

# POINTWISE CONVERGENCE OF AVERAGES ALONG CUBES II

I. ASSANI

ABSTRACT. Let  $(X, \mathcal{B}, \mu, T)$  be a measure preserving system. We prove the pointwise convergence of averages along cubes of  $2^k - 1$  bounded and measurable functions for all  $k$ .

## 1. INTRODUCTION

Let  $(X, \mathcal{B}, \mu, T)$  be a dynamical system where  $T$  is a measure preserving transformation on the measure space  $(X, \mathcal{B}, \mu, T)$ . In [1] we proved the pointwise convergence of the averages

$$\frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

and of similar averages with seven bounded functions  $f_i$ . We also showed that if  $T$  is weakly mixing then similar averages for  $2^k - 1$  bounded functions converge a.e to the product of the integrals of the functions  $f_i$ . The averages of three functions were used in [3] to generalize Khintchine recurrence result [5]. In [2] B.Host and B.Kra proved that the averages of  $2^k - 1$  bounded functions converge in  $L^2$  norm. To achieve this result they identified increasing factors  $Z_k$ ,  $k = 0, 1, 2, \dots$  of ergodic dynamical systems and showed the following

- The averages of  $2^k - 1$  bounded functions converge a.e if each function belongs to the factor  $Z_{k-1}$ . They used for that a result of A. Leibman [7].
- The averages of  $2^k - 1$  functions converge in  $L^2$  norm

One consequence of their method is that for each  $k$  the factor  $Z_{k-1}$  is characteristic for the  $L^2$  norm of the averages of  $2^{k-1}$  functions. Let us note that  $Z_1$  is the Kronecker factor and

$Z_2$  the CL factor. The notion of characteristic factor is due to H. Furstenberg and can be found explicitly stated in [4]. Our main results are the following

**Theorem 1.** *Let  $(X, \mathcal{B}, \mu, T)$  be a measure preserving system . Then the averages of the cubes of  $2^k - 1$  functions converge a.e.*

One consequence of the path we use is the following

**Theorem 2.** *Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic dynamical system. For each  $k \geq 1$  the factors  $Z_{k-1}$  is characteristic for the pointwise convergence of the averages along the cubes of  $2^k - 1$  bounded and measurable functions.*

## 2. INEQUALITIES FROM THE AVERAGES OF THREE OR SEVEN FUNCTIONS

As shown in [2] the factors  $Z_k$  can be defined inductively by using the seminorms  $||| \cdot |||_k$  where

$$(1) \quad |||f|||_1 = \left| \int f d\mu \right|$$

$$(2) \quad \text{For every } k \geq 1$$

$$|||f|||_{k+1}^{2^{k+1}} = \lim_H \frac{1}{H} \sum_{h=1}^H |||f \cdot f \circ T^h|||_k^{2^k}$$

With the help of these semi norms factors are built with the property that for all  $f \in L^\infty$  we have  $\mathbb{E}(f|Z_k) = 0$  if and only if  $|||f|||_{k+1} = 0$ .

We mention a few inequalities that were used in [1] in the proof of the pointwise convergence of the averages of three and seven bounded functions. The constant  $C$  may change from one line to the other. But it will depend only at time on the  $L^\infty$  norm of the functions  $f_j$ . For all bounded functions  $f_i$ ,  $1 \leq i \leq 7$  we have

$$(1) \quad \left( \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} f_2(T^m x) f_3(T^{n+m} x) \right|^2 \right) \leq C \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_3(T^{m'} x) e^{2\pi i m' t} \right|^2 \|f_2\|_\infty^2.$$

$$(2) \quad \limsup_N \sup_t \left| \frac{1}{N} \sum_{n=0}^{N-1} f(T^n x) e^{2\pi i n t} \right|^2 \leq C \|f\|_2^2.$$

$$(3) \quad \frac{1}{N^2} \sum_{p=0}^{N-1} \sum_{n=0}^{N-1} \|f_1\|_\infty^2 \|f_2\|_\infty^2 \|f_3\|_\infty^2 \left| \frac{1}{N} \sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m} x) f_6(T^{p+m} x) f_7(T^{p+n+m} x) \right|^2 \\ \leq C \prod_{i=1}^5 \|f_i\|_\infty^2 \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \right|^2$$

$$(4) \quad \limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m=0}^{N-1} f_1(T^m x) f_2(T^{n+m} x) e^{2\pi i m t} \right|^2 \leq C \text{Min}[\|f_1\|_3^2, \|f_2\|_3^2].$$

