DUALITY AND THE ONE-SIDED ERGODIC HILBERT TRANSFORM

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ABSTRACT. Let X_n be an iid sequence of random variables with finite p-moment, $1 < p < \infty$ and zero expectation defined on a probability measure space $(\Omega, \mathcal{A}, P)$ We prove that we can find a set of full measure Ω' such that for $\omega \in \Omega'$, for each $1 < r \leq \infty$, the random series

$$\sum_{n=1}^{\infty} \frac{X_n(\omega) g(S^n y)}{n}$$

converges a.e ν for all dynamical system $(Y, \mathcal{G}, \nu, S)$ and all $g \in L^r(\nu)$. We also characterize the functions $f \in L^1$ for which the averages

$$\sum_{n=1}^{\infty} \frac{f(T^n x) X_n(\omega)}{n}$$

converge a.e.(with respect to ω).

1. INTRODUCTION

In [A1] we proved that if X_n is an iid sequence of symmetric random variables with finite p-moment , for some $1 < p < \infty$ then for P a.e. (ω) the averages

$$\frac{1}{N} \sum_{n=1}^{N} X_n(\omega) g(S^n y)$$

converge a.e ν for every dynamical system $(Y, \mathcal{A}, \nu, S)$ for all $g \in L^r(\nu)$, $1 < r \leq \infty$. We noticed later in [A2] that the symmetry assumption could be removed.

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One of the novelties of the result in [A1] was the "break of duality". We mean by this the fact that the sequence X_n providing the random weight $X_n(\omega)$ and the function g belong to spaces L^p and L^r that go beyond the duality normally imposed by Holder's inequality. (we can have the pointwise convergence even if $1 < r < \frac{p}{p-1}$).

Here we look at the corresponding problem for the one-sided ergodic Hilbert transform given by the series

$$\sum_{n=1}^{\infty} \frac{X_n(\omega)g(S^n y)}{n}$$

This is motivated in part by a question raised in [CJR]. They asked if one could still have the break of duality for this one sided ergodic Hilbert transform. We will show in theorem 1 that the situation in this case is different from the Cesaro averages: we do not have a.e. convergence for $p = 1$ and $r = \infty$.

Theorem 1. *There exist symmetric L^1 random variables for which one can not find a set of ω with full measure for which the following series*

$$\sum_{n=1}^{\infty} \frac{X_n(\omega)g(S^n y)}{n}$$

converges a.e. ν for all dynamical systems $(Y, \mathcal{G}, \nu, S)$ and all $g \in L^\infty$.

Our second result is the following theorem that deals with the remaining values of p and r.

Theorem 2. *Let X_n be an iid sequence of random variables with finite p-moment, $1 < p < \infty$ and zero expectation defined on a probability measure space $(\Omega, \mathcal{A}, P)$. We can find a set* *of full measure* Ω' *such that for* $\omega \in \Omega'$ *, for each* $1 < r \leq \infty$*, the random series*

$$\sum_{n=1}^{\infty} \frac{X_n(\omega) g(S^n y)}{n}$$

converges a.e ν *for all dynamical system* $(Y, \mathcal{G}, \nu, S)$ *and all* $g \in L^r(\nu)$*,* $1 < r \leq \infty$*.*

We will give a proof of this result in the following section. In the final section we consider the "dual case" of the convergence of the series $\sum_{n=1}^{\infty} \frac{f(T^n x)}{n} X_n(\omega)$ where $f \in L^1(\mu)$ and X_n is a sequence of L^1 iid random variables with zero expectation. We will show that in this case we do not have convergence for all function $f \in L^1$. However the almost everywhere convergence of the series $\sum_{n=1}^{\infty} \frac{f(T^n x)}{n} X_n(\omega)$ holds as soon as $X_1 \in L^p$ for some $1 < p \leq \infty$.

2. PROOF OF THEOREM 1

If theorem 1 was false then by applying such a result to the family of rotations $(\mathbb{T}, \mathcal{B}(\mathbb{T}), m, S_\alpha)$ on the one dimensional torus $\mathbb{T}$ where $S_\alpha(y) = y + \alpha$ and to the function $e^{2\pi i y}$ we would have the a.e. convergence for all t of the random Fourier series

$$\sum_{n=1}^{\infty} \frac{X_n(\omega) e^{2\pi i n t}}{n}$$

. By Billard result [Ka] we would have the uniform convergence a.e. of this random Fourier series. But such a conclusion has been shown to be false in [T] as soon as

$$E(|X_1| \log\log(\max e^e, |X_1|) = \infty$$

.

