POINTWISE CONVERGENCE OF AVERAGES ALONG CUBES

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Abstract. Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. We prove the pointwise convergence of the averages

$$\frac{1}{N^2}\sum_{n,m=0}^{N-1} f_1(T^n x)f_2(T^m x)f_3(T^{n+m}x)$$

and of similar averages with seven bounded functions.

1. Introduction

In [3], V. Bergelson generalized Khintchine's theorem [6] by proving the L^2 convergence of the averages

$$\frac{1}{N^2}\sum_{n,m=0}^{N-1} f_1(T^n x)f_2(T^m x)f_3(T^{n+m}x)$$

where the functions f_i are bounded measurable and $(X, \mathcal{B}, \mu, T)$ is a measure preserving system. In [1], B. Host and B. Kra extended his result by proving the L^2 convergence of the following averages

$$\frac{1}{N^3}\sum_{m,n,p=0}^{N-1} f_1(T^m x)f_2(T^n x)f_3(T^{m+n}x)f_4(T^p x)f_5(T^{m+p}x)f_6(T^{n+p}x)f_7(T^{m+n+p}x)$$

They also proved that if T is ergodic and all functions f_i are in the CL factor for T then the averages of these seven functions converge a.e.. The pointwise convergence on such factors is a consequence of A. Leibman's result [8]

We want to show that these averages actually converge a.e. by showing the a.e. convergence when one of the functions f_i belongs to $CL^\perp$.

Theorem 1. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system. If the functions f_i, $1 \le i \le 7$, are all bounded then the averages*

$$\frac{1}{N^3} \sum_{m,n,p=0}^{N-1} f_1(T^m x) f_2(T^n x) f_3(T^{m+n} x) f_4(T^p x) f_5(T^{m+p} x) f_6(T^{n+p} x) f_7(T^{m+n+p} x)$$

converge a.e.

A corollary of our method of proof is the following result.

Theorem 2. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system. Then*

1. *Its Kronecker factor is characteristic for the pointwise convergence of the averages* $\displaystyle \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$
2. *Its CL factor is characteristic for the pointwise convergence of the averages*

$$\frac{1}{N^3} \sum_{m,n,p=0}^{N-1} f_1(T^m x) f_2(T^n x) f_3(T^{m+n} x) f_4(T^p x) f_5(T^{m+p} x) f_6(T^{n+p} x) f_7(T^{m+n+p} x).$$

The notion of characteristic factor is originally due to H. Furstenberg. It is explicitly stated in [5]. In the weakly mixing case we have the following result.

Theorem 3. *Let $(X, \mathcal{B}, \mu, T)$ be a weakly mixing dynamical system. The averages $M_N(f_1, f_2, ..., f_{2^k-1})$ of $2^k - 1$ bounded functions f_i converge a.e. to $\displaystyle \prod_{i=1}^{2^k-1} \int f_i d\mu$ for all $k \ge 1$.*

2. Proofs

In the subsequent inequalities the constant C may change from one line to the other. It will depend only at time on the L^∞ norm of the functions f_j.

2.1. **Pointwise convergence for the averages of three functions.** We start by proving the pointwise convergence of the averages

$$M_N(f_1, f_2, f_3)(x) = \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

for f_i bounded and measurable functions. This will help illustrate the method. We assume without loss of generality that T is ergodic. We recall Bourgain's uniform Wiener Wintner ergodic result announced in [4].

Lemma 1. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and f a function in the ortho-complement of the Kronecker factor. Then for a.e. x we have $\lim_N \sup_t |\frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi int}| = 0$.*

Using this lemma we can prove the following

Theorem 4. *Let $(X, \mathcal{B}, \mu, T)$ be a measure preserving system and f_i, $1 \leq i \leq 3$ three bounded functions then the averages*

$$M_N(f_1, f_2, f_3)(x) = \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

converge a.e.