With these inequalities we can explain the first induction step allowing to get the convergence of the averages of seven functions from the inequalities obtained for the averages of three functions. We hope that these explanations will make the proof of theorem 1 more transparent. The square of the averages of three functions

$$M_N(f_1, f_2, f_3)(x) = \frac{1}{N^2} \sum_{n,m=0}^{N-1} f_1(T^n x) f_2(T^m x) f_3(T^{n+m} x)$$

is bounded by

$$C \left( \frac{1}{N} \sum_{n=0}^{N-1} \left| \frac{1}{N} \sum_{m=0}^{N-1} f_2(T^m x) f_3(T^{n+m} x) \right|^2 \right).$$

which by the equation (1) is bounded by

$$C \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_3(T^{m'} x) e^{2\pi i m' t} \right|^2 \|f_2\|_\infty^2.$$

The equation (2) guarantees that the lim sup of this last quantity is equal to zero if

$$\|f_3\|_2 = 0$$

. This is the same of saying that the function  $f_3 \in \mathcal{K}^\perp$ . By using a similar path for the functions  $f_1$  and  $f_2$  one can see that the Kronecker factor is pointwise characteristic for the averages of three bounded functions.

The square of the averages of seven bounded functions  $f_i$ ,  $1 \leq i \leq 7$ , is bounded by

$$\frac{1}{N^2} \sum_{p=0}^{N-1} \sum_{n=0}^{N-1} \|f_1\|_\infty^2 \|f_2\|_\infty^2 \|f_3\|_\infty^2 \left| \frac{1}{N} \sum_{m=0}^{N-1} f_4(T^m x) f_5(T^{n+m} x) f_6(T^{p+m} x) f_7(T^{p+n+m} x) \right|^2.$$

By using the equation (3) this term is bounded by

$$\leq C \prod_{i=1}^5 \|f_i\|_\infty^2 \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \right|^2$$

Then the equation (4) applied to  $f_6$  and  $f_7$  shows that

$$\limsup_N \frac{1}{N} \sum_{n=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{m'=0}^{N-1} f_6(T^{m'} x) f_7(T^{n+m'} x) e^{2\pi i m' t} \right|^2 = 0,$$

if one of the functions  $f_6$  or  $f_7$  belongs to  $CL^\perp$ . The equation (4) is a consequence of the equation (1) and of the van der Corput's inequality. In fact it is by averaging with  $h$  means of functions of the form  $f \cdot f \circ T^h$  that the semi norms  $\|f\|_3$  appear. We are going to follow a similar path to prove theorem 1.

### 3. PROOF OF THEOREM 1

We will prove theorem 1 by induction on  $k$ . In [1] we proved that the averages of seven functions converge a.e. We showed that the  $Z_2 = CL$  factor was characteristic for the

pointwise convergence of seven functions. This establishes the first step of the induction process. We will use the same notation and some of the remarks made in [1].

- For each  $k \geq 4$  we denote by  $M_N(f_1, f_2, \dots, f_{2^k-1})$  the averages of  $2^k - 1$  bounded functions. Without loss of generality we assume that the functions are bounded by 1 in absolute value.
- The functions  $f_j$  are listed in such a way that those depending on the index  $i_k$  are indexed by those  $j$ ,  $2^{k-1} \leq j \leq 2^k - 1$ . The product of these terms depending on  $i_k$  is denoted by  $S_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x)$ . Each term  $S_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x)$  is the product of two groups of  $2^{k-2}$  functions denoted by

$$A_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, f_{2^{k-1}+1}, \dots, f_{3 \cdot 2^{k-2}}(x))$$

and

$$B_{N,(i_1,i_2,\dots,i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(x)$$

where the powers of  $T$  associated with each function in the second group are those appearing in the first group shifted by the index  $i_1$ . We have

$$B_{N,(i_1,i_2,\dots,i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(x) = A_{N,(i_1,i_2,\dots,i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(T^{i_1}x)$$

- We have also the inequality

$$(5) \quad |M_N(f_1, f_2, \dots, f_{2^k-1})(x)|^2 \leq \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_1, \dots, i_{k-1}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=0}^{N-1} S_{N,(i_1,i_2,\dots,i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x) \right|^2.$$

### Induction Assumption

We make the following assumption

For all bounded functions  $g_j$ ,  $3 \cdot 2^{k-2} + 1 \leq j \leq 2^k - 1$  we have

$$(6) \quad \limsup_N \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(g_{3 \cdot 2^{k-2}+1}, \dots, g_{2^k-1})(x) \right|^2 \\ \leq C \cdot \text{Min}_{\{3 \cdot 2^{k-2}+1 \leq j \leq 2^k-1\}} \|g_j\|_{k-2}^2.$$

As indicated above this assumption is shown to be true for  $k = 3, 4$  in [1]. We want to show that it also holds for  $k$ . To this end we have the following extension of lemma 4 in [1].