3. PROOF OF THEOREM 2

We list in the following proposition and the next theorem some of the elements we will use.

Proposition 3. *Let X_n be a iid sequence of random variables with mean zero such that $E(|X_1|^p) < \infty$ for some p, $1 < p < \infty$. Define the variables $X_n^{'} = X_n \mathbf{1}_{\{|X_n| \leq n\}} - \mathbb{E}[X_n \mathbf{1}_{\{|X_n| \leq n\}}]$ Then for β large enough we have the following:*

1. *For P a.e the sequence $\frac{1}{N^\beta} \sum_{n=N^\beta}^{(N+1)^\beta} |X_n^{'}(\omega)||g|(S^n y)$ converges to zero for all dynamical systems $(Y, \mathcal{G}, \nu, S)$ and all $g \in L^r(\nu)$, $1 < r \leq \infty$.*
2. *For a function $g \in L^r(\nu)$ we have $\sum_{N=1}^{\infty} \sum_{i=[N^{\frac{1}{r-1}}]}^{\infty} i\nu A_i < \infty$ where $A_i = \{y : i - 1 < g \leq i\}$.*

Proof. In [A1] by using lemma 5 we proved that for P a.e. the sequence $\frac{1}{N^\beta} \sum_{n=N^\beta}^{(N+1)^\beta} |X_n(\omega)||g|(S^n y)$ converges to zero for all dynamical systems $(Y, \mathcal{G}, \nu, S)$ and all $g \in L^r(\nu)$, $1 < r \leq \infty$. The details are given on page 148 of the paper.

As $|X_n \mathbf{1}_{\{|X_n| \leq n\}}| \leq |X_n|$ we only need to show that

$$\frac{1}{N^\beta} \sum_{N^\beta}^{(N+1)^\beta} |\mathbb{E}[X_n \mathbf{1}_{\{|X_n| \leq n\}}]||g(S^n y)|$$

converges ν a.e. to zero. The variables X_n having zero mean we have

$$|\mathbb{E}[X_n \mathbf{1}_{\{|X_n| \leq n\}}]| = |\mathbb{E}[X_n \mathbf{1}_{\{|X_n| > n\}}]|$$

. By Holder and Chebyshev inequalities the last absolute value is less than

$$\|X_n\|_p \frac{1}{n^{\frac{p}{q}}} \|X_n\|_p^{\frac{p}{q}} = \|X_1\|_p^p \frac{1}{n^{\frac{p}{q}}}$$

Therefore we can bound the $L^1(\nu)$ norm of

$$\frac{1}{N^\beta}\sum_{N^\beta}^{(N+1)^\beta}|\mathbb{E}[X_n\mathbf{1}_{\{|X_n|\leq n\}}]||g(S^n y)|$$

by

$$C\|g\|_1\frac{1}{N^\beta}\,\frac{1}{N^{\beta\frac{p}{q}}}\,N^{\beta-1}=C\frac{\|g\|_1}{n^{\beta\frac{p}{q}+1}}$$

Part (1) follows now from the summability of the series.

Part(2) can be obtained by partial summation. It is done on page 144 of [A1]. □

One of the main ingredients for the proof of theorem 1 is a consequence of the proof of theorem 4.3.1 in Salem-Zygmund [SZ]

Theorem 4. *There exists a finite constant C such that for each positive integer T for all Y_n independent sequence, mean zero L^2 random variables we have*

$$\|\sup_t|\sum_{n=1}^{T}Y_n e^{2\pi int}|\|_2\leq C\sqrt{(\sum_{n=1}^{T}\mathbb{E}(|Y_n|^2)\sqrt{\log T}} \tag{1}$$

A proof of this theorem was shown to us by M. Weber and is indicated in [A1].

We proceed now with the proof of theorem 2.

