

# SPECTRAL CHARACTERIZATION OF WIENER WINTNER DYNAMICAL SYSTEMS

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ABSTRACT. Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system on the finite measure space  $(X, \mathcal{B}, \mu, T)$  and  $\mathcal{K}$  its Kronecker factor. We denote by  $U$  the restriction of  $T$  onto  $\mathcal{K}^\perp$  the orthocomplement of  $\mathcal{K}$ . We give a spectral characterization in  $L^2$  of Wiener Wintner functions in terms of the capacity of the support of the maximal spectral type of  $U$  and the a.e. continuity of the fractional rotated ergodic Hilbert transform. The study of the  $L^2$  case leads to new classes of dynamical systems.

## 1. INTRODUCTION

In [A1] we introduced and studied ergodic dynamical systems that we called Wiener Wintner dynamical systems of power type in  $L^p$ . We defined a function  $f$  to be a Wiener Wintner function in the following way:

**Definition 1:** Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system. A function  $f$  is a Wiener-Wintner function of power type  $\alpha$  in  $L^p$  if there exist finite positive constants  $C_f$  and  $\alpha$  such that

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right| \right\|_p \leq \frac{C_f}{N^\alpha} \text{ for all positive integers } N.$$

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We also said that an ergodic dynamical system is a Wiener Wintner dynamical system of power type  $\alpha$  in  $L^p$  if we could find in the orthocomplement of its Kronecker factor  $\mathcal{K}$  a dense set (for the  $L^2$  norm) of Wiener Wintner functions of power type  $\alpha$  in  $L^p$ . It was shown that a difficult theorem such as the a.e double recurrence [B1] is much easier to prove for such systems. Many known systems have this property (see [A1] for examples and some of their structural properties and [AN] and [L] for examples with singular spectrum on  $\mathcal{K}^\perp$ ). At the present time it is not known if all ergodic dynamical systems are Wiener Wintner dynamical systems of power type  $\alpha$  in  $L^1$ .

In this paper we will focus on the spectral properties of Wiener Wintner dynamical systems in  $L^2$ . We will characterize Wiener Wintner functions in  $L^2$  by the help of the a.e continuity of the Random Fourier series  $H_\gamma^\varepsilon(f)(x) = \sum_{k=-\infty}^{\infty} \frac{f(T^k x) \text{sgn} k}{|k|^\gamma} e^{2\pi i k \varepsilon}$  that we call the fractional rotated ergodic Hilbert transform. (the notation  $\sum_{k=-\infty}^{\infty}$  is used to delete the term corresponding to  $k=0$  in the sum). It will be shown in section 1 that a  $L^2$  function is a Wiener Wintner function of power type  $\alpha$  in  $L^2$  if and only if for a.e  $x$   $H_\gamma^\varepsilon(f)(x)$  is a continuous function of  $\varepsilon$ . This answers one of the questions in [A1].

We study the cases where the  $L^p$  norm is used instead of the  $L^1$  norm as well as a logarithmic growth of convergence. In this paper we will show that the study of the  $L^2$  case leads to apparently new classes of dynamical systems.

We will get these new classes by considering the apparently weaker conditions

$$\sup_{\varepsilon} \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_p \leq \frac{C_f}{N^\alpha}$$

, ( $1 \leq p \leq \infty$ ) and slower rates of convergence of the form  $\frac{C_f}{[\ln(N+1)]^{1+\alpha}}$ . To be more precise we introduce the following definitions:

**Definition 2:** Consider  $(X, \mathcal{B}, \mu, T)$  an ergodic dynamical system. We say that a function  $f$  is a weak Wiener Wintner function (WWW- function) of logarithmic type  $\alpha$  in  $L^p$  if there exist finite positive constants  $C_f$  and  $\alpha$  such that

$$\sup_{\varepsilon} \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_p \leq \frac{C_f}{[\ln(N+1)]^{1+\alpha}}$$

for all positive integer  $N$ .

**Definition 3:** An ergodic dynamical system  $(X, \mathcal{B}, \mu, T)$  is said to be a weak Wiener Wintner dynamical system (WWW dynamical system) of logarithmic type  $\alpha$  in  $L^p$  if there exists in  $\mathcal{K}^\perp$  a dense set of WWW functions of logarithmic type  $\alpha$  in  $L^p$ .

Similar definitions can be given if one considers instead rates of the form  $\frac{C_f}{N^\alpha}$  ( WWW-functions of power type  $\alpha$ , WWW Dynamical Systems of power type  $\alpha$ ). It is clear that a WWW function of power type  $\alpha$  is also of logarithmic type  $\alpha$ . One can note that the range of  $\alpha$  is  $(-1, \infty)$  for the logarithmic type.

The two rates -power type- or -logarithmic type- are in fact linked to two well known capacities; the  $\alpha$  and logarithmic capacities,  $C_\alpha$  and  $C_{\ln^\gamma}$ . For more on these notions one may consult ([Z],[SK],[K] or [SZ]). Let us denote by  $U$  the restriction of  $T$  onto the orthocomplement of the Kronecker factor of  $T$  namely  $\mathcal{K}^\perp$  and by  $S$  the support of the maximal spectral type of  $U$  on  $\mathbb{T}$ . (For references on the maximal spectral type, see [CFS] for instance). We will prove in section 3 that if  $T$  is a weak Wiener Wintner dynamical system of logarithmic type  $\alpha$  in  $L^2$  then the  $\gamma$ -logarithmic capacity of  $S$  is positive for all

$0 < \gamma < 2(1 + \alpha)$ . An analogous result will be given for *WWW – dynamical – system* of power type  $\alpha$  in  $L^2$ . Related to these notions we will introduce the  $\gamma$ -logarithmic (resp.  $\alpha$ ) capacity of  $U$ ,  $\mathcal{C}_{\ln^\gamma}(U)$  (resp.  $\mathcal{C}_\alpha(U)$ ):

$$\mathcal{C}_{\ln^\gamma}(U) = \left( \inf_{\nu \sim \sigma} \sup_{x \in [0,1]} \int_0^1 \left( \log \left( \frac{1}{|t-x|} \right) \right)^\gamma d\nu \right)^{-1}$$

,

$$\mathcal{C}_\alpha(U) = \left( \inf_{\nu \sim \sigma} \sup_{x \in [0,1]} \int_0^1 \left| \frac{1}{|t-x|^\alpha} \right| d\nu \right)^{-1}$$

, where  $\sigma$  denotes the maximal spectral type of  $U$  and  $\nu(\mathbb{T}) = \sigma(\mathbb{T}) = 1$ .

We will show that  $(X, \mathcal{B}, \mu, T)$  is a *WWW – dynamical – system* in  $L^2$  of logarithmic type  $\alpha$  if  $\mathcal{C}_{\ln^{2(1+\alpha)}}(U) > 0$ . We will also study the structural properties of such systems. Using this characterization we will provide an example of a WWW dynamical system which is of logarithmic type  $-1/2$  in  $L^2$  and is not of power type  $\beta$  for all  $0 < \beta < 1$ . This example shows at least that Wiener Wintner dynamical systems of logarithmic type  $\alpha$  in  $L^2$  form a strictly larger class of dynamical systems than the class of Wiener Wintner dynamical system of power type  $\alpha$ . While this paper was written an example was given in [L] of a weak Wiener-Wintner dynamical system which is not of power class  $\alpha$  in  $L^2$  for any  $0 < \alpha < 1$ . These two examples were obtained independently.