Proof. It is enough to show this convergence for ergodic measure preserving systems (using the ergodic decomposition). We have the following inequalities.

$$
\begin{aligned}
&|M_N(f_1, f_2, f_3)(x)|^2\\
&\leq \|f_1\|_\infty^2 \Big(\frac{1}{N}\sum_{n=0}^{N-1} \big|\frac{1}{N}\sum_{m=0}^{N-1} f_2(T^m x) f_3(T^{n+m}x)\big|^2\Big)\\
&\leq \|f_1\|_\infty^2 \frac{1}{N}\sum_{n=0}^{N-1}\left|\int \big(\sum_{m=0}^{N-1} f_2(T^m x)e^{-2\pi imt}\big)\big(\frac{1}{N}\sum_{m'=0}^{2(N-1)} f_3(T^{m'}x)e^{2\pi im't}\big).e^{2\pi int}dt\right|^2\\
&\leq \|f_1\|_\infty^2 \frac{1}{N}\int \left|\sum_{m=0}^{N-1} f_2(T^m x)e^{-2\pi imt}\right|^2 \left|\frac{1}{N}\sum_{m'=0}^{2(N-1)} f_3(T^{m'}x)e^{2\pi im't}\right|^2 dt\\
&\leq \frac{C}{N}\sup_t \left|\frac{1}{N}\sum_{m'=0}^{N-1} f_3(T^{m'}x)e^{2\pi im't}\right|^2 \int \big|\sum_{m=0}^{N-1} f_2(T^m x)e^{-2\pi imt}\big|^2 dt\\
&\leq C\sup_t \left|\frac{1}{N}\sum_{m'=0}^{N-1} f_3(T^{m'}x)e^{2\pi im't}\right|^2 \frac{1}{N}N\|f_2\|_\infty^2
\end{aligned}
$$

With the help of lemma 1 we can conclude that for f_3 in the orthocomplement of the Kronecker factor the averages $M_N(f_1, f_2, f_3)$ converge a.e. to zero.

If f_3 is one of the eigenfunctions for T with eigenvalue $e^{2\pi i\theta}$ then

$$M_N(f_1, f_2, f_3) = f_3\big(\frac{1}{N}\sum_{n=0}^{N-1} f_1(T^n x)e^{2\pi in\theta}\big)\big(\frac{1}{N}\sum_{m=0}^{N-1} f_2(T^m x)e^{2\pi im\theta}\big).$$

The convergence in this case follows from Birkhoff 's theorem applied to the product of T and the rotation θ. The convergence for a general function f_3 in the Kronecker factor follows now by linearity and approximation.

□

Remarks 1

- The proof of theorem 4 shows that if f_1 and f_2 are bounded functions and $P_{\mathcal{K}}$ denotes the projection onto the Kronecker factor of T then

$$\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\Big|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{m+n}x)\Big|^2 = \limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\Big|\frac{1}{N}\sum_{m=0}^{N-1} P_{\mathcal{K}}(f_1)(T^m x) P_{\mathcal{K}}(f_2)(T^{m+n}x)\Big|^2 \tag{1}$$

- The proof of theorem 4 actually shows that

$$\left(\frac{1}{N}\sum_{n=0}^{N-1}\Big|\frac{1}{N}\sum_{m=0}^{N-1} f_2(T^m x) f_3(T^{n+m}x)\Big|^2\right) \leq C \sup_t \left|\frac{1}{N}\sum_{m'=0}^{N-1} f_3(T^{m'}x)e^{2\pi i m' t}\right|^2 \|f_2\|_\infty^2. \tag{2}$$

  A similar estimate can be obtained with $\sup_t \left|\frac{1}{N}\sum_{m'=0}^{N-1} f_2(T^{m'}x)e^{2\pi i m' t}\right|^2$ if we focus instead on the function f_2.