We first introduce some notations; We fix $1<p<\infty$, $1<r\leq 2$ (it is enough to prove the theorem for r in this range) and consider a sequence X_n of iid, mean zero random variables with $\mathbb{E}(|X_1|^p)=1$. We introduce the variables $X_n^{'}=X_n\mathbf{1}_{\{|X_n|\leq n\}}-\mathbb{E}[X_n\mathbf{1}_{\{|X_n|\leq n\}}]$ and the partial sums $S_M^{'}(t)=\displaystyle\sum_{k=1}^{M}\frac{X_k^{'}}{k}e^{2\pi ikt}$ and $S_M(t)=\displaystyle\sum_{k=1}^{M}\frac{X_k}{k}e^{2\pi ikt}$. We fix β large enough.

Lemma 1. *There exists a finite constant C such that for all positive integer N we have*

$$\left\|\sup_t\left|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{X_n'}{n}e^{2\pi int}\right|\right\|_1 \leq \frac{C}{N^{\beta\frac{p-1}{2}+1/2}}\sqrt{\log N} \tag{2}$$

Proof. We can apply (1) to the variables Y_n defined as $Y_n = \frac{X_n'}{n}$ for $[N^\beta] < n \leq [(N+1)^\beta]$ and $Y_n = 0$ for $1 \leq n \leq [N^\beta]$. We have

$$\left\|\sup_t|\sum_{n=[N^\beta]}^{[(N+1)^\beta]}Y_ne^{2\pi int}|\right\|_1 \leq C\sqrt{\sum_{n=[N^\beta]}^{[(N+1)^\beta]}\frac{\mathbb{E}[|X_n'|^2]}{n^2}}\sqrt{\log N}$$

. As

$$\mathbb{E}[|X_n'|^2] \leq 2n^{2-p}\mathbb{E}[|X_n|^p] \leq 2n^{2-p}$$

we have

$$\left\|\sup_t|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{X_n'}{n}e^{2\pi int}|\right\|_1 \leq C(\sum_{n=[N^\beta]}^{[(N+1)^\beta]}\frac{1}{n^p})^{1/2}\sqrt{\log N}$$

The right hand side of the previous inequality being less than

$$\frac{C}{N^{p\beta/2}}N^{\frac{\beta-1}{2}}\sqrt{\log N}$$

the lemma follows. □

Lemma 2. *For $\frac{\beta(p-1)+1}{2} > \gamma+1$ we have for P a.e. ω*

$$\sum_{N=1}^{\infty}N^\gamma\sup_t|S'_{[(N+1)^\beta]}(t) - S'_{[N^\beta]}(t)| < \infty \tag{3}$$

Proof. Using lemma 1 we have

$$\sum_{N=1}^{\infty}\left\|\left\|\sup_t|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{X_n'}{n}e^{2\pi int}\right\|\right\|_1 \leq \sum_{N=1}^{\infty}\frac{C}{N^{\beta\frac{p-1}{2}+1/2}}N^\gamma\sqrt{\log N}$$

The assumption $\frac{\beta(p-1)+1}{2} > \gamma+1$ makes the right hand side summable. □

We consider now a dynamical system $(Y, \mathcal{G}, \nu, S)$ and a fixed nonnegative function $g \in L^r(\nu)$. We write g as $g = g_N^1 + g_N^2$ where $g_N^1 = g \wedge N^{\frac{1}{r-1}}$. We denote by $\sigma_{g_N^1}$ the spectral measure of the function g_N^1.

We have for ω fixed in the set of full measure given by lemma 2 and for $\gamma = \frac{2-r}{2(1-r)}$.

$$\begin{aligned}
&\sum_{N=1}^{\infty} \left\| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{X_n'(\omega)}{n} g_N^1(S^n y) \right\|_1 \\
&\qquad \leq \sum_{N=1}^{\infty} \left\| N^{\frac{1}{r-1}} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{X_n'(\omega)}{n} g_N^1(S^n y) \right\|_2 \\
&\qquad = \sum_{N=1}^{\infty} \left(\int |N^{\frac{1}{r-1}} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{X_n'(\omega)}{n} e^{2\pi i n t}| d\sigma_{g_N^1}(t) \right)^{1/2} \\
&\qquad \leq \sum_{N=1}^{\infty} \sup_t \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{X_n'(\omega)}{n} e^{2\pi i n t} \right| \|g_N^1\|_2 \\
&\qquad = \sum_{N=1}^{\infty} \sup_t \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{X_n'(\omega)}{n} e^{2\pi i n t} \right| \left(\int |g_N^1(y)|^2 d\nu(y) \right)^{1/2} \\
&\qquad \leq \sum_{N=1}^{\infty} N^{\frac{2-r}{2(1-r)}} \sup_t \left| \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{X_n'(\omega)}{n} e^{2\pi i n t} \right| \|g\|_r^{r/2} \\
&\qquad < \infty
\end{aligned}$$