Since we announced the results of this paper in [A3] we showed in [A5] that for each ergodic dynamical system there exists a continuous positive function  $g$  satisfying  $\lim_N g(N) = \infty$  and a dense set of functions  $f \in \mathcal{K}^\perp$  such that  $\sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_2 \leq \frac{C_f}{g(N)}$  for all positive integer. Thus there exists always a growth of convergence.

In the fourth section of the paper we will show that for any ergodic dynamical system, f a function in  $L^p$  and  $g \in L^\infty$ , the  $L^1$  norm of the averages  $\frac{1}{N} \sum_{n=1}^N f \circ T^n g \circ T^{2n}$  can be controlled by the sup  $\| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon} \|_p$ . This motivates in part the study of *WWW – dynamical – system* in  $L^p$ .

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## 2. SPECTRAL CHARACTERIZATION OF WWDS IN $L^2$ .

We will use several known facts on the spectral measure of a unitary operator. One may consult [Q] or [N] for more details on spectral measures. If we denote by  $E$  the spectral measure then a family of mesures  $\sigma_{f,g}$  can be defined by the formula  $\sigma_{f,g}(B) = \langle f, E(B)g \rangle$  for any borelian set  $B$  of  $\mathbb{T}$ . If  $U$  is the unitary operator on  $H$  (Hilbert space) then the n Fourier coefficients of this measure is given by  $\hat{\sigma}_{f,g}(n) = \langle U^n f, g \rangle$ . When  $f=g$  the measure  $\sigma_f = \sigma_{f,f}$  is nonnegative. If  $\sigma_f$  and  $\sigma_g$  are absolutely continuous with respect to the same measure  $\nu$  then the Radon Nykodim derivative  $h$  of  $\sigma_{f,g}$  satisfies the pointwise inequality  $|h(x)| \leq \frac{d\sigma_f}{d\nu}^{\frac{1}{2}} \frac{d\sigma_g}{d\nu}^{\frac{1}{2}}$ . We have the equation  $\sigma_{f+g} = \sigma_f + \sigma_g + \sigma_{f,g} + \sigma_{g,f}$ . If  $\sigma_f$  and  $\sigma_g$  are mutually singular then  $\sigma_{f+g} = \sigma_f + \sigma_g$ . We assume that  $T$  is invertible.

One way to access directly the spectral measure is through the helical transform or rotated ergodic Hilbert transform . By using as in [C] the functional calculus starting with the equation  $T = \int_{\mathbb{T}} e^{2\pi i t} dE(t)$  we get the following equation

$$(1) \quad H_\varepsilon f = \sum_{n=-\infty}^{\infty} \frac{f \circ T^n}{n} e^{2\pi i n \varepsilon} = i\varepsilon f + 2iE(-\varepsilon, 0)f + H_0 f$$

for a function  $f$  in  $\mathcal{K}^\perp$ . (The notation  $\sum_{n=-\infty}^{\infty}$  is used to eliminate the term corresponding to  $n=0$  in the sum). This can be seen by writing  $H_\varepsilon f - H_0 f = \int_{\mathbb{T}} S(t + \varepsilon) - S(t) dE(t)$ , where  $S$  is the odd function which on  $[0, \frac{1}{2})$  takes the value  $1 - 2t$  and  $S(0) = 0$ . One of the equivalent conditions for a function  $f$  to be in  $\mathcal{K}^\perp$  is to have a continuous spectral measure (i.e. no atoms) (see [A2], for instance). Hence this eliminates the terms  $E(\lambda)f$  in the difference  $H_\varepsilon f - H_0 f$ . An easy consequence of this difference equation is the following that we will use

$$(2) \quad 2iE(-b, -a)f = \sum_{n=-\infty}^{\infty} \frac{f(T^n)}{n} (e^{2\pi i n b} - e^{2\pi i n a}) - i(b - a)f.$$

To prove one of the implications of the next theorem we will need the following proposition.

**Proposition 1.** *Given  $0 < \gamma < 1$  there exist  $C_\gamma$  and  $0 < r < 1$  such that*

$$\int_{\delta}^{1-\delta} \left| \sum_{k=1}^{\infty} \frac{\cos 2\pi k(\varepsilon - \delta)}{k^{1-\gamma}} - \sum_{k=1}^{\infty} \frac{\cos 2\pi k(\varepsilon + \delta)}{k^{1-\gamma}} \right| d\varepsilon \leq C_\gamma(\delta^r)$$

for all  $0 < \delta < \frac{1}{2}$ .

*Proof.* We will use the known formula

$$\int_0^{\infty} t^{-\gamma} e^{-it} dt = \Gamma(1 - \gamma) e^{\frac{1}{2}\pi i(1-\gamma)}$$

in order to compute the Fourier coefficients of the function  $t^{-\gamma}$ . We have

$$c_n = \int_0^1 t^{-\gamma} e^{-2\pi i n t} dt = \frac{1}{(2\pi n)^{1-\gamma}} \Gamma(1 - \gamma) e^{\frac{1}{2}\pi i(1-\gamma)} - \frac{1}{(2\pi n)^{1-\gamma}} \int_{2\pi n}^{\infty} \frac{e^{-it}}{t^\gamma} dt$$

, for  $n > 0$  and  $c_0 = \frac{1}{2} \frac{1}{1-\gamma}$ . We have

$$\int_{2\pi n}^{\infty} \frac{e^{-it}}{t^\gamma} dt = \lim_L \int_{2\pi n}^{2\pi L} \frac{e^{-it}}{t^\gamma} dt$$

Using two integrations by parts and taking the limit with  $L$  we obtain

$$-\frac{1}{(2\pi n)^{1-\gamma}} \int_{2\pi n}^{\infty} \frac{e^{-it}}{t^\gamma} dt = -\frac{1}{2\pi in} - \frac{\gamma}{(2\pi n)^2} + \frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}} \int_{2\pi n}^{\infty} \frac{e^{-it}}{t^{(\gamma+2)}} dt$$

Hence

$$c_n = \frac{1}{(2\pi n)^{1-\gamma}} \Gamma(1-\gamma) e^{\frac{1}{2}\pi i(1-\gamma)} + \frac{1}{2\pi in} - \frac{\gamma}{(2\pi n)^2} + \frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}} \int_{2\pi n}^{\infty} \frac{e^{-it}}{t^{(\gamma+2)}} dt$$

We can write  $c_n$  as  $a_n + ib_n$  where

$$c_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n x) + b_n \sin(2\pi n x)$$

converges a.e. to the function  $x^{-\gamma}$ . Using the change of variable  $x \rightarrow (1-x)$  we obtain

$$c_0 + \sum_{n=1}^{\infty} a_n \cos(2\pi n x) = \frac{1}{2} (x^{-\gamma} + (1-x)^{-\gamma})$$

for a.e  $x$  in  $[0,1)$ . As  $a_n$  is the real part of  $c_n$  it is equal to

$$\frac{1}{(2\pi n)^{1-\gamma}} \Gamma(1-\gamma) \cos\left(\frac{1}{2}\pi(1-\gamma)\right) - \frac{\gamma}{(2\pi n)^2} + \frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}} \int_{2\pi n}^{\infty} \frac{\cos(t)}{t^{(\gamma+2)}} dt$$

