, Lebesgue's monotone convergence theorem and Corollary 4.5 we have

$$\left\| \sum_{k=1}^N \frac{T^k f}{k} \right\|^2 = \int_{-1}^0 \left( \sum_{k=1}^N \frac{t^k}{k} \right)^2 d\sigma_f(t) + \int_0^1 \left( \sum_{k=1}^N \frac{t^k}{k} \right)^2 d\sigma_f(t) \leq$$

$$\int_{-1}^0 |t|^2 d\sigma_f(t) + \int_0^1 \left( \sum_{k=1}^\infty \frac{t^k}{k} \right)^2 d\sigma_f(t) \leq$$

$$C + \int_0^1 \lim_{\alpha \downarrow 0} \left( \sum_{k=1}^{\infty} \frac{t^k}{k^{1+\alpha}} \right)^2 d\sigma_f(t) \leq C + \lim_{\alpha \downarrow 0} \left\| \sum_{k=1}^{\infty} \frac{T^k f}{k^{1+\alpha}} \right\|^2 = C + \|Gf\|^2.$$

Hence $\sup_N \left\| \sum_{k=1}^N \frac{T^k f}{k} \right\| < \infty$, so by Propositions 4.2 and 4.1 $\sum_{k=1}^N \frac{T^k f}{k}$ converges weakly to $-Gf$.

The estimate (18), Lebesgue's bounded convergence theorem and (17) yield

$$\int_{-1}^0 \left( \sum_{k=1}^{N} \frac{t^k}{k} \right)^2 d\sigma_f(t) \underset{N\to\infty}{\longrightarrow} \int_{-1}^0 \left( \sum_{k=1}^{\infty} \frac{t^k}{k} \right)^2 d\sigma_f(t) =$$

$$\int_{-1}^0 \lim_{\alpha \downarrow 0} \left( \sum_{k=1}^{\infty} \frac{t^k}{k^{1+\alpha}} \right)^2 d\sigma_f(t) = \lim_{\alpha \downarrow 0} \int_{-1}^0 \left( \sum_{k=1}^{\infty} \frac{t^k}{k^{1+\alpha}} \right)^2 d\sigma_f(t),$$

and the monotone convergence theorem yields

$$\int_0^1 \left( \sum_{k=1}^{N} \frac{t^k}{k} \right)^2 d\sigma_f(t) \underset{N\to\infty}{\longrightarrow} \int_0^1 \left( \sum_{k=1}^{\infty} \frac{t^k}{k} \right)^2 d\sigma_f(t) =$$

$$\int_0^1 \lim_{\alpha \downarrow 0} \left( \sum_{k=1}^{\infty} \frac{t^k}{k^{1+\alpha}} \right)^2 d\sigma_f(t) = \lim_{\alpha \downarrow 0} \int_0^1 \left( \sum_{k=1}^{\infty} \frac{t^k}{k^{1+\alpha}} \right)^2 d\sigma_f(t).$$

Combining these and using again Corollary 4.5 we obtain

$$\lim_{N\to\infty} \left\| \sum_{k=1}^{N} \frac{T^k f}{k} \right\|^2 = \lim_{\alpha \downarrow 0} \left\| \sum_{k=1}^{\infty} \frac{T^k f}{k^{1+\alpha}} \right\|^2 = \| - Gf\|^2.$$

Hence $\sum_{k=1}^N \frac{T^k f}{k}$ converges to $-Gf$ in norm, since we already have weak convergence. □

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University of North Carolina, Chapel Hill, NC 27599, USA
*E-mail address*: assani@math.unc.edu

Ben-Gurion University of the Negev, Beer-Sheva, Israel
*E-mail address*: lin@math.bgu.ac.il