We can also estimate the same partial sums with the sequence g_N^2 Using part (2) of Proposition 3, we obtain:

$$\begin{aligned}
&\sum_{N=1}^{\infty}\left\|\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{X'_n(\omega)}{n}g_N^2(S^ny)\right\|_1 \\
&\qquad \leq \sum_{N=1}^{\infty}\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}\frac{|X'_n(\omega)|}{n}\int g_N^2(S^ny)d\nu(y) \\
&\qquad \leq \left(\sup_N \frac{\sum_{n=[N^\beta]+1}^{[(N+1)^\beta]}|X'_n(\omega)|}{N^\beta}\right)\sum_{N=1}^{\infty}\sum_{i=[N^{\frac{1}{r-1}}]}^{\infty} i\nu(A_i) \\
&\qquad < \infty
\end{aligned}$$

From these two series of estimates we can conclude that the sequence

$$H^\omega_{[M^\beta]} = \sum_{n=1}^{[M^\beta]}\frac{X'_n(\omega)}{n}g(S^ny)$$

converges ν a.e. .

To establish the a.e. convergence of the series

$$\sum_{n=1}^{\infty}\frac{X'_n(\omega)}{n}g(S^ny)$$

we just need to show that

$$\lim_N \sup_{[N^\beta]+1\leq n<[(N+1)^\beta]}|H_n - H_{[N^\beta]}| = 0$$

ν a.e.. But using part 1 of Proposition 2 we have

$$\begin{aligned} \lim_N \sup_{[N^\beta]+1 \le n < [(N+1)^\beta]} |H_n - H_{[N^\beta]}| & \\ & \le \lim_N \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} \frac{|X_n^{'}(\omega)|}{n} |g(S^n y)| \\ & \le \lim_N \frac{1}{N^\beta} \sum_{n=[N^\beta]+1}^{[(N+1)^\beta]} |X_n^{'}(\omega)| |g(S^n y| \\ & = 0 \end{aligned}$$

Therefore we have shown that for a.e ω the series

$$\sum_{n=1}^{\infty} \frac{X_n^{'}(\omega)}{n} g(S^n y)$$

converges ν a.e for all dynamical systems $(Y, \mathcal{G}, \nu, S)$ and all $g \in L^r(\nu)$.

To conclude it remains then to prove that the series

$$\sum_{n=1}^{\infty} \frac{X_n(\omega) - X_n^{'}(\omega)}{n} g(S^n y)$$

converges ν a.e.. It will be enough to show that each one of the following series converges ν a.e..

$$\sum_{n=1}^{\infty} \frac{X_n(\omega) \mathbf{1}_{\{|X_n| > n\}}}{n} g(S^n y) \tag{4}$$

and

$$\sum_{n=1}^{\infty} \frac{(-\mathbb{E}[X_n \mathbf{1}_{|X_n| > n}])}{n} g(S^n y) \tag{5}$$

As the series $\sum_{n=1}^{\infty} P(\{|X_n| > n\})$ converges because $\mathbb{E}[|X_1|] < \infty$, P a.e. ω belongs then only to finitely many of the sets $\{|X_n| > n\}$. Hence the series in (4) converges ν a.e. as we will only have finitely non zero terms in the series.

The series in (5) is absolutely summable because it is dominated in absolute value by the series

$$C \sum_{n=1}^{\infty} \frac{|g(S^n y|}{n^{1+\frac{p}{q}}}$$

(we showed in the proof of part 1 of proposition 2 that $|\mathbb{E}[X_n \mathbf{1}_{\{|X_n|>n\}}| \leq \frac{C}{n^{1+\frac{p}{q}}}$).

This ends the proof of theorem 1.

Remarks

(1) One can also obtain the pointwise convergence for the fractional one sided ergodic Hilbert transform

$$\sum_{n=1}^{\infty} \frac{X_n(\omega)}{n^{\delta}} g(S^n y)$$

where $0 < c(p,r) < \delta \leq 1$ and $c(p,r)$ is a constant depending only on r and p.