From this we can now estimate what we wanted , i.e.  $\sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{1-\gamma}}$ . We have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^{1-\gamma}} \\ &= C_\gamma \left( \frac{1}{2} \left( \frac{1}{x^\gamma} + \frac{1}{(1-x)^\gamma} \right) - \frac{1}{1-\gamma} + \sum_{n=1}^{\infty} \cos(2\pi n x) \left( \frac{\gamma}{(2\pi n)^2} + \frac{\gamma(\gamma+1)}{(2\pi n)^{1-\gamma}} \int_{2\pi n}^{\infty} \frac{\cos(t)}{t^{(\gamma+2)}} dt \right) \right) \end{aligned}$$

where  $C_\gamma$  is an absolute constant depending only on  $\gamma$ . From this we can finish the proof of this proposition. We have

$$\begin{aligned} & \int_{\delta}^{1-\delta} \left| \sum_{k=1}^{\infty} \frac{\cos 2\pi k(\varepsilon - \delta)}{k^{1-\gamma}} - \sum_{k=1}^{\infty} \frac{\cos 2\pi k(\varepsilon + \delta)}{k^{1-\gamma}} \right| d\varepsilon \\ & \leq \int_{\delta}^{1-\delta} C_\gamma \left| \frac{1}{2} \left( \frac{1}{(\varepsilon - \delta)^\gamma} - \frac{1}{(\varepsilon + \delta)^\gamma} + \frac{1}{(1 - \delta + \varepsilon)^\gamma} - \frac{1}{(1 - \delta - \varepsilon)^\gamma} \right) \right| d\varepsilon \\ & \quad + \int_{\delta}^{1-\delta} C'_\gamma \left| \sum_{n=1}^{\infty} \frac{\sin(2\pi n\delta)}{n^{1+d}} \right| d\varepsilon \end{aligned}$$

where  $C'_\gamma$  and  $d$  are positive constant depending only on  $\gamma$ . The second integral satisfies the estimate of the form  $C(\delta^r)$  for some  $r > 0$  as  $|\sin(2\pi k\delta)| \leq (2\pi k\delta)^b$  for  $0 < b < d$ . The first integral satisfies also the same kind of estimate as we can split the integral into two where we group the first two terms then the next two. The nature of the functions allows to compare them, to eliminate the absolute values and to integrate.  $\square$

**Theorem 2.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system and  $f$  a  $L^2$  function in  $\mathcal{K}^\perp$ .*

*The following are equivalent*

- (1) *There exists  $0 < r < 1$  and a finite constant  $C_r$  such that for all positive integer  $N$  we have*

$$\left\| \sup_{1 \leq j \leq N} \left| E\left(-\delta - \frac{j}{N}, \delta - \frac{j}{N}\right) f \right| \right\|_2 \leq C_r(\delta^r)$$

*for all  $0 < \delta < \frac{1}{2}$*

- (2) *The function  $f$  is a Wiener – Wintner function of power type  $\alpha$  in  $L^2$  for some  $0 < \alpha < 1$ .*

- (3) *There exists  $0 < \gamma < 1$  such that the random Fourier series  $\sum_{k=-\infty}^{\infty} \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k \varepsilon}$  is a.e a continuous function of  $\varepsilon$  and  $\sup_{\varepsilon} \left| \sum_{k=-\infty}^{\infty} \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k \varepsilon} \right| \in L^2$ .*

*Proof.* We first prove that (1) implies (2). First we notice that by Bernstein's theorem there exists an absolute constant  $K$  such that for all  $N'$  positive integer we have

$$\left\| \sup_{\varepsilon} \left| \frac{1}{N'} \sum_{n=1}^{N'} f \circ T^n e^{2\pi i n \varepsilon} \right| \right\|_2^2 \leq C \left\| \sup_{1 \leq j \leq N} \left| \frac{1}{N'} \sum_{n=1}^{N'} f \circ T^n e^{2\pi i n j / N} \right| \right\|_2^2$$

for  $N = KN'$ . This can be seen by applying Bernstein theorem pointwise to the trigonometric polynomial  $\frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon}$  and by using the mean value theorem on each interval of the form  $[\frac{j}{N}, \frac{j+1}{N}]$ . If we denote by  $U_{j/N}$  the unitary operator defined by  $U_{j/N} f = f \circ T e^{2\pi i j / N}$

we have for each  $j$ ,  $1 \leq j \leq N$ ;  $\frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n j / N} = \frac{1}{N} \sum_{n=1}^N U_{j/N}^n f(x)$ . We can write  $f$

in the following way

$$f = E_{j/N}(-\delta, \delta) f + f - E_{j/N}(-\delta, \delta) f$$

, where  $E_{j/N}$  denotes the spectral measure of the operator  $U_{j/N}$  restricted to  $\mathcal{K}^\perp$ . We will use the following three properties

a) We have  $E_{j/N}(-\delta, \delta) f = E(-\delta - \frac{j}{N}, \delta - \frac{j}{N}) f$ , where  $E$  is the spectral measure of  $T$  on  $\mathcal{K}^\perp$

b) There exists a function  $g_{\delta, j}$  such that  $f - E_{j/N}(-\delta, \delta) f = g_{\delta, j} - U_{j/N} g_{\delta, j}$

c) On  $(I - E_{j/N}(-\delta, \delta))(\mathcal{K}^\perp)$ , the operator  $I - U_{j/N}$  is invertible and we have

$$\|(I - U_{j/N})^{-1}\| \leq C \frac{1}{|\sin \pi \delta|}.$$

The properties b) and c) were observed in [K]. Let us see quickly why b) and c) are true.

To show b) first we notice that

$$\sup_M \left\| \sum_{n=0}^M [U_{j/N}]^n (f - E_{j/N}(-\delta, \delta) f) \right\|_2 < \infty$$

and this follows easily by using the spectral measure of  $f$  with respect to  $U_{j/N}$ . Such evaluation gives us a bound in the order of  $\frac{C}{|\sin(\pi\delta)|} \|f\|_2$ . Now we can find a function  $g_{\delta,j}$  and a subsequence of integers  $M_k$  that we denote by  $K$  for convenience such that the averages

$$\frac{1}{K} \sum_{k=1}^K \sum_{n=0}^k [U_{j/N}]^n (f - E_{j/N}(-\delta, \delta)f)$$

converge weakly to the function  $g_{\delta,j}$ .

Applying  $U_{j/N}$  to these averages we see that  $g_{\delta,j} - U_{j/N}g_{\delta,j}$  is the weak limit of

$$f - \frac{1}{K} \sum_{k=1}^K [U_{j/N}]^{k+1} (f - E_{j/N}(-\delta, \delta)f)$$

. As the second term converges to zero in norm we have b). Furthermore  $\|g_{\delta,j}\|_2 \leq \frac{C}{|\sin(\pi\delta)|} \|f\|_2$ .

To prove c) we only have to show that  $I - U_{j/N}$  is one to one. If this was not true then we would have on  $\mathcal{K}^\perp$  an eigenfunction for  $T$  for the eigenvalue  $e^{2\pi ij/N}$  and this is impossible on  $\mathcal{K}^\perp$  by definition. Thus b) and c) are true.