2.2. **Pointwise convergence for the averages of seven functions.** As T is ergodic there exists in $\mathcal{K}$ an orthonormal basis of eigenfunctions g_j with modulus 1 corresponding to the eigenvalue $e^{2\pi i\theta_j}$ so that any function $G \in \mathcal{K}$ can be written as

$$G = \sum_{j=1}^{\infty}\Big(\int G.\overline{g_j}\,d\mu\Big)g_j. \tag{3}$$

In [2] it is shown that the CL factor is characteristic for the convergence in L^2 norm of the averages of seven functions. Functions in this factor are characterized by the seminorm $|||.|||_3$ such that

$$|||f|||_3^8 = \lim_H \frac{1}{H}\sum_{h=0}^{H-1} |||f.f\circ T^h|||_2^4 \tag{4}$$

where

$$|||f|||_2^4 = \lim_H \frac{1}{H}\sum_{h=0}^{H-1}\Big|\int f.f(T^h)\,d\mu\Big|^2. \tag{5}$$

A function $f \in CL^{\perp}$ if and only $|||f|||_3 = 0$.

Lemma 2. *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic dynamical system and $f \in L^\infty(\mu)$ then for all H positive integer we have*

$$\limsup_N \sup_t \Big|\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)e^{2\pi i n t}\Big|^2 \leq C\Big(\frac{1}{H} + \frac{1}{H}\sum_{h=1}^{H}\Big|\int f.\overline{f\circ T^h}d\mu\Big|\Big)$$

In particular we have

$$\limsup_N \sup_t \Big|\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)e^{2\pi i n t}\Big|^2 \leq C|||f|||_2^2. \tag{6}$$

Proof. Without loss of generality we can assume that the function f takes only real values. We apply van der Corput's inequality ([7]) . For $H < N$ we get

$$\sup_t \Big|\frac{1}{N}\sum_{n=0}^{N-1} f(T^n x)e^{2\pi i n t}\Big|^2 \leq C\Big(\frac{1}{H} + \frac{1}{H}\sum_{h=1}^{H}\Big|\frac{1}{N}\sum_{n=0}^{N-h} f(T^n x)f(T^{n+h}x)\Big|\Big)$$

Birkhoff's pointwise ergodic theorem allows us to obtain the first part of the lemma. For the second part we can use Cauchy Schwartz inequality to write that

$$\frac{1}{H}\sum_{h=1}^{H}\Big|\int f.f\circ T^h d\mu\Big| \leq \Big(\frac{1}{H}\sum_{h=1}^{H}\Big|\int f.f\circ T^h d\mu\Big|^2\Big)^{1/2}.$$

Now using the definition of $|||f|||_2$, (see (5)), we can end the proof of this lemma. □

The lemma that replaces the uniform Wiener Wintner ergodic theorem in the case of the averages of seven functions is the following.

Lemma 3. *If f_1 or f_2 is in $CL^\perp$ then for a.e. x*

$$\lim_N \frac{1}{N}\sum_{n=0}^{N-1}\sup_t\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{n+m}x)e^{2\pi i m t}\right|^2 = 0 \tag{7}$$

Proof. We can assume without loss of generalities that the functions are uniformly bounded by one. We use again van der Corput's inequality, [7]. For $(H+1)^2 < N$ we get

$$\begin{aligned}
&\sup_t \left| \frac{1}{N} \sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{n+m} x) e^{2\pi i m t} \right|^2 \\
&\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \left| \frac{1}{N} \sum_{m=0}^{N-h-1} f_1(T^m x) f_2(T^{m+n} x) \overline{f_1(T^{m+h} x) f_2(T^{m+n+h} x)} \right|
\end{aligned}$$