(2) In summary we have shown that we do not have pointwise convergence for $p = 1$ and for $r = \infty$. On the other hand theorem 1 shows that we have pointwise convergence if $1 < p \leq \infty$ and $1 < r \leq \infty$. At the present time we do not know if we have convergence for $1 < p \leq \infty$ and $r = 1$.

(3) G. Cohen pointed out to us that he obtained with similar arguments a proof of theorem 2.

4. The dual case

In this section we look at the convergence of the series $\sum_{n=1}^{\infty}\frac{f(T^nx)X_n(y)}{n}$. In this case one can obtain a more complete characterization of those values of p and r for which the convergence holds as the next theorem shows.

Theorem 5. *Let $(X,\mathcal{B},\mu,T)$ be an ergodic measure preserving system and $1\le p\le\infty$*

1. ***The Case** $p=1$*

   There exists $f\in L^1(\mu)$ for which one can not find a set of x with full measure off which the series $\sum_{n=1}^{\infty}\frac{f(T^nx)X_n(\omega)}{n}$ converges for all iid sequence X_n of L^1 random variables with zero expectation.

2. ***The case** $1<p\le\infty$*

   For μ a.e x, for all $X_n\in L^p$ iid random variables with finite expectation, the series $\sum_{n=1}^{\infty}\frac{f(T^nx)X_n(\omega)}{n}$ converges for a.e ω

Proof. **The case** $p=1$.

The proof of theorem 8 in [A3] actually shows that the following are equivalent for a nonnegative sequence c_n such that $\lim_n\frac{c_n}{n}=0$.

1. $\sup_n\frac{\{\#k;\frac{c_k}{k}\ge\frac{1}{n}\}}{n}<\infty$
2. for any iid sequence X_n of L^1 random variables the sequence $\frac{c_nX_n(\omega)}{n}$ converges a.e. to zero.

The equivalence is stated for any dynamical system $(Y,\mathcal{G},\nu,S)$ but a close look at the proof shows that we used shifted random variables of X_1 (given at the bottom of page 243), thus L^1 iid random variables.

In [ABM] we showed that we could find in any ergodic dynamical system $(X, \mathcal{A}, \mu, T)$ a function $f \in L^1$ for which $\sup_n \dfrac{\{\#k; \frac{f(T^k x)}{k} \geq \frac{1}{n}\}}{n}$ is not finite a.e.. By the remark above this means that for each x in a set of positive measure X_f we can find a sequence Y_n^x of L^1 iid random variables such that

$$\frac{f(T^n x) Y_n^x(\omega)}{n}$$

does not converge a.e. (ω) to zero. As a consequence of this and the a.e. convergence to zero of the sequence $\frac{f(T^n x)}{n}$ we can conclude that the sequence

$$\frac{f(T^n x) X_n^x(\omega)}{n}$$

does not converge a.e. (ω) to zero, where $X_n^x = Y_n^x - \mathbb{E}(Y_n^x)$. This implies that the series

$$\sum_{n=1}^{\infty} \frac{f(T^n x) X_n^x(\omega)}{n}$$

does not converge a.e., as its generic term does not converge to zero.

The case $1 < p \leq \infty$. We will use for this the three series theorem. By this theorem we will have the pointwise convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{f(T^n x) X_n(\omega)}{n}$ if for each $c > 0$ the following series converge

(1) $\displaystyle\sum_{n=1}^{\infty} P[|\frac{f(T^n x) X_n(\omega)}{n}| > c]$

(2) $\displaystyle\sum_{n=1}^{\infty} \int_{\{|X_n(\omega) < \frac{c.n}{|f(T^n x)|}\}} X_n(\omega). \frac{f(T^n x)}{n} dP(\omega)$

(3) $\displaystyle\sum_{n=1}^{\infty} \int_{\{|X_n(\omega) < \frac{c.n}{|f(T^n x)|}\}} [X_n(\omega). \frac{f(T^n x)}{n}]^2 dP(\omega)$

We can observe that if some of the terms $f(T^n x)$ are zero then we would have less terms to consider in these series. So without loss of generality we can assume that $|f(T^k x)| > 0$ for each k.

We will also use the result in [J] which says that for each $f \in L^1(\mu)$ for each $1 < s < \infty$, for a.e. x, the series

$$\sum_{n=1}^{\infty} [\frac{f(T^n x)}{n}]^s \tag{6}$$

converges.