Now we proceed with the proof. We have

$$\left\| \sup_{1 \leq j \leq N'} \frac{1}{N'} \left| \sum_{n=1}^{N'} (U_{j/N})^n f \right\|_2 \right\|_2^2 \leq C \frac{1}{N'^2} \sum_{j=1}^{N'} \|g_{\delta,j}\|_2^2 + \left\| \frac{1}{N'} \sum_{n=1}^{N'} \left| \sup_{1 \leq j \leq N} E(-\delta - j/N, \delta - j/N) f \circ T^n \right\|_2 \right\|_2^2$$

Hence using (1) and (c) we can conclude that

$$\left\| \sup_{1 \leq j \leq N} \frac{1}{N'} \left| \sum_{n=1}^{N'} (U_{j/N})^n f \right\|_2 \right\|_2^2 \leq C(\delta^{2r} + \frac{1}{N\delta^2})$$

Now we can pick  $\delta$  (as a function of  $N$ ) appropriately to make  $f$  a *Wiener–Wintner function* of power type  $\alpha$ . This proves the implication (1)  $\implies$  (2).

The implication (2)  $\implies$  (3) can be shown by using the first part of the proof of Theorem 3 in [AN]. Note that as stated (3) does not assume the convergence of the series for each  $\varepsilon$  off a null set of  $x$ . The continuity of the limit is obtained by the uniform convergence of the subsequence of the partial sums

$$\sum_{k=-N^\delta}^{N^\delta} \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k \cdot e^{2\pi i k \varepsilon}$$

for some  $\delta > 0$ . The same argument shows that

$$\sup_\varepsilon \left| \sum_{k=-\infty}^{\infty} \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k \varepsilon} \right| \in L^2$$

It remains to prove (3)  $\implies$  (1). We will use for this the proposition and the equation (2)

linking the ergodic Hilbert transform to the spectral measure of  $T$  on  $\mathcal{K}^\perp$ . We have

$$2iE(-\delta + j/N, \delta + j/N)f = \sum_{n=-\infty}^{\infty} \frac{f \circ T^n}{n} (e^{2\pi i n j/N} 2i \sin(2\pi k \delta)) - 2i\delta f$$

so we only have to estimate the series. We need an estimate for a.e.  $x$ . By the pointwise convergence of the ergodic Hilbert transform in  $L^p$  for invertible measure preserving transformation (see [Co]) we can notice that for each  $\delta$  and each  $N$  the series

$$\sum_{n=-\infty}^{\infty} \frac{f(T^n x)}{n} (e^{2\pi i n j/N} 2i \sin(2\pi n \delta))$$

converges a.e.. This follows by applying the pointwise convergence result to the ergodic Hilbert transform of the transformation  $(x, y) \rightarrow (Tx, y + j/N + \delta)$  and to the function  $f \otimes e^{2\pi i y}$ . Hence to estimate the series pointwise we can write

$$\left| \sum_{k=-\infty}^{\infty} \frac{f(T^k x)}{k} (e^{2\pi i k j/N} 2i \sin(2\pi k \delta)) \right|$$

$$\begin{aligned}
&= \left| \int_0^1 \left( \sum_{k=-\infty}^{\infty} \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} (e^{2\pi i k j/N} e^{-2\pi i k t}) \right) \left( \sum_{k=-\infty}^{\infty} \sin(2\pi k \delta) \frac{e^{2\pi i k t}}{|k|^{1-\gamma}} \right) dt \right| \\
&\leq \sup_{\theta} \left| \left( \sum_{k=-\infty}^{\infty} \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k \theta} \right) \int_0^1 \left| \left( \sum_{k=-\infty}^{\infty} \sin(2\pi k \delta) \frac{e^{2\pi i k t}}{|k|^{1-\gamma}} \right) \right| dt \right| \\
&= \sup_{\theta} \left| \left( \sum_{k=-\infty}^{\infty} \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k \theta} \right) \int_0^1 \left| \sum_{k=1}^{\infty} \frac{\cos 2\pi k(t-\delta)}{|k|^{1-\gamma}} - \sum_{k=1}^{\infty} \frac{\cos 2\pi k(t+\delta)}{|k|^{1-\gamma}} \right| dt \right|
\end{aligned}$$

The first term in the last expression is in  $L^2$  and independent of  $j$ . For the second term we can split the integral from 0 to 1 into three integrals ;  $\int_0^\delta$ ,  $\int_\delta^{1-\delta}$  and the  $\int_{1-\delta}^1$ . The proposition 1 tells us that the second integral is less than  $C_\gamma(\delta^r)$  for some  $0 < r < 1$  for all  $\delta > 0$ . The first and the third term can be handled similarly by using the pointwise estimate  $\left| \sum_{k=0}^{\infty} \frac{\cos(2\pi k x)}{k^{1-\gamma}} \right| \leq C_\gamma x^{-\gamma}$ . This estimate can be found in [Z] p. 191. This shows that there exists a constant  $s$  depending on  $\gamma$  such that

$$E(-\delta + j/N, \delta + j/N) f(x) \leq \sup_{\theta} \left| \left( \sum_{k=-\infty}^{\infty} \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k \theta} \right) \right| C_\gamma \delta^s.$$

As

$$\sup_{\theta} \left| \left( \sum_{k=-\infty}^{\infty} \frac{f(T^k x) \operatorname{sgn} k}{|k|^\gamma} e^{2\pi i k \theta} \right) \right|$$

is in  $L^2$  by assumption we can conclude that

$$\left\| \sup_{1 \leq j \leq N} E(-\delta + j/N, \delta + j/N) f \right\|_2 \leq C_\gamma \delta^s$$

for some  $s > 0$  and this ends the proof of the theorem.  $\square$

**Remark**

If one assumes that  $f \in L^\infty$  then one can add to (3) in theorem 2 the a.e convergence for all  $\varepsilon$  of the series  $\sum_{k=-\infty}^{\infty} \frac{f(T^k x)}{|k|^\gamma} \operatorname{sgn} k e^{2\pi i k \varepsilon}$ . We do not know at the present time if one can reach the same conclusion for  $f \in L^2$ . Our method of proof uses proposition 1 in [AN] which seems to restrict the domain of  $f$ .

### 3. WEAK WIENER WINTNER DYNAMICAL SYSTEMS

We will study first in a subsection some structural properties of *WWW – dynamical – system*. In the second subsection deals with links between capacities and *WWW – dynamical – system*. The third provides the example of *WWW – dynamical – system* not of power type but of logarithmic type. The fourth deals with estimates for the double recurrence.

**3.1. Structural properties of Weak Wiener Wintner Dynamical Systems.** We begin with a simple proposition on spectral measures, showing the link between Bochner's theorem characterizing the continuity of a measure on  $\mathbb{T}$  and the concept of *WWW – dynamical – system*.