So recalling that the constant C may change from one line to another but remains an absolute constant we have,

$$\begin{aligned}
&\frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{n+m} x) e^{2\pi i m t} \right|^2 \\
&\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{m=0}^{N-h-1} f_1(T^m x) f_2(T^{m+n} x) \overline{f_1(T^{m+h} x) f_2(T^{m+n+h} x)} \right| \\
&\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{m+n} x) \overline{f_1(T^{m+h} x) f_2(T^{m+n+h} x)} \right. \\
&\left. - \sum_{m=N-h}^{N-1} f_1(T^m x) f_2(T^{m+n} x) \overline{f_1(T^{m+h} x) f_2(T^{m+n+h} x)} \right| \\
&\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{m+n} x) \overline{f_1(T^{m+h} x) f_2(T^{m+n+h} x)} \right| + \frac{C}{H} \sum_{h=1}^{H} \frac{1}{N} \sum_{n=0}^{N-1} \frac{h}{N} \\
&\leq \frac{C}{H} + \frac{C}{H} \sum_{h=1}^{H} \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{m+n} x) \overline{f_1(T^{m+h} x) f_2(T^{m+n+h} x)} \right|.
\end{aligned}$$

Thus using the inequality (or Cauchy Schwartz's inequality)

$$|\frac{1}{P} \sum_{p=1}^{P} u_p| \leq (\frac{1}{P} \sum_{p=1}^{P} |u_p|^2)^{1/2} \tag{8}$$

we obtain

$$\frac{1}{N}\sum_{n=0}^{N-1}\sup_t\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{n+m}x)e^{2\pi imt}\right|^2$$

$$\le \frac{C}{H}+\left(\frac{C}{H}\sum_{h=1}^{H}\left(\frac{1}{N}\sum_{n=0}^{N-1}\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{m+n}x)\overline{f_1(T^{m+h}x)f_2(T^{m+n+h}x)}\right|^2\right)\right)^{1/2}$$

Finally by applying the inequality (2) made after the Remarks 1 to the function $f_1.\overline{f_1\circ T^h}$ we get

$$\frac{1}{N}\sum_{n=0}^{N-1}\sup_t\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{n+m}x)e^{2\pi imt}\right|^2$$

$$\le \frac{C}{H}+\left(\frac{C}{H}\sum_{h=1}^{H}\left(\sup_t\left|\frac{1}{N}\sum_{m'=0}^{N-1}(f_1.\overline{f_1\circ T^h})(T^{m'}x)e^{2\pi im't}\right|^2\right)\right)^{1/2}$$

Now by using Lemma 2 and the inequality $\frac{1}{H}\sum_{h=1}^{H}|u_h|^2 \le \big(\frac{1}{H}\sum_{h=1}^{H}|u_h|^4\big)^{1/2}$ we obtain

$$\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\sup_t\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{n+m}x)e^{2\pi imt}\right|^2$$

$$\le \frac{C}{H}+\left(\frac{C}{H}\sum_{h=1}^{H}\limsup_N\sup_t\left|\frac{1}{N}\sum_{m'=0}^{N-1}(f_1.\overline{f_1\circ T^h})(T^{m'}x)e^{2\pi im't}\right|^2\right)^{1/2}$$

$$\le \frac{C}{H}+\left(\frac{C}{H}\sum_{h=1}^{H}|||f_1.\overline{f_1\circ T^h}|||_2^2\right)^{1/2}$$

$$\le \frac{C}{H}+\left(\frac{C}{H}\sum_{h=1}^{H}|||f_1.\overline{f_1\circ T^h}|||_2^2\right)^{1/2}$$

$$\le \frac{C}{H}+\left(\frac{C}{H}\sum_{h=1}^{H}|||f_1.\overline{f_1\circ T^h}|||_2^4\right)^{1/4}$$

Taking now the limit when H tends to ∞ we get the following estimate

$$\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\sup_t\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{n+m}x)e^{2\pi imt}\right|^2 \le C|||f_1|||_3^2 \tag{9}$$

Thus if we assume that $f_1 \in CL^{\perp}$ then $|||f_1|||_3 = 0$ and we obtain the equation (7). We have the same conclusion if one assumes that $f_2 \in CL^{\perp}$. $\square$

Using Lemma 3 we can now give a proof of theorem 1.