For the first series (1) we have the following estimate

$$\begin{aligned} &\sum_{n=1}^{\infty} P\big[|X_n(\omega)||\frac{f(T^n x)}{n}| > c\big] \\ &\qquad\qquad \leq \sum_{n=1}^{\infty} P\big[|X_n(\omega)| > \frac{c.n}{|f(T^n x)|}\big] \\ &\qquad\qquad \leq \sum_{n=1}^{\infty} [\frac{f(T^n x}{n}]^p \|X_1\|_p^p < \infty \end{aligned}$$

by using Chebyshev 's inequality in L^p and (6).

We can write the second series as

$$\sum_{n=1}^{\infty} \int X_n(\omega) \frac{f(T^n x)}{n} - \sum_{n=1}^{\infty} \int_{\{|X_n(\omega) \geq \frac{c.n}{|f(T^n x)|}\}} X_n(\omega). \frac{f(T^n x)}{n} dP(\omega).$$

As the first term converges because the variables X_n have zero expectation, we can focus only on the second term. We have for any $1 < r < \infty$, ($r* = \frac{r}{r-1}$.)

$$\sum_{n=1}^{\infty} \int_{\{|X_n(\omega)| \geq \frac{c.n}{|f(T^n x)|}\}} \frac{f(T^n x) X_n(\omega)}{n} dP(\omega) = \sum_{n=1}^{\infty} \frac{f(T^n x)}{n} \int_{\{|X_n(\omega)| \geq \frac{c.n}{|f(T^n x)|}\}} X_n(\omega) dP(\omega)$$

$$\leq \big(\sum_{n=1}^{\infty} |\frac{f(T^n x)}{n}|^{r*}\big)^{1/r*} \big(\sum_{n=1}^{\infty} |\int_{\{|X_n(\omega)| \geq \frac{c.n}{|f(T^n x)|}\}} X_n(\omega) dP(\omega)|^r\big)^{1/r}$$

by using Holder's inequality. Again by (6) we just need to focus our attention on

$$\big(\sum_{n=1}^{\infty} |\int_{\{|X_n(\omega)| \geq \frac{c.n}{|f(T^n x)|}\}} X_n(\omega) dP(\omega)|^r\big)^{1/r}.$$

By using this time Holder's inequality for integrals and Chebyshev's inequality in L^p we have

$$
\begin{aligned}
&\Big(\sum_{n=1}^{\infty}\Big|\int_{\{|X_n(\omega)|\geq \frac{c.n}{|f(T^n x)|}\}} X_n(\omega)dP(\omega)\Big|^r\Big)^{1/r} \\
&\qquad\leq \Big(\sum_{n=1}^{\infty}\big(P\big[|X_1| > \frac{c.n}{|f(T^n x|}\big]\big)^{\frac{r(p-1)}{p}}\|X_1\|_p^r\Big)^{1/r} \\
&\qquad\leq \Big(\big(\sum_{n=1}^{\infty}\frac{|f(T^n x)|}{cn}\|X_1\|_p\big)^{r(p-1)}\Big)^{1/r}\|X_1\|_p \\
&\qquad= \Big(\sum_{n=1}^{\infty}\big(\frac{|f(T^n x)|}{cn}\big)^{r(p-1)}\Big)^{1/r}\|X_1\|_p^p
\end{aligned}
$$

which is finite if we choose r large enough so that $r(p-1) > 1$. This choice of r is consistent with the implicit assumption made on $r*$ namely that $r* > 1$.

Finally for the third series we have

$$
\sum_{n=1}^{\infty}\int_{\{|X_n(\omega)<\frac{c.n}{|f(T^n x)|}\}}\big[X_n(\omega).\frac{f(T^n x)}{n}\big]^2 dP(\omega)
$$

$$
=\sum_{n=1}^{\infty}\big(\frac{f(T^n x)}{n}\big)^2\int_{\{|X_n(\omega)<\frac{c.n}{|f(T^n x)|}\}}|X_n|^p|X_n|^{2-p}dP(\omega)
$$

$$
\leq \sum_{n=1}^{\infty}\big(\frac{f(T^n x)}{n}\big)^2\frac{c^{2-p}n^{2-p}}{|f(T^n x)|^{2-p}}\int|X_n|^p dP(\omega) = \sum_{n=1}^{\infty}\big|\frac{f(T^n x)}{n}\big|^p c^{2-p}\|X_1\|_p^p < \infty
$$

This ends the proof of this theorem. □

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