**Proposition 3.** (1) *If  $f$  is a WeakWiener – Wintner function in  $L^2$  of power type  $\alpha$ ,*

$$\text{then for all } N \sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| \leq \frac{C}{N^{\alpha/2}}.$$

(2) *If  $\sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| \leq \frac{C}{N^\alpha}$  for all  $N$ , then  $f$  is a WeakWiener – Wintner function in  $L^2$  of power type  $\alpha$ .*

(3) *If  $f$  is a WeakWiener – Wintner function in  $L^2$  of logarithmic type  $\alpha$ , then*

$$\sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| \leq \frac{C}{(\ln(N+1))^{\frac{1+\alpha}{2}}} \text{ for all } N.$$

(4) *If  $\sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| \leq \frac{C}{(\ln(N+1))^{1+\alpha}}$  for all  $N$ , then  $f$  is a WeakWiener – Wintner function in  $L^2$  of logarithmic type  $\alpha$ .*

*Proof.* We only prove (1) and (2). The proofs for (3) and (4) are similar. Assume that  $f$  is a *WeakWiener – Wintner function* in  $L^2$  of power type  $\alpha$ . By using Fubini and Holder's

inequality we have

$$\begin{aligned}
\frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)|^2 &= \frac{1}{N} \sum_{n=1}^N \widehat{\sigma_{f,g}}(n) \overline{\widehat{\sigma_{f,g}}(n)} \\
&= \left| \frac{1}{N} \sum_{n=1}^N \int \int \bar{g}(x) f(T^n x) \bar{g}(y) f(T^n y) d(\mu \otimes \mu)(x, y) \right| \\
&= \left| \int \int \bar{g}(x) \bar{g}(y) \frac{1}{N} \sum_{n=1}^N f(T^n x) f(T^n y) d(\mu \otimes \mu)(x, y) \right| \\
&= \left| \int \bar{g}(x) \left( \int \bar{g}(y) \frac{1}{N} \sum_{n=1}^N f(T^n x) f(T^n y) d\mu(y) \right) d\mu(x) \right| \\
&\leq \int \left[ |\bar{g}(x)| \|g\|_2 \left\| \frac{1}{N} \sum_{n=1}^N f(T^n x) f(T^n y) \right\|_2 \right] d\mu(x) \\
&= \|g\|_2 \int \left[ \bar{g}(x) \left( \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon} \right|^2 d\sigma_f(\varepsilon) \right)^{1/2} \right] d\mu(x) \\
&\leq \|g\|_2^2 \left( \int \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon} \right|^2 d\sigma_f(\varepsilon) d\mu(x) \right)^{1/2}
\end{aligned}$$

This last term is less or equal to

$$\begin{aligned}
&\|g\|_2^2 \left( \int \left( \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon} \right|^2 d\mu(x) \right) d\sigma_f(\varepsilon) \right)^{1/2} \\
&\leq \|g\|_2^2 \|f\|_2 \sup_{\varepsilon} \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_2 \leq \frac{C}{N^\alpha}
\end{aligned}$$

where the last inequality follows by assumption. Hence,

$$\begin{aligned}
\frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| &\leq \left( \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)|^2 \right)^{1/2} \\
&\leq \frac{C}{N^{\alpha/2}}
\end{aligned}$$

(2)  $\implies$  (1)

Assume that

$$\sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| \leq \frac{C}{N^\alpha}$$

Then, by duality we get that

$$\begin{aligned} \sup_{\varepsilon} \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_2 &= \sup_{\varepsilon} \sup_{\|g\|_2 \leq 1} \left| \int g(x) \left( \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon} \right) d\mu(x) \right| \\ &= \sup_{\varepsilon} \sup_{\|g\|_2 \leq 1} \left| \frac{1}{N} \sum_{n=1}^N e^{2\pi i n \varepsilon} \int g(x) f(T^n x) d\mu(x) \right| \\ &\leq \sup_{\varepsilon} \sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N \left| \int g(x) f(T^n x) d\mu(x) \right| \\ &= \sup_{\|g\|_2 \leq 1} \frac{1}{N} \sum_{n=1}^N |\widehat{\sigma_{f,g}}(n)| \\ &\leq \frac{C}{N^\alpha} \end{aligned}$$

where the last inequality follows by assumption.  $\square$

The following theorem lists some stability properties.

**Theorem 4.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system . Then*

- (1) *There exists in  $\mathcal{P}^\perp$ , the orthocomplement of the Pinsker  $\sigma$  algebra of  $T$ , a dense set of WeakWiener – Wintner function of logarithmic type  $\alpha$  in  $L^2$ .*
- (2) *If  $(X, \mathcal{B}, \mu, T)$  is a WWW – dynamical – system of logarithmic type  $\alpha$  in  $L^2$  then anyone of its factors is a WWW – dynamical – system of logarithmic class  $\alpha$ . Furthermore any closed  $T$  invariant vector subspace of  $\mathcal{K}^\perp$  contains a dense set of WWW – dynamical – system of logarithmic type  $\alpha$ .*
- (3) *If we decompose  $L^2(X)$  into the direct sum  $H_s + H_l$ , where  $H_s$  represents the functions with spectral measure singular with respect to Lebesgue measure and  $H_l$  those*

with spectral measure absolutely continuous with respect to Lebesgue measure, then  $H_1$  contains a dense set of *WeakWiener – Wintner function* of power type  $\frac{1}{2}$  in  $L^2$ .

- (4) Assume  $(X, \mathcal{B}, \mu, T)$  is a *WWW – dynamical – system* of logarithmic type  $\alpha$  in  $L^2$  and its product with the *WWW – dynamical – system* of logarithmic type  $\alpha$ ,  $(Y, \mathcal{C}, \nu, S)$ , is ergodic. Then this product is a *Weak Wiener Wintner Dynamical System* of logarithmic type  $\alpha$  in  $L^2$ .

*Proof.* The proof of (1) and the first part of (2) follows the same line as in [A1]. We omit it. The second part of (2) follows by using the projection of  $\mathcal{K}^\perp$  onto the closed  $T$  invariant vector space.

The property (3) follows immediately by considering the dense set of functions  $f$  whose spectral measure  $\sigma_f$  is of the form  $hdm$  where  $h \in L^\infty(m)$ ,  $m$  being Lebesgue measure. For the proof of (4) we can remark that the orthocomplement of the Kronecker factor of  $T \times S$  is spanned by the finite linear combinations of product of functions  $f \otimes g$  where  $f \in \mathcal{K}_T^\perp$  or  $g \in \mathcal{K}_S^\perp$ . It is enough then to show that the product  $f \otimes g$  is a *WeakWiener – Wintner function* of logarithmic class  $\alpha$  in  $L^2(\mu \otimes \nu)$  if one of the functions is a *WeakWiener – Wintner function* of logarithmic type  $\alpha$  in  $L^2$ . We have

$$\begin{aligned} & \sup_\varepsilon \int \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) g(S^n y) e^{2\pi i n \varepsilon} \right|^2 d\mu(x) \otimes \nu(y) \\ &= \sup_\varepsilon \int \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n(t+\varepsilon)} \right|^2 d\sigma_g(\varepsilon) d\mu(x) \\ &\leq \int \sup_\theta \int \left| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \theta} \right|^2 d\mu(x) d\sigma_g(t) \leq \frac{C}{(\ln(N+1))^{2(1+\alpha)}} \end{aligned}$$

□

**Remarks.** Similar results can be obtained for *WWW – dynamical – system* of power type  $\alpha$ .

**3.2. Capacity and WWW – dynamical – systems.** We recall a few things about capacities that one can find in [Z] for instance. Let  $\lambda$  be a real-valued, periodic, integrable, even, continuous function on the torus,  $\mathbb{T}$ , such that  $\lim_{x \rightarrow 0} \lambda(x) = \infty$ . Let  $\mu$  be a positive measure on the Borel  $\sigma$ -algebra,  $\mathcal{B}(\mathbb{T})$ , on  $\mathbb{T}$ , and  $F$  a Borel set such that  $\mu(F) = 1$ . Define a new function  $l : \mathbb{T} \rightarrow \mathbb{R}$  by  $l(x) = l(x, \mu) = \int_0^1 \lambda(x - t) d\mu(t)$ . Let  $L(\mu) = \sup_{x \in [0,1]} l(x, \mu)$ .