Proof. **Theorem 1**

$$|M_N(f_1, f_2, ..., f_7)|^2$$

$$= \left| \frac{1}{N^3} \sum_{p=0}^{N-1} f_1(T^p x) \sum_{n=0}^{N-1} f_2(T^n x) f_3(T^{p+n} x) \big(\sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m} x) f_6(T^{p+m} x) f_7(T^{n+m+p} x) \big) \right|^2$$

$$\leq \frac{1}{N^2} \sum_{p=0}^{N-1} \sum_{n=0}^{N-1} \|f_1\|_\infty^2 \|f_2\|_\infty^2 \|f_3\|_\infty^2 \left| \frac{1}{N} \sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m} x) f_6(T^{p+m} x) f_7(T^{p+n+m} x) \right|^2$$

$$= \frac{1}{N^2} \prod_{i=1}^{3} \|f_i\|_\infty^2 .$$

$$\sum_{n=0}^{N-1} \sum_{p=0}^{N-1} \left| \int \big(\sum_{m=0}^{(N-1)} f_4(T^m x) f_5(T^{n+m} x) e^{-2\pi i m t} \big) \big(\frac{1}{N} \sum_{m'=0}^{2(N-1)} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \big) . e^{2\pi i p t} dt \right|^2$$

$$\leq \frac{1}{N^2} \prod_{i=1}^{3} \|f_i\|_\infty^2 \sum_{n=0}^{N-1} \int \left| \sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m} x) e^{-2\pi i m t} \big) \big(\frac{1}{N} \sum_{m'=0}^{2(N-1)} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \big) \right|^2 dt$$

$$\leq \frac{C}{N^2} \prod_{i=1}^{3} \|f_i\|_\infty^2 \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \right|^2 N \prod_{j=4}^{5} \|f_j\|_\infty^2$$

$$= C \prod_{i=1}^{5} \|f_i\|_\infty^2 \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \right|^2$$

With the help of lemma 3 one can conclude that if f_6 or f_7 belong to $CL^{\perp}$ then the averages of these seven functions converge to zero. By using the symmetry of the sum of the averages with respect to n, m and p one can see that the averages will converge to zero if one of the functions $f_i \in CL^{\perp}$, $1 \leq i \leq 7$.

$\square$

Remarks 2

- The last steps of the proof of theorem 1 show that for bounded functions f_i, $4 \leq i \leq 7$ if we denote by $P_{CL}(f_i)$ their projection onto the CL factor then we have

$$\limsup_N \frac{1}{N^2}\sum_{n,p=0}^{N-1}\left|\frac{1}{N}\sum_{m=0}^{N-1} f_4(T^m x)f_5(T^{n+m}x)f_6(T^{p+m}x)f_7(T^{p+n+m}x)\right|^2$$
$$= \limsup_N \frac{1}{N^2}\sum_{n,p=0}^{N-1}\left|\frac{1}{N}\sum_{m=0}^{N-1} P_{CL}(f_4)(T^m x)P_{CL}(f_5)(T^{n+m}x)P_{CL}(f_6)(T^{p+m}x)P_{CL}(f_7)(T^{p+n+m}x)\right|^2. \tag{10}$$

- The proof of lemma 3 gives the following estimate

$$\limsup_N \frac{1}{N}\sum_{n=0}^{N-1}\sup_t\left|\frac{1}{N}\sum_{m=0}^{N-1} f_1(T^m x)f_2(T^{n+m}x)e^{2\pi imt}\right|^2 \leq CMin[|||f_1|||_3^2, |||f_2|||_3^2]. \tag{11}$$

2.3. **Proof of Theorem 2.** The proof is a consequence of the path used in establishing theorem 1. We have shown that if one of the functions $f_i \in CL^{\perp}$, $1 \leq i \leq 7$, then the averages converge pointwise to zero. This shows that the CL factor is characteristic for the pointwise convergence. For the averages of three functions the Kronecker factor is characterisitc for the pointwise convergence for the same reason.

2.4. **Proof of Theorem 3.** We list some properties and some notations. They may seem a bit complicated at first reading. So the reader may wish to first translate all these properties to the case of 15 functions.