**Lemma 1.** *If one of the  $2^{k-2}$  functions  $f_j$ ,  $3 \cdot 2^{k-2} + 1 \leq j \leq 2^k - 1$  is in  $Z_{k-1}^\perp$  then*

$$(7) \quad \lim_N \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(x) e^{2\pi i i_k t} \right|^2 = 0$$

*Proof.* We use now the same path as in [1]. With Van der Corput lemma applied to each term

$$\sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(x) e^{2\pi i i_k t} \right|^2,$$

we have then for each  $(H + 1) \ll N$

$$\begin{aligned}
& \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2+1}}, \dots, f_{2^{k-1}})(x) e^{2\pi i i_k t} \right|^2 \\
& \leq C \cdot \left( \frac{1}{H} + \frac{1}{H} \sum_{h=1}^H \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \right. \\
& \quad \left. \left| \frac{1}{N} \sum_{i_k=0}^{N-h-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2+1}} \cdot f_{3 \cdot 2^{k-2+1}} \circ T^h, \dots, f_{2^{k-1}} \cdot f_{2^{k-1}} \circ T^h)(x) \right| \right) \\
& \leq C \cdot \left( \frac{1}{H} + \frac{1}{H} \sum_{h=1}^H \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \right. \\
& \quad \left. \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2+1}} \cdot f_{3 \cdot 2^{k-2+1}} \circ T^h, \dots, f_{2^{k-1}} \cdot f_{2^{k-1}} \circ T^h)(x) \right| \right) \\
& \leq C \cdot \left( \frac{1}{H} + \left( \frac{1}{H} \sum_{h=1}^H \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \right. \right. \\
& \quad \left. \left. \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2+1}} \cdot f_{3 \cdot 2^{k-2+1}} \circ T^h, \dots, f_{2^{k-1}} \cdot f_{2^{k-1}} \circ T^h)(x) \right|^2 \right)^{1/2} \right)
\end{aligned}$$

So by the induction assumption we have

$$\begin{aligned}
& \limsup_N \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2+1}}, \dots, f_{2^{k-1}})(x) e^{2\pi i i_k t} \right|^2 \\
& \leq C \cdot \left( \frac{1}{H} + \left( \frac{1}{H} \sum_{h=1}^H \limsup_N \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \right. \right. \\
& \quad \left. \left. \left| \frac{1}{N} \sum_{i_k=1}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2+1}} \cdot f_{3 \cdot 2^{k-2+1}} \circ T^h, \dots, f_{2^{k-1}} \cdot f_{2^{k-1}} \circ T^h)(x) \right|^2 \right)^{1/2} \right) \\
& \leq C \cdot \left( \frac{1}{H} + \left( \frac{1}{H} \sum_{h=1}^H \text{Min}_{\{3 \cdot 2^{k-2+1} \leq j \leq 2^{k-1}\}} \|f_j f_j \circ T^h\|_{k-2}^2 \right)^{1/2} \right)
\end{aligned}$$

By using the monotonicity in  $\alpha$  of the fractions  $(\frac{1}{P} \sum_{h=1}^H |u_h|^\alpha)^{1/\alpha}$ , we have

$$\begin{aligned} & \limsup_N \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^{k-1}})(x) e^{2\pi i i_k t} \right|^2 \\ & \leq C \cdot \left( \frac{1}{H} + \left( \frac{1}{H} \sum_{h=1}^H \text{Min}_{\{3 \cdot 2^{k-2}+1 \leq j \leq 2^{k-1}\}} \|f_j f_j \circ T^h\|_{k-2}^{2^{k-2}} \right)^{1/2^{k-2}} \right) \end{aligned}$$

By taking now the  $\limsup_H$  of the last term we get

$$\begin{aligned} (8) \quad & \limsup_N \frac{1}{N^{k-2}} \sum_{i_1, \dots, i_{k-2}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i_k=0}^{N-1} A_{N, (i_1, i_2, \dots, i_{k-2}, i_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^{k-1}})(x) e^{2\pi i i_k t} \right|^2 \\ & \leq C \cdot \text{Min}_{\{3 \cdot 2^{k-2}+1 \leq j \leq 2^{k-1}\}} \|f_j\|_{k-1}^2 \end{aligned}$$

Thus if one of the functions  $f_j$  belongs to  $\mathcal{Z}_{k-1}^\perp$  then the limit in the equation (7) is equal to zero. □

### End of the proof of theorem 1

We just need to finish the induction process the same way we did in [1] by proving the induction assumption for  $k$ . We consider the averages of  $2^k - 1$  functions  $f_j$ ,  $M_N(f_1, f_2, \dots, f_{2^k-1})(x)$ . With the inequality (5) we have

$$\begin{aligned} & |M_N(f_1, f_2, \dots, f_{2^k-1})(x)|^2 \\ & \leq \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_1, \dots, i_{k-1}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=0}^{N-1} S_{N, (i_1, i_2, \dots, i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x) \right|^2. \end{aligned}$$