**Definition 4.** The  $\lambda$  capacity of  $F$  is

$$C(F) = \left( \inf_{\{\mu: \mu(F)=1\}} L(\mu) \right)^{-1}$$

**Definition 5.** For  $\gamma > 0$  the  $\gamma$ -Logarithmic Capacity of  $F$ , denoted by  $C_{\ln^\gamma}(F)$  is the  $\lambda$  capacity of  $F$  when  $\lambda(x) = \left( \ln \left( \frac{1}{|x|} \right) \right)^\gamma$ .

**Definition 6.** For  $0 < \alpha < 1$ , the  $\alpha$ -Capacity of  $F$ , denoted by  $C_\alpha$  is the  $\lambda$  capacity of  $F$  when  $\lambda(x) = \frac{1}{|x|^\alpha}$ .

Connected to these notions we introduce the following definition.

**Definition 7.** Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system and  $U$  its restriction to the orthocomplement of the Kronecker factor. The  $\gamma$ -logarithmic capacity of  $U$  (resp. the

$\alpha$ -capacity of  $U$ ) is defined as  $\mathcal{C}_{\ln^\gamma}(U)$  (resp.  $\mathcal{C}_\alpha(U)$ );

$$\mathcal{C}_{\ln^\gamma}(U) = \left( \inf_{\nu \sim \sigma} \sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|t-x|} \right) \right)^\gamma d\nu \right)^{-1}$$

, resp.

$$\mathcal{C}_\alpha(U) = \left( \inf_{\nu \sim \sigma} \sup_{x \in [0,1]} \int_0^1 \left( \frac{1}{|t-x|^\alpha} \right) d\nu \right)^{-1}$$

, where  $\sigma$  is the maximal spectral type of  $U$  and  $\nu$  and  $\sigma$  are probability measures.

The following lemma gives a characterization in  $L^2$  of *WeakWiener – Wintner function*.

**Lemma 1.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system and  $f$  a function in  $\mathcal{K}^\perp$ . The function  $f$  is a *WeakWiener – Wintner function of logarithmic type  $\alpha$*  (resp. *of power type  $\alpha$* ) if and only if  $\sup_{x \in [0,1]} \sigma_f \star \delta_x([- \delta, \delta]) \leq \frac{C}{|\ln(\delta)|^{2(1+\alpha)}}$  (resp.  $\sup_{x \in [0,1]} \sigma_f \star \delta_x([- \delta, \delta]) \leq \frac{C}{\delta^{2\alpha}}$ ) for all  $\delta > 0$ .*

*Proof.* The proof uses part of the method of theorem 3 in [K]. We sketch it in the logarithmic case. By periodicity of the functions on the torus one can use the interval  $(-1/2, 1/2]$  instead of  $(0, 1]$ . If  $f$  is a *WeakWiener – Wintner function* of logarithmic type  $\alpha$  in  $L^2$  then

$$\sup_x \int_{-1/2}^{1/2} \left| \sum_{n=1}^N e^{2\pi i n t} \right|^2 d\sigma_f \star \delta_x(t) \leq \frac{C_f N^2}{(\ln N)^{2(1+\alpha)}}$$

and this shows that  $\sup_x \sigma_f \star \delta_x([- \delta, \delta]) \leq \frac{C}{|\ln \delta|^{2(1+\alpha)}}$ .

For the reverse implication assuming that  $\sup_x \sigma_f \star \delta_x([- \delta, \delta]) \leq \frac{C}{|\ln \delta|^{2(1+\alpha)}}$  and following the decomposition made in [K](p.664), we just need to dominate  $\int_{|t| \geq \pi/n} \frac{1}{x^2} d\sigma_f \star \delta_x(t)$ . Writing  $\sigma_f \star \delta_x\{\pi/k + 1 \leq |t| \leq \pi/k\}$  as  $s_k - s_{k+1}$  where  $s_k = \sigma_f \star \delta_x\{0 < |t| \leq \pi/k\}$  and

using partial summation, we are left with controlling  $\sum_{k=2}^n \frac{k}{|\ln(k)|^{2(1+\alpha)}}$ . We dominate this term by  $C'n + \sum_{k=\lfloor \sqrt{n} \rfloor}^n \frac{k}{|\ln(k)|^{2(1+\alpha)}}$ . This quantity is less than  $C'' \frac{n^2}{|\ln(n)|^{2(1+\alpha)}}$ .  $\square$

**Proposition 5.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system and  $f$  a function in  $\mathcal{K}^\perp$ .*

(1) *If  $f$  is a WeakWiener – Wintner function of logarithmic type  $\alpha$  in  $L^2$  then*

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^\beta d\sigma_f(t) < \infty$$

, for all  $0 < \beta < 2(1 + \alpha)$ .

(2) *If*

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^{2(1+\alpha)} d\sigma_f(t) < \infty$$

, then  $f$  is a Wiener – Wintner function of logarithmic type  $\alpha$  in  $L^2$ .

(3) *If the function  $f$  is a WeakWiener – Wintner function of power type  $\alpha$  in  $L^2$  then*

$$\sup_{x \in [0,1]} \int_0^1 \left( \frac{1}{|x-t|^\beta} \right) d\sigma_f(t) < \infty$$

, for all  $0 < \beta < 2\alpha$ .

(4) *If*

$$\sup_{x \in [0,1]} \int_0^1 \left( \frac{1}{|x-t|^{2\alpha}} \right) d\sigma_f(t) < \infty$$

then  $f$  is a WeakWiener – Wintner function of power type  $\alpha$ .