(1) For each $k \geq 4$ we denote by

$$M_N(f_1, f_2, ..., f_{2^k-1})(x)$$

the averages of $2^k - 1$ bounded functions . We number the functions f_j so that those with $2^{k-1} \leq j \leq 2^k - 1$ are depending of the index i_k. For instance in the sum of 7 functions, the functions are f_j, $4 \leq j \leq 7$ and they appear in the sum $\sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m}x) f_6(T^{p+m}x) f_7(T^{p+n+m}x)$. In the case of 15 functions if we denote by p, n, k, m the indices i_1, i_2, i_3, i_4 then they appear in the sum

$$\sum_{m=0}^{N-1} f_8(T^m x) f_9(T^{n+m}x) \ldots f_{15}(T^{p+n+k+m}x)$$

We denote by $S_{N,(i_1,i_2,\ldots,i_k)}(f_{2^{k-1}}, \ldots, f_{2^k-1})(x)$ these terms depending on i_k. We can observe that each term $S_{N,(i_1,i_2,\ldots,i_k)}(f_{2^{k-1}}, \ldots, f_{2^k-1})(x)$ is the product of two groups of 2^{k-2} functions,

$$A_{N,(i_2,\ldots,i_{k-1},i_k)}(f_{2^{k-1}}, f_{2^{k-1}+1}, \ldots, f_{3.2^{k-2}})(x)$$

and

$$B_{N,(i_1,i_2,\ldots,i_k)}(f_{3.2^{k-2}+1}, \ldots, f_{2^k-1})(x)$$

such that the powers of T associated with each function in the second group are exactly those associated with the functions in the first group shifted by the index i_1. Similar decompositions can be obtained if one focus on shifted blocks by another index. One can observe that we could write

$$B_{N,(i_1,i_2,\ldots,i_k)}(f_{3.2^{k-2}+1}, \ldots, f_{2^k-1})(x) = A_{N,(i_2,\ldots,i_{k-1},i_k)}(f_{3.2^{k-2}+1}, \ldots, f_{2^k-1})(T^{i_1}x) \tag{12}$$

The interest in those terms in the numerator of $M_N(f_1, f_2, \ldots, f_{2^k-1})(x)$ rests also in the following

$$\begin{aligned} &|M_N(f_1, f_2, \ldots, f_{2^k-1})(x)|^2 \\ &\leq \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_1,\ldots,i_{k-1}=0}^{N-1} \Big|\frac{1}{N} \sum_{i_k=0}^{N-1} S_{N,(i_1,i_2,\ldots,i_k)}(f_{2^{k-1}}, \ldots, f_{2^k-1})(x)\Big|^2. \end{aligned} \tag{13}$$

(2) When T is weakly mixing the Kronecker and CL factors are trivial. Thus we have $P_K f_i = P_{CL}(f_i) = \int f_i d\mu$.

We want to prove theorem 3 by induction on k. We formulate our induction assumption.

Induction Assumption

We assume that the following properties hold for all bounded functions f_j, $1 \leq j \leq k-1$.

(1)

$$\limsup_N \frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \left|\frac{1}{N}\sum_{i_{k-1}=0}^{N-1} S_{N,(i_1,i_2,\dots,i_{k-1})}(f_{2^{k-2}},\dots,f_{2^{k-1}-1})(x)\right|^2$$
$$= \limsup_N \frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \prod_{j=2^{k-2}}^{2^{k-1}-1} \left|\int f_j d\mu\right|^2$$
$$= \prod_{j=2^{k-2}}^{2^{k-1}-1} \left|\int f_j d\mu\right|^2$$

(Compare these equalities to the equations (1) and (10) in the remarks after the proofs for three terms and seven terms).

(2) The averages of $2^{k-1}-1$ bounded functions converge a.e. to the product of the integrals of these functions.