By using the same method used to derive the equations (1) and (3) we get

$$\begin{aligned}
& \prod_{j=1}^{2^{k-1}-1} \|f_j\|_\infty^2 \frac{1}{N^{k-1}} \sum_{i_1, \dots, i_{k-1}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=0}^{N-1} S_{N, (i_1, i_2, \dots, i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x) \right|^2 \\
& \leq C \frac{1}{N^{k-2}} \sum_{i_2, \dots, i_{k-1}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i'_k=0}^{N-1} A_{N, (i_2, \dots, i_{k-1}, i'_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(x) e^{2\pi i'_k t} \right|^2
\end{aligned}$$

By using lemma 1 and (8) one concludes that

$$\begin{aligned}
& \limsup_N \frac{1}{N^{k-1}} \sum_{i_1, \dots, i_{k-1}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=0}^{N-1} S_{N, (i_1, i_2, \dots, i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x) \right|^2 \\
& \leq C \frac{1}{N^{k-2}} \sum_{i_2, \dots, i_{k-1}=0}^{N-1} \sup_t \left| \frac{1}{N} \sum_{i'_k=0}^{N-1} A_{N, (i_2, \dots, i_{k-1}, i'_k)}(f_{3 \cdot 2^{k-2}+1}, \dots, f_{2^k-1})(x) e^{2\pi i'_k t} \right|^2 \\
& \leq C \cdot \text{Min}_{\{3 \cdot 2^{k-2}+1 \leq j \leq 2^k-1\}} \|f_j\|_{k-1}^2
\end{aligned}$$

By symmetry on the indices  $i_1, i_2, \dots, i_k$  one obtains the following inequality for the  $2^{k-1}$  functions  $f_j$

$$\begin{aligned}
& \limsup_N \frac{1}{N^{k-1}} \sum_{i_1, \dots, i_{k-1}=0}^{N-1} \left| \frac{1}{N} \sum_{i_k=0}^{N-1} S_{N, (i_1, i_2, \dots, i_k)}(f_{2^{k-1}}, \dots, f_{2^k-1})(x) \right|^2 \\
& \leq C \text{Min}_{\{2^{k-1} \leq j \leq 2^k-1\}} \|f_j\|_{k-1}^2
\end{aligned}$$

By applying this last inequality to any set of  $2^{k-1}$  functions  $g_j$  that we can label from  $3 \cdot 2^{k-1} + 1$  to  $2^{k+1} - 1$  instead of 1 to  $2^k - 1$  we obtain our induction assumption for  $k$ . Thus the averages  $M_N(f_1, f_2, \dots, f_{2^k-1})(x)$  converge a.e. to zero if one of the functions  $f_j \in Z_{k-1}^\perp$  (using the symmetry of the indices). Combining this result with the pointwise convergence when all functions are in  $Z_{k-1}$  mentioned in [2], (see [7]) this ends the proof of theorem 1.

## 4. PROOF OF THEOREM 2

The proof of theorem 2 follows from the path we used. We showed that if one of the functions  $f_j$  is in the orthocomplement of the  $Z_{k-1}$  factor then the averages of these  $2^k - 1$  functions converge a.e to zero. Thus the limit is given by the pointwise convergence when all functions are in the factor  $Z_{k-1}$ .

## REFERENCES

- [1] **I. Assani**: "Pointwise convergence of averages along cubes", preprint.
- [2] **B. Host and B. Kra**: "Nonconventional ergodic averages and nilmanifolds", to appear in *Annals of Math.*
- [3] **V. Bergelson**: "The multifarious Poincare Recurrence theorem," *Descriptive Set Theory and Dynamical Systems*, Eds M. Foreman, A.S. Kechris, A. Louveau, B. Weiss. Cambridge University Press, New York (2000), 31-57.
- [4] **H. Furstenberg and B. Weiss**: "A mean ergodic theorem for  $\frac{1}{N} \sum_{n=1}^N f(T^n x)g(T^{n^2} x)$ ", *Convergence in Ergodic Theory and Probability*, Eds: Bergelson/March/Rosenblatt, Walter de Gruyter-Co, Berlin, New York (1996), 193-227.
- [5] **A. Y. Khintchine**: "Eine Verschärfung des Poincareschen "Wiederkehrrsatzes", " *Comp. Math.*, 1, (1934), 177-179.
- [6] **L. Kuipers and H. Niederreiter**: *Uniform Distribution of Sequences*. John Wiley & Sons, 1974.
- [7] **A. Leibman**: *Pointwise convergence of ergodic averages for polynomial sequences of rotations of a nilmanifold*: <http://www.math.ohio-state.edu/~leibman/preprints/>