*Proof.* We only prove the logarithmic case, the power case using a similar path. If  $f$  is a *WeakWiener – Wintner function* of logarithmic type  $\alpha$  in  $L^2$  then by the lemma we have

$$\sup_x \sigma_f \star \delta_x[\delta, \delta] \leq \frac{C}{|\ln(\delta)|^{2(1+\alpha)}}. \text{ Elementary computations shows that this implies}$$

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^\beta d\sigma_f(t) < \infty$$

, for all  $0 < \beta < 2(1 + \alpha)$ . This shows (1). Part (2) follows without difficulty from the lemma.  $\square$

**Theorem 6.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system. Let  $U$  be its restriction to  $\mathcal{K}^\perp$ . Then*

- (1) *If  $(X, \mathcal{B}, \mu, T)$  is a WWW – dynamical – system of logarithmic type  $\alpha$  in  $L^2$  then  $\mathcal{C}_{(\ln)^\beta}(U) > 0$ , for all  $0 < \beta < 2(1 + \alpha)$ .*
- (2) *If  $\mathcal{C}_{(\ln)^{2(1+\alpha)}}(U) > 0$  then  $(X, \mathcal{B}, \mu, T)$  is a WWW-dynamical-system of logarithmic type  $\alpha$  in  $L^2$ .*
- (3) *If  $(X, \mathcal{B}, \mu, T)$  is a WWW – dynamical – system of power type  $\alpha$  in  $L^2$  then  $\mathcal{C}_\beta(U) > 0$  for all  $0 < \beta < 2\alpha$ .*
- (4) *If  $\mathcal{C}_{2\alpha}(U) > 0$  then  $(X, \mathcal{B}, \mu, T)$  is a WWW – dynamical – system of power type  $\alpha$  in  $L^2$ .*

*Proof.* Again we only show the logarithmic part. The notion of support that we will use in the proof (and later) is meant in the measurable sense with respect to the maximal spectral type and not as the topological support. We assume that we have in  $\mathcal{K}^\perp$  a dense set of *WeakWiener – Wintner function* of logarithmic type  $\alpha$  in  $L^2$ . We want to show that there

exists a *WeakWiener – Wintner function* of the same logarithmic type in  $L^2$  that realizes the maximal spectral type of  $U$ . We know that there exists a function  $g \in \mathcal{K}^\perp$  that realizes the maximal spectral type. We can assume that  $\|g\|_2 = 1$ . Let us consider a sequence of *WeakWiener – Wintner function* functions  $f_k$  of logarithmic type  $\alpha$  in  $L^2$  which converge to  $g$ . The union of the support of  $\sigma_{f_k}$  is equal to  $S$  the support of  $\sigma_g$ . Furthermore by the proposition 5 for each  $k$  we have a constant  $C_k$  that we can assume to be greater than 1, such that

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^\beta d\sigma_{f_k}(t) < C_k < \infty$$

, for all  $0 < \beta < 2(1 + \alpha)$ . The function

$$F = f_1 + \sum_{k=2}^{\infty} \frac{1}{C_k 2^k} E(\text{supp}(\sigma_{f_k}) \setminus (\cup_{j=1}^{k-1} \text{supp}(\sigma_{f_j}))) [f_k]$$

has a spectral measure  $\sigma_F$  whose support is equal to the union of the support of the measures  $\sigma_{f_k}$ . (Note that the spectral measures in this sum have disjoint support). To conclude we just have to observe that for a *WeakWiener – Wintner function*  $g$  of logarithmic type  $\alpha$ , for each borelian set  $B$  on  $\mathbb{T}$ , the function  $E(B)g$  is also a *WeakWiener – Wintner function* of the same type. The spectral measure of the function  $G = \frac{F}{\|F\|_2}$  is a probability measure realizing the maximal spectral type of  $U$ . One verifies easily that

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^\beta d\sigma_G(t) < \infty$$

This proves that the  $\beta$ -logarithmic capacity of  $U$  is finite. For the reverse implication let us assume that  $\mathcal{C}_{\ln 2(1+\alpha)}(U)$  is positive. Then by definition there exists a positive measure

$\rho$  supported on a set  $S$  such that

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^{2(1+\alpha)} d\rho(t) < \infty$$

This measure  $\rho$  by definition is equivalent to the maximal spectral type  $\sigma$  of  $U$ . The functions  $f$  with spectral measure of the form  $h d\rho$  where  $h \in L^\infty(\rho)$  are dense in  $\mathcal{K}^\perp$ . For each one of these functions we have

$$\sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^{2(1+\alpha)} d\sigma_f(t) \leq \|h\|_\infty \sup_{x \in [0,1]} \int_0^1 \left( \ln \left( \frac{1}{|x-t|} \right) \right)^{2(1+\alpha)} d\rho.$$

By the proposition 5 those functions  $f$  are *WeakWiener – Wintner function* of logarithmic type  $\alpha$  in  $L^2$ . This ends the proof of the theorem.

□

An immediate consequence of this theorem is the following corollary.

**Corollary 1.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system. Let  $S$  be support of the maximal spectral type of  $U = T|_{\mathcal{K}}$ . Then*

- (1) *If  $(X, \mathcal{B}, \mu, T)$  is a WWW – dynamical – system of logarithmic type  $\alpha$  in  $L^2$  then the  $\beta$ -logarithmic capacity of  $S$ ,  $C_{\ln^\beta}(S)$  is positive for all  $0 < \beta < 2(1 + \alpha)$ .*
- (2) *If  $(X, \mathcal{B}, \mu, T)$  is a WWW – dynamical – system of power type  $\alpha$  in  $L^2$  then the  $\beta$  capacity of  $S$ ,  $C_\beta(S)$  is positive for all  $0 < \beta < 2\alpha$ .*

**3.3. An ergodic dynamical system which is a  $WWW$  – dynamical – system of logarithmic type  $-1/2$  in  $L^2$  but is not of power type  $\beta$  for all  $0 < \beta < 1$ .**

**Theorem 7.** *There exists a weakly mixing system which is a Wiener–Wintner–dynamical–system of logarithmic type  $-1/2$  in  $L^2$  which is not of power type  $\beta$  for all  $0 < \beta < 1$ .*

*Proof.* The example is a Gauss dynamical system. We start by exhibiting a set  $P$  of Hausdorff measure zero but of positive logarithmic capacity. The starting point for the construction of such a set is standard (see [SZ] or [KS]) . It consists of a symmetric Cantor type subset of  $[0,1]$  formed of points of the form

$$x = (t_1(1 - a_1) + t_2a_1(1 - a_2) + \dots + t_k a_1 \dots a_{k-1}(1 - a_k) + \dots$$

where the  $0 < a_k < \frac{1}{2}$  are the successive ratios of dissection of the intervals in three parts proportional to  $(a_k, 1 - 2a_k, a_k)$  and the  $t_k$  are 0 or 1. In [SZ] the following properties are shown;

- a) The logarithmic capacity of  $P$  is positive if and only if the series

$$\sum_1^{\infty} \frac{1}{2^k} \log\left(\frac{1}{a_k}\right)$$

converges.

- b) The  $\alpha$  capacity of  $P$  is positive if and only if the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k (a_1 \dots a_k)^\alpha}$$

converges.

Following these properties if we pick  $a_k = \frac{1}{k+2}$  then simple calculations show that the series

$$\sum_1^{\infty} \frac{1}{2^k} \log\left(\frac{1}{a_k}\right)$$

converges while the series

$$\sum_{k=1}^{\infty} \frac{1}{2^k (a_1 \dots a_k)^\alpha}$$

has a generic term equivalent to  $\frac{[(k+2)!]^\alpha}{2^k}$  and thus diverges for all  $\alpha$ . Hence we have  $C_\alpha(P) = 0$  for all  $0 < \alpha < 1$  and  $C_{\ln}(P) > 0$

By Frostman's theorem (see [KS] for instance ) the Hausdorff dimension of  $P$  is also equal to zero as it is equal to the sup of those  $\beta$  for which  $C_\beta(P) > 0$ .