We want to show that the same assumptions hold then for k. We can assume that all functions are real valued. First we want to establish the following lemma

Lemma 4. *If one of the* 2^{k-2} *functions* f_j, $3.2^{k-2}+1 \leq j \leq 2^k - 1$ *has zero integral then*

$$\lim_N \frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \sup_t \left|\frac{1}{N}\sum_{i_k=0}^{N-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1},\dots,f_{2^k-1})(x)e^{2\pi i i_k t}\right|^2 = 0 \tag{14}$$

Proof. As previously we apply Van der Corput lemma to each term

$$\sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1},\dots,f_{2^k-1})(x) e^{2\pi i i_k t} \right|^2$$

We have then for each $H < N$

$$\frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1},\dots,f_{2^k-1})(x) e^{2\pi i i_k t} \right|^2$$

$$\leq \frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} C.\Bigg(\frac{1}{H} + \frac{1}{H} \sum_{h=1}^{H}$$

$$\left| \frac{1}{N} \sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1} \circ T^h,\dots,f_{2^k-1}.f_{2^k-1} \circ T^h)(x) \right| \Bigg)$$

$$\leq C.\Bigg(\frac{1}{H} + \frac{1}{H} \sum_{h=1}^{H} \frac{1}{N^{k-2}}$$

$$\sum_{i_1,\dots,i_{k-2}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1} \circ T^h,\dots,f_{2^k-1}.f_{2^k-1} \circ T^h)(x) \right| \Bigg)$$

$$\leq C.\Bigg(\frac{1}{H} + \frac{1}{H} \sum_{h=1}^{H} \frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1}$$

$$\left| \frac{1}{N} \sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1} \circ T^h,\dots,f_{2^k-1}.f_{2^k-1} \circ T^h)(x) \right| \Bigg)$$

Then we estimate

$$\frac{1}{H} \sum_{h=1}^{H} \limsup_N \frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1} \circ T^h,\dots,f_{2^k-1}.f_{2^k-1} \circ T^h)(x) \right|$$

which by the equation (8) (in the proof of lemma 3) is less than

$$\frac{1}{H} \sum_{h=1}^{H} \limsup_N$$

$$\left(\frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1} \circ T^h,\dots,f_{2^k-1}.f_{2^k-1} \circ T^h)(x) \right|^2 \right)^{1/2}$$

Now using the first induction assumption we conclude that

$$\limsup_N \Big(\frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1}$$

$$\Big|\frac{1}{N}\sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1}\circ T^h, \dots, f_{2^k-1}.f_{2^k-1}\circ T^h)(x)\Big|^2\Big)^{1/2}$$

$$= (\prod_{j=2^{k-2}}^{2^{k-1}-1} |\int f_j.f_j\circ T^h|^2)^{1/2}$$

As one of the functions f_j let us say $g = f_{j_0}$ has integral zero and T is weakly mixing then the spectral measure σ_g is continuous . Thus we have

$$\lim_H \frac{1}{H}\sum_{h=1}^H |\int g.g\circ T^h d\mu\Big|^2 = 0$$

As the functions are bounded

$$\frac{1}{H}\sum_{h=1}^H \limsup_N$$

$$\Big(\frac{1}{N^{k-2}} \sum_{i_1,\dots,i_{k-2}=0}^{N-1} \Big|\frac{1}{N}\sum_{i_k=1}^{N-h-1} A_{N,(i_1,i_2,\dots,i_{k-2},i_k)}(f_{3.2^{k-2}+1}.f_{3.2^{k-2}+1}\circ T^h, \dots, f_{2^k-1}.f_{2^k-1}\circ T^h)(x)\Big|^2\Big)^{1/2}$$

$$\leq C.\frac{1}{H}\sum_{h=1}^h |\int g.g\circ T^h d\mu|$$

Taking now the limit with H we obtain a proof of the lemma. □

Remark 3 In the case of the averages of 15 functions the equation (14) in lemma 4 is

$$\lim_N \frac{1}{N^2}\sum_{p=0}^{N-1}\sum_{n=0}^{N-1}\sup_t \Big|\frac{1}{N}\sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m}x) f_6(T^{p+m}x) f_7(T^{p+n+m}x) e^{2\pi i m t}\Big|^2 = 0$$

End of the proof of theorem 3

We just need to prove the induction at step $l = k$. We consider then the averages of $2^k - 1$ functions f_j and we use the previous observations to write

$$
\begin{aligned}
&|M_N(f_1, f_2, ..., f_{2^k-1})(x)|^2 \\
&\leq \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_1,\dots,i_{k-1}=0}^{N-1} \Big|\frac{1}{N} \sum_{i_k=0}^{N-1} S_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, ..., f_{2^k-1})(x)\Big|^2
\end{aligned}
$$

Using the equation (14) we can write

$$
\begin{aligned}
&\Big|\frac{1}{N} \sum_{i_k=0}^{N-1} S_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, ..., f_{2^k-1})(x)\Big|^2 \\
&= \Big| \int \Big(\frac{1}{N} \sum_{i_k=0}^{N-1} A_{N,(i_2,\dots,i_{k-1},i_k)}(f_{2^{k-1}}, f_{2^{k-1}+1}, ..., f_{3.2^{k-2}})(x) e^{-2\pi i_k t}\Big) \\
&\Big(\sum_{i'_k=0}^{N-1} A_{N,(i_2,\dots,i_{k-1},i'_k)}(f_{3.2^{k-2}+1}, ..., f_{2^k-1})(x) e^{2\pi i'_k t}\Big).e^{2\pi i i_1 t} dt\Big|^2
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&|M_N(f_1, f_2, ..., f_{2^k-1})(x)|^2 \\
&\leq \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_1,\dots,i_{k-1}=0}^{N-1} \Big|\frac{1}{N} \sum_{i_k=0}^{N-1} S_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, ..., f_{2^k-1})(x)\Big|^2 \\
&\leq \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_2,\dots,i_{k-1}=0}^{N-1} \int \Big| \Big(\sum_{i_k=0}^{N-1} A_{N,(i_2,\dots,i_{k-1},i_k)}(f_{2^{k-1}}, f_{2^{k-1}+1}, ..., f_{3.2^{k-2}})(x) e^{-2\pi i_k t}\Big) \\
&\Big(\frac{1}{N} \sum_{i'_k=0}^{N-1} A_{N,(i_2,\dots,i_{k-1},i'_k)}(f_{3.2^{k-2}+1}, ..., f_{2^k-1})(x) e^{2\pi i'_k t}\Big)\Big|^2 dt \\
&\leq C \frac{1}{N^{k-2}} \sum_{i_2,\dots,i_{k-1}=0}^{N-1} \sup_t \Big|\frac{1}{N} \sum_{i'_k=0}^{N-1} A_{N,(i_2,\dots,i_{k-1},i'_k)}(f_{3.2^{k-2}+1}, ..., f_{2^k-1})(x) e^{2\pi i'_k t}\Big|^2
\end{aligned}
$$

By using Lemma 4 one can conclude that the averages $M_N(f_1, f_2, ..., f_{2^k-1})(x)$ converge a.e to zero when one of the functions f_j has a zero integral. (using the symmetry on the indices). From this one derives that the averages of $2^k - 1$ bounded functions converge to the product of the integral of the functions. This is part (2) of the induction assumption at level k. To end the proof of the theorem we just need to observe that the proof given for $l = k$ proves also the first assumption for k.

Remark 4

If one considers instead the averages

$$\frac{1}{(N-M)^2} \sum_{n,m=M}^{N} f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

where $(N - M)$ tends to ∞ then we do not have a.e. convergence in general while as shown in [3] and [1] we do have convergence in L^2 norm. For instance it is shown in [9] that for $\beta \geq 3$ the averages

$$\frac{1}{N^{\beta-1}} \sum_{n=N^\beta}^{(N+1)^\beta} f(T^n x)$$

do not converge a.e. even if f is the characteristic function of a set of positive measure. So in this case the Kronecker factor is characteristic for the L^2 norm but not for the pointwise convergence.

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