As the logarithmic capacity of  $P$  is positive there exists a positive continuous probability measure  $m$  concentrated on  $P$  (i.e. with support  $M$  a subset of  $P$ ) such that

$$\sup_{x \in [0,1]} \int_0^1 \ln \left( \frac{1}{|t-x|} \right) dm < \infty$$

We can symmetrize the measure  $m$  to obtain a measure  $\sigma$  with a symmetric support  $Q$ . We normalize  $\sigma$  to make it a probability measure. As  $Q$  is a subset of  $P$  its  $\alpha$  capacity is equal to zero for all  $0 < \alpha < \frac{1}{2}$  while its logarithmic capacity is positive.

The construction of the system  $(X, \mathcal{B}, \mu, T)$  is classical. One considers the Gaussian stationary process generated by the measure  $\sigma$  (see [CFS] for more details). The maximal spectral type of this Gaussian system on the orthocomplement of the constant functions is given by

$$e^\sigma = \sum_{n=1}^{\infty} \sigma^{(n)} / n!$$

(The Kronecker factor is trivial in this case) Its support is  $S = \bigcup_{n=1}^{\infty} Q_n$  where  $Q_n = Q + \dots + Q$ ,  $n$  times. We consider the closed subset  $H_{\sigma}$  of functions  $f$  whose spectral measures is absolutely continuous with respect to  $\sigma$ . None of these functions can be of power type  $\alpha$  in  $L^2$ ; if not we could find a positive measure  $\sigma_f$  with support included in  $Q$  for which

$$\sup_{x \in [0,1]} \int_0^1 \left( \frac{1}{|x-t|^{\beta}} \right) d\sigma_f(t) < \infty$$

for all  $0 < \beta < 2\alpha$ . In view of the definition 4 this would yield a contradiction with the zero capacity of the set  $Q$ . Thus for each  $0 < \beta < 1$  we can not find in  $\mathcal{K}^{\perp}$  a dense set of Weak Wiener-Wintner function of power type  $\beta$  in  $L^2$ . If not the projection of such functions onto  $H_{\sigma}$  would create *WeakWiener – Wintner function* of the same power type. Thus the system is not a *WWW – dynamical – system* of power type  $\beta$  for all  $\beta$ .

It remains to show that the logarithmic capacity of this system is positive. What we know so far is that  $C_{\ln}(P)$  is positive . This implies that  $C_{\ln}(S)$  is also positive as  $S$  contains  $P$ . But this does not say that the logarithmic capacity of  $T$  is positive. To be able to make such a statement we need to find a function  $F$  whose spectral measure  $\sigma_F$  realizes the maximal spectral type and such that

$$\sup_{x \in [0,1]} \int_0^1 \ln \left( \frac{1}{|t-x|} \right) d\sigma_F < \infty$$

In the definitions of capacities one can notice that one can take as well the periodic distance  $|x-t|^{*} = \inf_{s \in \mathbb{Z}} |x-t-s|$  . The definition of convolution gives immediately for  $f$  non-negative  $\int_0^1 f(|x-t|^{*}) d\sigma^{(n)}(t) \leq \sup_y \int_0^1 f(|y-t|^{*}) d\sigma(t)$  as  $\sigma$  is a probability measure.

We conclude that

$$\int_0^1 \ln\left(\frac{1}{|x-t|^*}\right) de^\sigma(t) \leq e \cdot \sup_y \int_0^1 \ln\left(\frac{1}{|y-t|^*}\right) d\sigma(t),$$

and

$$\sup_x \int_0^1 \ln\left(\frac{1}{|x-t|^*}\right) de^\sigma(t) \leq e \cdot \sup_y \int_0^1 \ln\left(\frac{1}{|y-t|^*}\right) d\sigma(t).$$

This shows that the logarithmic capacity of  $T$  is positive as  $e^\sigma = \sigma_F$  for some function  $F$ .

□

### Remarks

Frostman's theorem is used in [L] to produce an example of a weakly mixing dynamical system (a Gauss system) which is not of power type  $\alpha$  in  $L^2$  for any  $0 < \alpha < 1$ . Our example is not of power type  $\beta$  in  $L^2$  for any  $0 < \beta < 1$  but it is of logarithmic class  $\alpha$  in  $L^2$ . These two examples were obtained independently.

### 3.4. WWW – dynamical – systems of power type $\alpha$ in $L^p$ and a.e double recurrence.

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In this subsection we want to give an estimate of the double recurrence averages in terms

of the quantity  $\sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_p$

**Theorem 8.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system and  $p$  a real number,  $1/2 \leq p < \infty$  and  $a_N = \sup_\varepsilon \left\| \frac{1}{N} \sum_{n=1}^N f(T^n x) e^{2\pi i n \varepsilon} \right\|_{2p}$ . Then for all  $f_1 \in \mathcal{K}^\perp \cap L^{2p}$  and  $f_2 \in L^\infty$ ,  $\|f_2\|_\infty \leq 1$ , we have*

$$\left\| \frac{1}{N} \sum_{n=1}^N f_1 \circ T^n f_2 \circ T^{2n} \right\|_1 \leq C \cdot (N^{\frac{1}{2p}} \cdot a_N)^{\frac{2}{3}}$$

for each positive integer  $N$ . In particular if  $f_1$  is a *WeakWiener–Wintner function* in  $L^{2p}$  of power type  $\alpha$  such that  $2p > \frac{1}{\alpha}$  then there exists  $\theta > 0$  such that  $\left\| \frac{1}{N} \sum_{n=1}^N f_1 \circ T^n f_2 \circ T^{2n} \right\|_1 \leq \frac{C}{N^\theta}$  for all positive integer  $N$ .

*Proof.* One can derive from the arguments used in the first lines of the proof of theorem 2 the following inequality

$$\left\| \sup_{\varepsilon} \left| \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right| \right\|_p^p \leq CN \sup_{\varepsilon} \left\| \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_p^p$$

. Combining it to formula (7) in [A1] we obtain easily a proof of theorem 9. □

### Remarks

(1) The inequality

$$\left\| \sup_{\varepsilon} \left| \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right| \right\|_p^p \leq CN \sup_{\varepsilon} \left\| \sum_{n=1}^N f \circ T^n e^{2\pi i n \varepsilon} \right\|_p^p$$

for any function  $f \in L^p$  is valid for  $p > 0$ . The estimate in theorem 9 seems only relevant when  $p > 1$ : this is because already for  $p=1$  there are no *WeakWiener – Wintner function* in  $L^2$  of power-type  $\alpha > \frac{1}{2}$ . This can be seen by the following inequalities

$$\sup_t \left\| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n t} \right\|_2 \geq \left( \int \int \left| \frac{1}{N} \sum_{n=1}^N f \circ T^n e^{2\pi i n t} \right|^2 d\mu dt \right)^{1/2} = \frac{\|f\|_2}{N^{1/2}}.$$

(2) It is easy to check that most examples already known of Wiener Wintner dynamical systems are *WWW – dynamical – system* of power type  $\alpha$  in  $L^{2p}$  with  $2p > \frac{1}{\alpha}$ . In fact all *WeakWiener – Wintner function* of power type  $\alpha$  in  $L^\infty$  trivially satisfy

the condition  $2p > \frac{1}{\alpha}$ . Examples of such systems are a family of skew products in [A1] and Morse dynamical system [L]. These systems are in fact Wiener Wintner dynamical systems of power  $\alpha$  in  $L^\infty$ .

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